# Integrability in the AdS/CFT Correspondence

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# Overview

AdS/CFT correspondence provides a fascinating link between conformal quantum field theories <u>without</u> gravity and string theory <u>with</u> (both classical and quantized) gravity

Major (recent) activities:

1. Integrability in AdS/CFT: Spectral problem solved (?)
2. Scattering amplitudes in maximally susy Yang-Mills, relation to light-like Wilson loops and dual superconformal symmetry
3. Novel well understood $AdS_4/CFT_3$ duality pair: IIA strings on $AdS_4 \times CP^3$ dual to max susy 3d Chern-Simons theory [Aharony,Bergmann,Jafferis,Maldacena '08]
4. "Applied" AdS/CFT: AdS/QCD and meson spectroscopy, applications to quark-gluon-plasma, condensed matter systems
5. Use AdS/CFT as tool to study quantum gravity
6. ...

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1. Integrability in AdS/CFT: Spectral problem solved (?)
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1: This talk!

2: E. Sokatchev's talk!

# $\mathcal{N}=4$ super Yang Mills: The simplest interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N\times N$ matrices
  - Gluons: $A_\mu$, $\mu=0,1,2,3$, $\Delta=1$
  - 6 real scalars: $\Phi_I$, $I=1,\dots,6$, $\Delta=1$
  - $4\times 4$ real fermions: $\Psi_{\alpha\,A}$, $\bar{\Psi}^{\dot\alpha}_{A}$ ,$\alpha,\dot\alpha=1,2$. $A=1,2,3,4$, $\Delta=3/2$
  - Covariant derivative: $\mathcal{D}_\mu=\partial_\mu-i[A_\mu,*]$, $\Delta=1$
- Action: Unique model completely fixed by SUSY

$$S=\frac{1}{{g_{\mathrm{YM}}}^2}\int d^4x\,\mathrm{Tr}\Big[\tfrac{1}{4}F^2_{\mu\nu}+\tfrac{1}{2}(D_\mu\Phi_i)^2-\tfrac{1}{4}[\Phi_I,\Phi_J][\Phi_I,\Phi_J]+$$
$$\bar{\Psi}^{A}_{\dot\alpha}\sigma^{\dot\alpha\beta}_{\mu}\mathcal{D}^\mu\Psi_{\beta\,A}-\tfrac{i}{2}\Psi_{\alpha\,A}\sigma^{AB}_{I}\epsilon^{\alpha\beta}\,[\Phi^I,\Psi_{\beta\,B}]-\tfrac{i}{2}\bar{\Psi}_{\dot\alpha\,A}\sigma^{AB}_{I}\epsilon^{\dot\alpha\dot\beta}\,[\Phi^I,\bar{\Psi}_{\dot\beta\,B}]\Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}}=0}$: Quantum Conformal Field Theory, 2 parameters: $N$ & $\lambda={g_{\mathrm{YM}}}^2N$
- Shall consider 't Hooft planar limit: $N\to\infty$ with $\lambda$ fixed.

# Most symmetric 4d gauge theory!

- Symmetry: $\mathfrak{so}(2,4)\otimes\mathfrak{so}(6)\subset\mathfrak{psu}(2,2|4)$

  Poincaré: $p^{\alpha\dot\alpha}=p_\mu\,(\sigma^\mu)^{\dot\alpha\beta},\quad m_{\alpha\beta},\quad \bar m_{\dot\alpha\dot\beta}$

  Conformal: $k_{\alpha\dot\alpha},\quad d\qquad (c$ : central charge$)$

  R-symmetry: $r_{AB}$

  Poincaré Susy: $q^{\alpha\,A},\bar q^{\dot\alpha}_A$ Conformal Susy: $s_{\alpha\,A},\bar s^A_{\dot\alpha}$

- $4+4$ Supermatrix notation $\bar A=(\alpha,\dot\alpha|A)$

$$J^{\bar A}{}_{\bar B}=\begin{pmatrix} m^\alpha{}_\beta-\frac12\,\delta^\alpha_\beta\,(d+\frac12 c) & k^\alpha{}_{\dot\beta} & s^\alpha{}_B\\ p^{\dot\alpha}{}_\beta & \overline{m}^{\dot\alpha}{}_{\dot\beta}+\frac12\,\delta^{\dot\alpha}_{\dot\beta}\,(d-\frac12 c) & \bar q^{\dot\alpha}{}_B\\ q^A{}_\beta & \bar s^A{}_{\dot\beta} & -r^A{}_B-\frac14\delta^A_B\,c\end{pmatrix}$$

- Algebra:

$$[J_i{}^{\bar A}{}_{\bar B}\,,\,J_j{}^{\bar C}{}_{\bar D}\}=\delta_{ij}[\delta^{\bar C}_{\bar B}\,J_i{}^{\bar A}{}_{\bar D}-(-1)^{(|\bar A|+|\bar B|)(\bar C|+|\bar D|)}\delta^{\bar A}_{\bar D}\,J_i{}^{\bar C}{}_{\bar B}]$$

# Observables

- **Local operators:** $\mathcal{O}_n(x) = \mathrm{Tr}[\mathcal{W}_1\,\mathcal{W}_2\ldots\mathcal{W}_n]$ with $\mathcal{W}_i \in \{\mathcal{D}^k\Phi, \mathcal{D}^k\Psi, \mathcal{D}^k F\}$

