Geometric flows: string cosmology, gravitational instantons and non-relativistic gravity

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Highlights

Framework and motivations

Reminder on Ricci flows

Strings, sigma models and RG flows

Gravitational instantons and geometric flows

Geometric flows in non-relativistic gravity

Last word

Geometric flows

Irreversible continuous evolution of Riemanian geometry $g_{ij}(t)$ *–*
non-linear parabolic equation driven by some tensor $S_{ij}\left[g_{ij}\right]$

$$\frac{\partial g_{ij}}{\partial t} = S_{ij}$$

- Plethora of flows: Ricci, Einstein, Calabi (even D), Cotton (3D), Bach (4D), . . .
- Plethora of behaviours: infinte or finite time, convergence towards canonical metrics, singular end-points with topology change, . . .

Ricci flows: $S_{ij} = -R_{ij}$

- Introduced by R. Hamilton in 1982 as a tool for proving Poincaré's (1904) and Thurston's (late 70s) 3D conjectures
- Driven by Hamilton's programme culminating in Perel'man's proof (2002–03)
- Appeared to be relevant in physics

Ricci flows and generalizations arise in problems related to gravity with time foliation

- In non-critical string theory Ricci flow can mimic time evolution as an RG flow towards the IR – cosmology
- In 4D self-dual gravitational instantons with homogeneous spatial sections: time evolution is a Ricci flow of the 3D homogeneous leaves
- In non-relativistic gravity with covariance explicitly broken to foliation-preserving diffeomorphisms and with detailed-balance dynamics: time evolution is a geometric flow of the 3D space

Aim: review these properties

Highlights

Basic features

Ricci-flow equations

$$\frac{\partial g_{ij}}{\partial t} = -R_{ij}$$

- Volume is not preserved along the flow
  $$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{2}\int \mathrm{d}^D x \sqrt{\det g}\, g^{ij}\frac{\partial g_{ij}}{\partial t} = -\frac{1}{2}\int \mathrm{d}^D x\sqrt{\det g}\, R$$
  Consequence:
  - positive curvature $\longrightarrow$ space contracts
  - negative curvature $\longrightarrow$ space expands
- Killing vectors are preserved in time: the isometry group can only grow or remain unaltered

Example

- At initial time: $R_{ij}^{(0)} = a g_{ij}^{(0)}$ with a constant
- Subsequent evolution: linear rescaling

$$g_{ij}(t) = (1 - at) g_{ij}^{(0)}$$
$$R_{ij}(t) = R_{ij}^{(0)}$$

- Properties
  - $a > 0 \Rightarrow$ uniform contraction $\rightarrow$ singularity at $t = {}^1/_a$
  - $a < 0 \Rightarrow$ indefinite expansion

Rôle in Poincaré's and Thurston's conjectures

Poincaré's conjecture: any closed 3-manifold with trivial fundamental group is homeomorphic to a 3-sphere

Similar theorem proven for $D > 4$ in the 60s and for $D = 4$ in 1982

Thurston's geometrisation conjecture: extension when the fundamental group is not trivial

- In direct relation with the classification of 3-manifolds (obtained e.g. as quotients by discrete isometry groups)
- Consequence: any 3-manifold can be decomposed in locally homogeneous components

Ricci flow is governed by a non-linear heat-like equation

Smoothes the initial geometry and brings it to a simpler form where the conjectures can be checked

Singularities and degeneracies can appear (e.g. pinching cycles) with topology changes

Must be kept under control: Perel'man's "surgery"

Remarks

To avoid the trivial singularity of shrinking volume: normalised flow

$$\frac{\partial g_{ij}}{\partial t} = -R_{ij} + \frac{\langle R \rangle}{D} g_{ij}$$

Consequence: $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ although the eqs. are identical to the ordinary ones after appropriate rescaling of g_{ij} and t

Other generalisations to the Ricci flow exist – Perel'man's flow is not pure Ricci: an extra scalar participates

The case of homogeneous 3-manifolds

Why?

- Appear in the final stage of Hamilton's programme for Thurston's geometrisation
- Building blocks for
  - Lorentzian 4-manifolds (cosmological solutions)
  - Euclidean 4-manifolds (gravitational instantons)
- Ricci flows can be studied by analytic methods – sometimes related to remarkable integrable systems

Spaces $\mathcal{M}_3$ admitting a group G_{hom} acting transitively [in 3D see Scott, 1983]

- Cosets or products thereof

$$H_3,\ H_2 \times S^1,\ S^2 \times S^1$$

  $H_3 = {}^{SL(2,\mathbb{C})}/{}_{SU(2)}$, $H_2 = {}^{SL(2,\mathbb{R})}/{}_{U(1)}$, $S^2 = {}^{SU(2)}/{}_{U(1)}$
  (dim $G_{\text{hom}} = 6, 4, 4 > 3 \Rightarrow$ action *multiply* transitive)