  2 point fct: $\langle \mathcal{O}_a(x_1)\,\mathcal{O}_b(x_2)\rangle = \dfrac{\delta_{ab}}{(x_1-x_2)^{2\,\Delta_a(\lambda)}}$ $\quad \Delta_a(\lambda)$ Scaling Dims

  3 point fct: $\Big\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_2)\Big\rangle = \dfrac{c_{abc}(\lambda)}{x_{12}^{\Delta_a+\Delta_b-\Delta_c}\,x_{23}^{\Delta_b+\Delta_c-\Delta_a}\,x_{31}^{\Delta_c+\Delta_a-\Delta_b}}$

  $n$-point functions follow from OPE

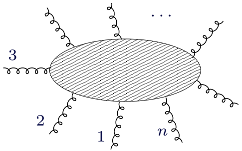
- **Wilson loops:**

$$\mathcal{W}_C = \Big\langle \mathrm{Tr}P\exp i\oint_C ds\,(\dot{x}^\mu A_\mu + i|\dot{x}|\,\theta^I\,\Phi_I)\Big\rangle$$

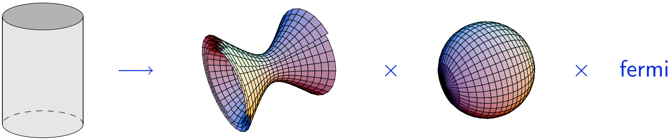
- **Scattering amplitudes:**

$$\mathcal{A}_n(\{p_i,h_i,a_i\};\lambda) = \left\{\begin{matrix}\text{UV-finite}\\ \text{IR-divergent}\end{matrix}\right\}$$

  helicities: $h_i \in \{0, \pm\frac{1}{2}, \pm 1\}$



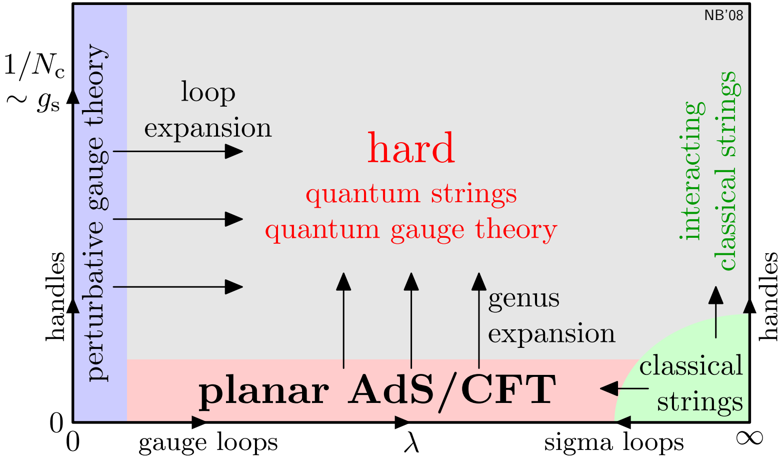
# Superstring in $AdS_5 \times S^5$




$$\boxed{I = \sqrt{\lambda} \int d\tau\, d\sigma \left[ G^{(\mathrm{AdS}_5)}_{mn}\, \partial_a X^m \partial^a X^n + G^{(\mathrm{S}_5)}_{mn}\, \partial_a Y^m \partial^a Y^n + \text{fermions} \right]}$$

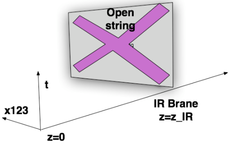
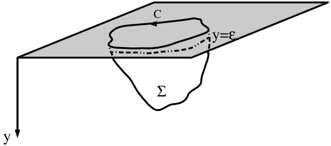
- $ds^2_{AdS} = R^2 \dfrac{dx^2_{3+1} + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, classical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- **Quantization unsolved!**
- String coupling constant $g_s = \frac{\lambda}{4\pi N} \to 0$ in 't Hooft limit
- Isometries: $\mathfrak{so}(2,4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$
- Include fermions: Formulate as $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ supercoset model [Metsaev,Tseytlin]

# The AdS/CFT landscape




(Picture by N. Beisert)

# Gauge Theory - String Theory Dictionary of Observables

| | | | |
|---|---|---|---|
| $\Delta_a(\lambda)$ spectrum of scaling dimensions | $\Leftrightarrow$ | $E(\lambda)$ string excitation spectrum | solved (?) |
| $c_{abc}(\lambda)$ structure constants | $(\Leftrightarrow)$ | Only SUGRA: $\mathcal{Z}_{AdS}[\phi\vert_{\partial \text{AdS}} = J] = \mathcal{Z}_{\text{CFT}}[J]$ | |
| $\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda)$ | $(\Leftrightarrow)$ |   | open string amplitude |
| Wilson loop $\mathcal{W}_C$ | $\Leftrightarrow$ |   | minimal surface |

# The spectral problem and integrability

# The spectral problem of AdS/CFT: Symmetry

$$\begin{array}{ccc} \text{string energy} & \leftrightarrow & \text{scaling dimension} \\ E(\lambda) & = & \Delta(\lambda) \end{array}$$

- String states resp. gauge theory local operators classified by conserved Cartan charges $(E, S_1, S_2)$ of $\mathfrak{so}(2,4)$ (energy and "spins") and $(J_1, J_2, J_3)$ of $\mathfrak{so}(6)$ ("angular momenta")
- Geometrical picture:

$$\begin{array}{ll} AdS_5 : & -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -R^2 \\ S^5 : & Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = R^2 \end{array}$$

$Z_0 + i\,Z_5 = \rho_3\, e^{i\,t}$, $Z_1 + i\,Z_2 = \rho_1\, e^{i\,\alpha_1}$, $Z_3 + i\,Z_4 = \rho_2\, e^{i\,\alpha_2}$:
3 angles $t$, $\alpha_1$, $\alpha_2$ $\longrightarrow$ 3 conserved quantities $E$, $S_1$, $S_2$. $E$ is the energy.

$Y_1 + i\,Y_2 = r_1\, e^{i\,\phi_1}$, $Y_3 + i\,Y_4 = r_2\, e^{i\,\phi_2}$, $Y_5 + i\,Y_6 = r_3\, e^{i\,\phi}$:
3 angles $\phi_1$, $\phi_2$, $\phi$ $\longrightarrow$ 3 conserved angular momenta $J_1$, $J_2$, $J_3$.

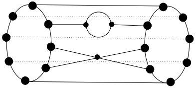
# The spectral problem of AdS/CFT: Expansions

- $AdS_5 \times S^5$ string spectrum

$$\widehat{H}\,|\psi\rangle_{\text{String}} = E(\lambda)\,|\psi\rangle_{\text{String}} \qquad E(\lambda) = ?$$

- Central observables in gauge theory: Correlation functions of composite operators
  e.g. $\mathcal{O}_\alpha(x) = \mathrm{Tr}[\,\Phi_{i_1}\,\Phi_{i_2}\ldots\Phi_{i_n}\,]$
  Two-point functions determined by scaling dimensions $\Delta(\lambda, N)$

$$\langle \mathcal{O}_\alpha(x)\,\mathcal{O}_\beta(y)\rangle = \frac{\delta_{\alpha\beta}}{(x-y)^{2\,\Delta(\lambda,N)}}$$



May be computed perturbatively in gauge theory: Loops $(\lambda)$ + genera $(1/N^2)$

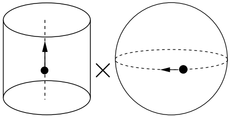
$$\boxed{\Delta = \Delta^0 + \lambda(\Delta_0^1 + \tfrac{1}{N^2}\Delta_1^1 + \ldots) + \lambda^2(\Delta_0^2 + \tfrac{1}{N^2}\Delta_1^2 + \ldots) + \ldots \quad \stackrel{!}{=} E(\lambda, g_s)}$$

$\uparrow$ here: $\Delta^0 = n = \#$ of scalars

# Spinning string solutions vs. Local Operators

- **Example 1:** Rotating point particle on $S^5$

$$\boxed{t=\kappa\,\tau \quad \rho=0 \quad \gamma=\frac{\pi}{2} \quad \phi_1=\kappa\,\tau \quad \phi_2=\phi_3=\psi=0}$$



$$ds^2_{\mathrm{AdS}_5}=d\rho^2-\cosh^2\rho\,dt^2+\sinh^2\rho\,d\Omega_3$$

$$ds^2_{\mathrm{S}_5}=d\gamma^2+\cos^2\gamma\,d\phi_3^2+\sin^2\gamma\,(d\psi^2+\cos^2\psi\,d\phi_1^2+\sin^2\psi\,d\phi_2^2)$$

- Solves eqs. of motion & Virasoro constraint (here $S_1, S_2, J_2, J_3=0$)

$$E=\sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\dot{X}_0=\sqrt{\lambda}\,\kappa \qquad \boxed{E=J} \quad \text{classical}$$

$$J_1=\sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,(Y_1\,\dot{Y}_2-Y_2\,\dot{Y}_1)=\sqrt{\lambda}\,\kappa=:J$$

- Dual gauge theory operator: $Z=\Phi_1+i\Phi_2$ [Berenstein,Madacena,Nastase]

$$\mathcal{O}_J=\mathrm{Tr}[Z^J] \qquad \text{with} \quad \Delta(\lambda)=\Delta(\lambda=0)=J$$

- Actually classical picture only good for $J\to\infty$

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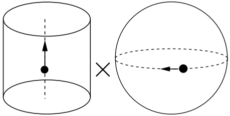

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$$J_1 = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,(Y_1\,\dot{Y}_2 - Y_2\,\dot{Y}_1) = \sqrt{\lambda}\,\kappa =: J$$

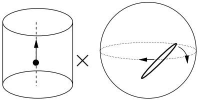
- Quantum fluctuations around solution: $\boxed{X^\mu = X^\mu_{\mathrm{sol}}(\tau) + \frac{1}{\lambda^{1/4}}\,x^\mu(\tau,\sigma)}$

  $\Rightarrow$ Energy: $\quad E = \sqrt{\lambda}\,\kappa + E_2(\kappa) + \dfrac{1}{\sqrt{\lambda}}E_4(\kappa) + \ldots$