- dim $G_{\text{hom}} = 3 \Rightarrow$ action *simply* transitive: $\mathcal{M}_3$ is locally G_{hom}
  - 3 linearly independent Killing vectors tangent to $\mathcal{M}_3$: $[\xi_i, \xi_j] = c^i{}_{jk}\xi_k$
  - left-invariant Maurer–Cartan forms σ^i: $\mathrm{d}\sigma^i = \frac{1}{2}c^i{}_{jk}\sigma^j \wedge \sigma^k$
  - $c^k{}_{ij} = -\epsilon_{ij\ell}m^{\ell k} + \delta^k_i a_j - \delta^k_j a_i \Rightarrow c^i{}_{ij} = 2a_j$

3D G_{hom} are Bianchi groups [Bianchi 1897; Taub, 1951]

Bianchi classes

class A unimodular: I, II, $\mathrm{VI}_{h=-1}$, $\mathrm{VII}_{h=0}$, VIII, IX
T_3, Heisenberg, $E_{1,1}$, E_2, $SL(2,\mathbb{R})$, $SU(2)$

class B non-unimodular: III, IV, V, $\mathrm{VI}_{h\neq -1}$, $\mathrm{VII}_{h\neq 0}$

Geometry

- The most general metric: $\mathrm{d}s^2 = g^{ij}\sigma^i\sigma^j = \delta_{ij}\theta^i\theta^j$
  $\theta^i = \Theta^i{}_j\sigma^j, g_{ij} = \delta_{k\ell}\Theta^k{}_i\Theta^\ell{}_j$ ($\Theta^i{}_j$ are coordinate-independent)
- Minimalistic (diagonal) ansatz $\Theta^i{}_j = \gamma_j\delta^i_j$:

$$\mathrm{d}s^2 = \sum_i \gamma_i \left(\sigma^i\right)^2$$

  not always the most general if the σ^is are in a canonical form

Behaviour under Ricci flow

Bianchi IX class: 3-spheres

- Left-invariant Maurer–Cartan forms of $G_{\text{hom}} \equiv SU(2)$

$$\begin{cases} \sigma^1 = \sin\vartheta \sin\psi \, \mathrm{d}\varphi + \cos\psi \, \mathrm{d}\vartheta \\ \sigma^2 = \sin\vartheta \cos\psi \, \mathrm{d}\varphi - \sin\psi \, \mathrm{d}\vartheta \\ \sigma^3 = \cos\vartheta \, \mathrm{d}\varphi + \mathrm{d}\psi \end{cases}$$

  with $0 \le \vartheta \le \pi, 0 \le \varphi \le 2\pi, 0 \le \psi \le 4\pi$ (Euler angles)
- $\mathrm{d}\sigma^i + \frac{1}{2}\epsilon^i{}_{jk}\sigma^j \wedge \sigma^k = 0$
- The diagonal ansatz is the most general
  - If *two* γs are equal the isometry group is promoted to $SU(2) \times U(1)$ (axial symmetry)
  - Full isotropy requires γs be *all* equal

The Ricci-flow equations ($\dot{} = \mathrm{d}/\mathrm{d}T = \gamma_1\gamma_2\gamma_3 \mathrm{d}/\mathrm{d}t$)

$$2\frac{\dot{\gamma}_1}{\gamma_1} = (\gamma_2 - \gamma_3)^2 - \gamma_1^2 \quad \text{and cyclic perms.}$$

Typical behaviour

At large T $\gamma_i \approx 1/\sqrt{T}$ $\Rightarrow$ convergence toward the round sphere of vanishing radius

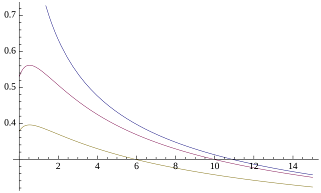

Figure: Behaviour of γ_is for Bianchi IX

Remarks

- If torsion is added ($H = \mathrm{d}B$), the flow

$$\frac{\partial g_{ij}}{\partial t} = -R^{(-)}_{ij}$$

  converges towards a finite-radius round sphere
- Other Bianchi classes of homegeneous 3-manifolds exhibit different behaviours (absence of convergence, pancake degeneracies, cigar degeneracies, ...) [Isenberg, Jackson, 1992]

A parenthesis: 19th century integrable systems

Bianchi IX Ricci-flow equations are a remarkable integrable system

[Sfetsos, unpubl.; Bakas, Orlando, Petropoulos, 2006; Bourliot, Estes, Petropoulos, Spindel, 2009]