- **Example 2:** Folded spinning string: $\boxed{J_1 \ \& \ J_2 \neq 0}$ [Frolov,Tseytlin]

Ansatz:

$$t = \kappa\,\tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2}$$



$$\phi_1 = \omega_1\,\tau \quad \phi_2 = \omega_2\,\tau \quad \phi_3 = 0 \quad \psi = \psi(\sigma)$$

- Solution yields Charges and Energy

$$J_1 = \sqrt{\lambda}\,\omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi}\,\cos^2\psi(\sigma) \qquad J_2 = \sqrt{\lambda}\,\omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi}\,\sin^2\psi(\sigma)\,.$$

$$E = J\left(1 + \frac{\lambda}{J^2}\,E_2 + \frac{\lambda^2}{J^4}\,E_4 + \dots\right) \qquad J = J_1 + J_2$$

where $E_2 = \frac{2}{\pi^2}\,K(q_0)\Big(E(q_0) - (1-q_0)\,K(q_0)\Big)$ with $\frac{J_2}{J} = 1 - \frac{E(q_0)}{K(q_0)}$

Similarly $E_{2l}$: $l$-loop gauge theory prediction.

- Dual gauge theory operator: $Z = \Phi_1 + i\Phi_2 \quad W = \Phi_3 + i\Phi_4$

$\boxed{\mathcal{O}_J = \mathrm{Tr}[Z^{J_1}W^{J_2}] + \dots}$ with $\Delta(\lambda) = J_1 + J_2 + \lambda\,\Delta_1(J_1,J_2) + \dots$

Indeed $\lim_{J\to\infty}\Delta_1(J_1,J_2) = \frac{\lambda}{J^2}\,E_2$!

# Operator mixing and the dilatation operator

- Composite operators are renormalized and operators with degenerate $(\Delta^0, S_1, S_2; J_1, J_2, J_3)$ charges mix:

$$\mathcal{O}^A_{\mathrm{ren}} = \mathcal{Z}^A{}_B\, \mathcal{O}^B_{\mathrm{bare}}$$

Mixing matrix (**dilatation operator** $\hat{=}\ d \in \mathfrak{psu}(2,2|4)$)

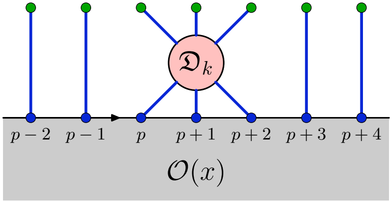
$$(\mathfrak{D})^A{}_B = (\mathcal{Z}^{-1})^A{}_C\, \frac{d}{\log\Lambda}\, \mathcal{Z}^C{}_B$$

- Acts on composite operators: $\mathcal{O}(x) = \mathrm{Tr}[\,\Phi_{i_1}\,\Phi_{i_2}\ldots\Phi_{i_n}\,]$

Eigenvalues yield scaling dims. $\boxed{\mathfrak{D}\circ\mathcal{O}(x) = \Delta_{\mathcal{O}}\,\mathcal{O}(x)}$ [Beisert,Kristjansen,Plefka,Staudacher]

- $\mathfrak{D}$ is perturbatively defined:

$$\mathfrak{D} = \Delta^0 + \sum_{l=1}^{\infty} \lambda^l\, \mathfrak{D}_{l+1} \qquad \mathfrak{D}_k = \sum_{p=1}^{L}$$

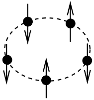


# The dilatation operator and spin chains

- For simplicity: Consider $\mathfrak{su}(2)$ subsector

$$Z = \Phi_1 + i\,\Phi_2 \qquad \text{and} \qquad W = \Phi_3 + i\,\Phi_4$$

  & consider operators $\mathcal{O} = \mathrm{Tr}(\text{word in } Z \;\&\; W)$

- Spin chain picture: Operator $\mathrm{Tr}(ZZWZW) \;\hat{=}\;$ State $|\downarrow\downarrow\uparrow\downarrow\uparrow\rangle \quad \hat{=}$ 

- One-loop structure: $\mathfrak{D}_2$ is Hamiltonian of the Heisenberg spin chain, an integrable system! [Minahan,Zarembo]

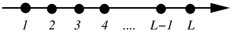
$$\boxed{\mathfrak{D}_2 = 2\sum_{l=1}^{L}(1 - P_{l,l+1})} \qquad P_{i,j} : \text{permutation operator}$$

- Ground state: $|\downarrow\downarrow\ldots\downarrow\rangle \;\hat{=}\; \mathrm{Tr}(Z^J)$ with $\Delta = 0$
- Excitations: "Magnons": $|m\rangle = |\underbrace{\uparrow\downarrow\ldots\downarrow\uparrow}_{m}\downarrow\rangle\rangle \;\hat{=}\; \mathrm{Tr}(WZ^mWZ^{J-m})$

# The coordinate Bethe Ansatz 1

- How to diagonalize $\widehat{\mathcal{D}}$? Open up the trace (no cyclicity)

$$\mathrm{Tr}(WZZW\ldots WZ) \longrightarrow |WZZW\ldots WZ\rangle$$




- Consider two-magnon states $|\psi\rangle = \sum_{1\leq x_1<x_2\leq L} \psi(x_1,x_2)\,|\ldots Z\underset{x \stackrel{\uparrow}{=} x_1}{W}Z\ldots Z\underset{x \stackrel{\uparrow}{=} x_2}{W}Z\ldots\rangle$

- One-loop Schrödinger eq. $\boxed{\sum_{i=1}^{L}(1-P_{i,i+1})\,|\psi\rangle = E_2\,|\psi\rangle}$ in "position space":

$x_2 > x_1+1$ $\qquad E_2\,\psi(x_1,x_2) = 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1+1,x_2)$

$\ldots ZWZ\ldots ZWZ\ldots$ $\qquad 2\,\psi(x_1,x_2) - \psi(x_1,x_2-1) - \psi(x_1,x_2+1)$

$x_2 = x_1+1$ $\qquad E_2\,\psi(x_1,x_2) = 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1,x_2+1)$

$\ldots ZWWZ\ldots$

# The coordinate Bethe Ansatz 2

$$x_2 > x_1 + 1 \qquad E_2\,\psi(x_1,x_2) = 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1+1,x_2)$$
$$2\,\psi(x_1,x_2) - \psi(x_1,x_2-1) - \psi(x_1,x_2+1) \qquad (1)$$

$$x_2 = x_1 + 1 \qquad E_2\,\psi(x_1,x_2) = 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1,x_2+1) \qquad (2)$$

- Solved by Bethe's ansatz (1931):

$$\boxed{\psi(x_1,x_2) = e^{i(p_1 x_1 + p_2 x_2)} + \underset{\downarrow\ \text{S-matrix}}{S(p_2,p_1)}\, e^{i(p_2 x_1 + p_1 x_2)}}$$

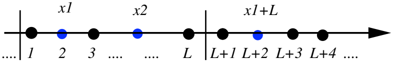
- Then (1) is solved for any $S(p_2,p_1)$ with $\boxed{E_2 = \sum_{k=1}^{M} 4\,\sin^2(\frac{p_k}{2})}$

  N.B. $2 - e^{-ip} - e^{ip} = 4\sin^2\frac{p}{2}$

- (2) determines S-matrix: $\boxed{S(p_2,p_1) = \frac{\varphi(p_1)-\varphi(p_2)+i}{\varphi(p_1)-\varphi(p_2)-i}}$ with $\varphi(p) = \frac{1}{2}\cot(\frac{p}{2})$

# Bethe equations 1

- Demand $\psi(x_1, x_2) = \psi(x_2, x_1 + L)$




$$\Rightarrow \qquad \boxed{e^{ip_1 L} = S(p_1, p_2) \quad \text{and} \quad e^{ip_2 L} = S(p_2, p_1)}$$

solve for $p_1$ & $p_2$ $\Rightarrow$ $E_2(p_1, p_2) = \sum_{k=1}^{2} 4 \sin^2 \frac{p_k}{2}$ spectrum!

- Big leap ($\Leftrightarrow$ factorized scattering from integrability): $M$-**body problem**

Total phase acquired by one magnon cycling around the chain:

$$\boxed{e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(p_k, p_i)} \qquad k = 1, \ldots, M$$

Scatters off all other magnons exactly once

# Bethe equations 2

- Energy additive:

$$E_2(p_1,\ldots,p_M)=\sum_{k=1}^{M}4\sin^2\tfrac{p_k}{2}$$

- Cyclicity of trace condition: $\sum_{k=1}^{M}p_k=0\Leftrightarrow$ vanishing total momentum
- Example: Two magnons: $p:=p_1=-p_2$

$$\begin{aligned} e^{ipL} &= \frac{\cot\frac{p}{2}+i}{\cot\frac{p}{2}-i}=e^{ip} \quad \Rightarrow \quad e^{ip(L-1)}=1 \quad \Rightarrow \quad p=\frac{2\pi\,n}{L-1} \\ E_2 &= 8\sin^2\left(\frac{\pi\,n}{L-1}\right) \quad \stackrel{L\to\infty}{\longrightarrow} \quad 8\pi^2\,\frac{n^2}{L^2}\,, \end{aligned}$$

Recall $\Delta_1=\frac{\lambda}{8\pi^2}E_2\to n^2\,\lambda/L^2$
Agrees with plane-wave string spectrum $E_{\mathrm{light-cone}}=2\sqrt{1+n^2\,\lambda/J^2}$

# Integrability

- Heisenberg spin chain is integrable: Existence of $L$ commuting charges $Q_n$: $\boxed{[Q_m, Q_n] = 0}$ $\forall (m,n)$!
- Spectrum determined by **Bethe equations**:

$$e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(p_k, p_i) \quad k = 1, \dots, M$$

With S-Matrix:

$$S(p_i, p_k) = \frac{x^+(p_i) - x^-(p_k)}{x^-(p_i) - x^+(p_k)} \quad \text{with} \quad x^\pm(p) = \frac{1}{2}(\cot(\frac{p}{2}) \pm i)$$

Energy (one loop scaling dimensions) additive:

$$\Delta = L + \lambda E_2 \qquad \text{with} \qquad E_2(p_1, \dots, p_M) = \sum_{k=1}^{M} 4 \sin^2 \frac{p_k}{2}$$

+ Cyclicity of trace condition: $\sum_{k=1}^{M} p_k = 0$

# The asymptotic Bethe Ansatz

What happens at higher loops?

$\lambda$ deformed variables: $\boxed{x^{\pm}(p) = \frac{e^{\pm i\,p/2}}{4\,\sin\frac{p}{2}}\,(1+\sqrt{1+\frac{\lambda}{\pi^2}\,\sin^2\frac{p}{2}}\,)}$ $\Leftrightarrow\; e^{ip} = \frac{x^+(p)}{x^-(p)}$

Asymptotic all loop conjecture: $x_k^{\pm} := x^{\pm}(p_k)$ [Beisert,Staudacher]

$$\boxed{\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j=1,j\neq k}^{M} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\,\frac{1-\frac{\lambda}{16\pi^2\,x_k^+\,x_j^-}}{1-\frac{\lambda}{16\pi^2\,x_k^-\,x_j^+}}\cdot S_0(\{p_k\},\lambda)^2}$$

$S_0$ : dressing factor

- Valid for $L>$loop order, completely fixed by $\mathfrak{psu}(2,2|4)$ symmetry up to $S_0$.
- Conjectured all loop form of $S_0$ exists [Beisert,Hernandez,Lopez;Beisert,Eden,Staudacher]
- Perturbatively: $S_0 \sim \mathcal{O}(\lambda^4)$ [Bern,Czakon,Dixon,Kosower,Smirnov]

Scaling dimensions then $\boxed{\Delta = \Delta_0 + \sum_{k=1}^{M}\sqrt{1+\frac{\lambda}{\pi^2}\,\sin^2\frac{p_k}{2}} - 1}$.

# Integrability in the AdS/CFT system

- $AdS_5 \times S^5$ string $\sigma$-model is classically integrable [Bena,Polchinski,Roiban]
  Can be solved completely in terms of algebraic curve

  [Kazakov,Marshakov,Minahan,Zarembo; Beisert,Kasazkov,Sakai,Zarembo]
- Full one-loop dilatation operator has been constructed in terms of an integrable super-spin chain and diagonalized by Bethe ansatz. [Minahan,Zarembo;Beisert,Staudacher]
  Super-magnon excitations scatter according to matrix Bethe equations:

$$\boxed{e^{ip_k L}\,|\Psi\rangle = \left(\prod_{j=1,j\neq i}^{M} S(p_k,p_j)\right)\cdot|\Psi\rangle\,, \qquad E = \sum_{k=1}^{M} q_2(p_k)\,.}$$

  (Asymptotic) S-matrix is assumed to be factorized. So far only proven at one-loop for all and up to four-loop for some operators.
- Wrapping problem: For finite size chains and long-range interactions not allowed to assume exactness of S-matrix!

# Full set of conjectured nested $\mathfrak{psu}(2,2|4)$ Bethe equations

[Beisert,Staudacher]

$$1 = \prod_{j=1}^{K_4} \frac{x^+_{4,k}}{x^-_{4,k}} \qquad \text{Spectral parameter: } x^\pm_{4,k} = \frac{1}{4}(\cot p_k/2 \pm i)\left(1 + \sqrt{1 + 16 g^2 \sin^2 p_k/2}\right)$$

magnon momentum

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i\eta_1}{u_{2,k} - u_{2,j} + i\eta_1} \prod_{j=1}^{K_3+K_1} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{3,j} - \frac{i}{2}\eta_1}$$

$$g := \sqrt{\lambda}/4\pi$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{3,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x^{+\eta_1}_{4,j} - x_{3,k}}{x^{-\eta_1}_{4,j} - x_{3,k}}$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^{L-\eta_1 K_1 - \eta_2 K_7} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{x^{+\eta_1}_{4,k} - x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k} - x^{+\eta_2}_{4,j}} \frac{1 - g^2/(x^+_{4,k} x^-_{4,j})}{1 - g^2/(x^-_{4,k} x^+_{4,j})} \mathbf{S_0^2}\right)$$

Dressing factor

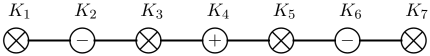
$$\times \prod_{j=1}^{K_3+K_1} \frac{x^{-\eta_1}_{4,k} - x_{3,j}}{x^{+\eta_1}_{4,k} - x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x^{-\eta_2}_{4,k} - x_{5,j}}{x^{+\eta_2}_{4,k} - x_{5,j}}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{5,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x^{+\eta_2}_{4,j} - x_{5,k}}{x^{-\eta_2}_{4,j} - x_{5,k}} \qquad u_{i,k} := x_{i,k} + g^2/x_{i,k}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i\eta_2}{u_{6,k} - u_{6,j} + i\eta_2} \prod_{j=1}^{K_5+K_7} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{5,j} - \frac{i}{2}\eta_2}\,.$$

with $\eta_1, \eta_2$ related to four different choices of $\mathfrak{psu}(2,2|4)$ Dynkin labels, e.g.

$\{\eta_1, \eta_2\} = \{+1, +1\}$:



# The AdS/CFT (internal) S-matrix

- Describes scattering of two super-magnons, should be unitary and satisfy Yang-Baxter equation:

  [Arutyunov,Frolov,Staudacher '04; Beisert, Staudacher '05 + '06; Beisert, Hernandez,Lopez '06, Beisert,Eden,Staudacher '06]

$$\boxed{S_{12}\, S_{21} = 1\,, \qquad S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}}$$

- Was (ad hoc) conjectured to possess crossing symmetry: [Janik, '06]

$$\boxed{S_{12}\, S_{\bar{1}2} = f_{12}^2}$$

  $\Rightarrow$ can be used to fix dressing factor $S_0$.
- AdS/CFT S-matrix has the structure [Beisert '05]

$$\boxed{S_{12} = \left(S_{12}^{\mathfrak{psu}(2|2)_L} \otimes S_{12}^{\mathfrak{psu}(2|2)_R}\right)\, S_0^2}$$

  Invariant under a residual $\mathfrak{J} \in \mathfrak{psu}(2|2)$ symmetry: $\boxed{[\mathfrak{J}_1 + \mathfrak{J}_2, S_{12}^{\mathfrak{psu}(2|2)_L}] = 0}$.
- First motivated from gauge theory spin chain, subsequently found in light-cone quantized string theory [Arutyunov,Frolov,Plefka,Zamaklar '06]

# Large Spin Limit of Twist Operators

- Consider twist operators: $S_1$: Spin $J_3$: "twist"

$$\boxed{\mathcal{O}_{S_1,J_3} = \mathrm{Tr}(\mathcal{D}^{S_1}\, Z^{J_3}) + \dots}$$

with $\mathcal{D} = \mathcal{D}_+$ covariant derivative in light-cone direction.

- General spin chain state of length $J_3$ is $\mathrm{Tr}\Big[(\mathcal{D}^{s_1} Z)\,(\mathcal{D}^{s_2} Z)\dots(\mathcal{D}^{s_{J_3}} Z)\Big]$ where $S_1 = s_1 + s_2 + \dots s_{J_3} =: M =$ Magnon number.
- Scaling dims in $S_1 \to \infty$ limit:

$$\boxed{\Delta_{\mathcal{O}_{S_1,J_3}} - S_1 - J_3 = \gamma(\lambda)\,\log S_1 + \mathcal{O}({S_1}^0)}$$

$\gamma(\lambda)$: Universal scaling function, aka cusp anomalous dimension.

- $\gamma(\lambda)$ also appears in 4 gluon MHV amplitudes $\mathcal{A}_{4,MHV}$ and in light-cone segmented Wilson loops $\mathcal{W}$! [Bern,Dixon,Smirnov]

$$\boxed{\mathcal{A}_{4,MHV}^{\text{all- loop}} \sim \exp\Big[\gamma(\lambda)\,\mathcal{A}_{4,MHV}^{\text{one-loop}}\Big]\,, \qquad \mathcal{A}_{4,MHV}^{\text{all- loop}} \sim \langle\mathcal{W}\rangle}$$

# The Beisert-Eden-Staudacher Integral Equation

- Asymptotic Bethe equations reduce in $S_1 \to \infty$, $L = J_3 \to \infty$ with $L \ll \log S_1$ to integral equation for density $\hat{\sigma}$ of Bethe roots: $(g = \sqrt{\lambda/4\pi})$

$$\boxed{\hat{\sigma}(t) \;=\; \frac{t}{e^{\,t}-1}\left[\;\hat{K}(2\,g\,t,0) - 4\,g^2\int_0^{\infty} dt'\;\hat{K}(2\,g\,t,2\,g\,t')\;\hat{\sigma}(t')\right].}$$

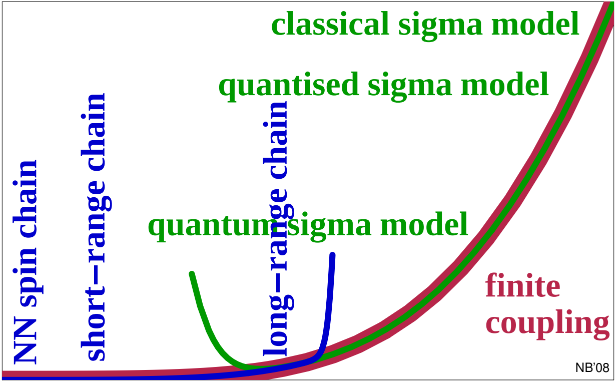
Cusp anomalous dimensions: $\boxed{\gamma(g) = 16\,g^2\,\hat{\sigma}(0)}$ **All loop prediction!**

- Solution yields weak and strong coupling predictions: [BES, Basso, Korchemsky, Kotanski '07]

$$\gamma(g) = \begin{cases} 8g^2 - \frac{8}{3}\pi^2\,g^4 + \frac{88}{45}\,\pi^4\,g^6 - 16(\frac{73}{630}\pi^6 + 4\,\zeta(3)^2)\,g^8 + \ldots & g \ll 1 \\ 4g - \frac{3\log 2}{\pi} - \frac{K}{4\pi^2}\,\frac{1}{g - 3\log 2/4\pi} - \frac{27\,\zeta(3)}{2^9\,\pi^3}\,\frac{1}{g^2} - \ldots & g \gg 1 \end{cases}$$

- Agrees with:
  1) Four loop gauge theory calculation [Bern,Czakon,Dixon,Kosower,Smirnov '06]
  2) 2 loop superstring calculation [Roiban,Tseytlin '07]

# Cusp anomalous dimension of $\mathcal{N} = 4$ SYM:

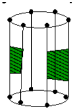



(Plot by N. Beisert)

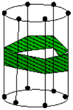
# Wrapping interactions

Asymptotic Bethe equations yield 'half' of the perturbative spectrum of $\mathcal{N}=4$ SYM:

| length \ loops | 1 | 2 | 3 | 4 | 5 | 6 | 7... |
|---|---|---|---|---|---|---|---|
| ⋮ | | | | | | | |
| 6 | √ | √ | √ | √ | √ | √ | √ |
| 5 | √ | √ | √ | √ | √ | √ | ? |
| 4 | √ | √ | √ | √ | √ | ? | ? |
| 3 | √ | √ | √ | √ | ? | ? | ? |
| 2 | √ | √ | √ | Janik's TBA | ? | ? | ? |



incorporated Feynman graphs



missing wrapping interactions

- Wrapping graphs contribute generically at order $g^{2L}$.
- Asymptotic Bethe eqs. describes $L \to \infty$ spin chain or string with worldsheet geometry $\mathbb{R}^2$ $\quad\Rightarrow\quad$ Exsistence of S-Matrix and asymptotic states

# Thermodynamic Bethe Ansatz

- Magnitude of finite size corrections: $\boxed{\sim e^{-E_{\mathsf{TBA}}(p_{\mathsf{TBA}})\,L}}$ with $E_{\mathsf{TBA}} = -ip$ and $p_{\mathsf{TBA}} = -iE$ in 'mirror' theory, i.e. original theory with space and time interchanged [Arutyunov,Frolov]

- Approach was successfully implemented by generalization of Lüscher's formulas for 2d Lorentz invariant FT: Computation of four loop scaling dimension of Konishi operator $\mathrm{Tr}([Z,W][Z,W])$ from asymptotic S-matrix [Bajnok,Janik '08]

- Agrees with perturbative four loop supergraph calculation!

  [Fiamberti,Santambrogio,Sieg,Zanon '08]

$$\Delta = \Delta_{\mathsf{aBE}} + \Delta_{\mathsf{wrapping}} \qquad \Delta_{\mathsf{wrapping}} = (324 + 864\zeta(3)1440\zeta(5))g^8$$

- Highly nontrivial test of AdS/CFT!!

# The Y system

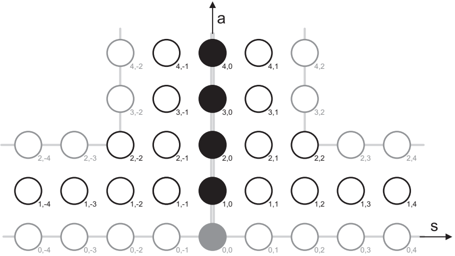
**Recent conjecture**: Implementation of TBA through a "Y-system" to describe planar AdS/CFT at finite size. Passes all known tests! [Gromov, Kazakov,Vieira '09]

## Result:

- Y-system

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\left[1+Y_{a+1,s}\right]}$$

$Y_{a,s}(u)$



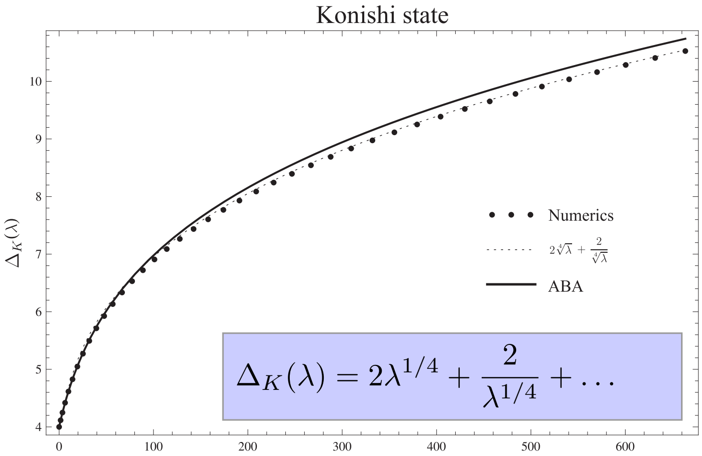

- Asymptotics

$$Y_{a,s\neq 0}(u\to\infty) \;\to\; \mathrm{const}_{a,s}$$

$$Y_{a,0}(u\to\infty) \;\to\; \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^{L}\times \mathrm{const}_{a}$$

(from talk of V. Kazakov at KITP 02/09)

# Konishi at *any* coupling




(from talk of P. Vieira at Strings 09)

# Conclusions

Great progress in our understanding of the maximally supersymmetric $AdS_5/CFT_4$ system due to **integrable structures**!

Spectral problem (close) to exact solution!

1. Exact $AdS_5 \times S^5$ closed string spectrum!
2. All loop form of two-point functions in 4d gauge theory!

Outlook:

- Next talk: Integrability in scattering amplitudes at higher loops?
- What can be said about gauge theory three-point functions?