- *Darboux* equations on "triply orthogonal surfaces" – solved by *Halphen* using modular forms [Darboux 1878; Halphen, 1881]
- Darboux–Halphen system studied extensively by mathematicians over the recent years [Takhtajan, 1992; Maciejewski, Strelcyn, 1995; Chakravarty, Halburd, Ablowitz, 2003]
- Darboux–Halphen appeared in the framework of Bianchi IX self-dual gravitational instantons and in the scattering of $SU(2)$ BPS monopoles [Gibbons, Pope, 1979; Manton, 1981; Atiyah, Hitchin, 1985; Gibbons, Manton, 1986]

Typical system appearing in general self-dual Yang-Mills reductions – in the late '70s all integrable systems were even thought to be SDYM reductions [Ward, 1985] – now in geometric flows

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String propagation in curved backgrounds

Non-linear sigma models (g_{MN} Lorentzian or Euclidean)

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z \, g_{MN}(x) \partial x^M \bar{\partial} x^N$$

potentially B_{MN} (axion), Φ (dilaton)

Not scale-invariant: g_{MN} flows under RG [Friedan, 1985]

- μ: RG 2D mass scale

$$\mu^{-1} \partial / \partial \mu^{-1} g_{MN} = -\beta \left[g_{MN} \right] \equiv -R_{MN} + \mathrm{O}(\alpha')$$

- $t = -\log \mu$: RG time $\begin{cases} \text{UV}: & t \to -\infty \\ \text{IR}: & t \to +\infty \end{cases}$

RG flow $\equiv$ Ricci flow

- $\exists$ fixed-points – possibly exact 2D CFTs (all-order α')

Application: string cosmology

Original aim: find time-dependent string backgrounds – exact or at $\mathrm{O}(\alpha')$ *– address the usual problems*

Keep in mind

- Around the big-bang: high curvature $\longrightarrow$ exact solutions
- After the inflation era: adiabatic evolution $\longrightarrow$ $\mathrm{O}(\alpha')$ enough – higher $\mathrm{O}(g_{\mathrm{s}})$ due to $V(\Phi)$

plus moduli problem, phase transitions, ...

The FRWL paradigm: assume $\mathcal{M}_4 = \mathcal{T} \times \mathcal{M}_3$ with $\mathcal{M}_3$ homogeneous (even maximally symmetric)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \exp 2\sigma(t)\mathrm{d}\Omega^2$$

- In general relativity: fluid, Λ_c, ... $\rightarrow$ FRWL eqs. for σ, p, ϱ
- In string theory: less freedom
  - moduli, fluxes, ...
  - internal manifold
  - curvature and dilaton set the validity of approximations

Example: linear-dilaton solution (exact)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\Omega^2_{S^3} + \mathrm{d}s^2_{\text{int}}$$
$$\Phi = \Phi_0 - \bar{Q}t, \ H = 2\omega_{S^3}$$

($a(\tau) = \bar{Q}^2\tau^2$ in Einstein frame)

Usually appears in limiting regimes

Central question

Could RG flow mimic cosmological evolution with IR fixed points being steady-state universes?

- Irreversibility and averaging hypothesis [Carfora, Piotrkowska, 1995]
- Hints from Liouville theory in non-critical strings [1989–1991: Das, David, Kawai, Polyakov, Sen, Wadia, . . .]
- Popular, not *a priori* justified
  - RG-flow equations are 1st-order in time: $\partial g_{ij}/\partial t = -\beta\left[g_{ij}\right]$
  - genuine evolution is second-order: $\beta\left[g_{\mu\nu}\right] = 0$

Analysed in various contexts to set under which circumstances this could be valid [Tseytlin, 1992; Schmidhuber, Tseytlin,1994; Bakas, Orlando, Petropoulos, 2006]

Strategy

Promote a flowing 3D spatial sigma model

$$g_{ij}\partial x^i \bar{\partial} x^j$$

into a scale-invariant 4D space–time sigma model with a genuine time coordinate along which the space–time is foliated

$$-\partial t \bar{\partial} t + g_{ij}(t)\partial x^i \bar{\partial} x^j + R_2 \Phi(t)$$

- Write $\beta\left[g_{\mu\nu}\right] = \beta\left[\Phi\right] = 0$ where $g_{\mu\nu} = \{-1, g_{ij}\}$
- Compare to $\frac{\partial g_{ij}}{\partial t} = -\beta\left[g_{ij}\right]$

Simplest situation: 3D space ≡ round sphere (isotropic Bianchi IX $\rightarrow \gamma_1 = \gamma_2 = \gamma_3 = L^2 \exp 2\sigma$*) with torsion* $H = {}^2/_L \omega_{S^3_L}$

- 3D: $\frac{d\sigma}{d \log \mu^{-1}} = -\frac{2e^{-2\sigma}}{L^2}\left(1 - e^{-4\sigma}\right) \Rightarrow$ IR fixed-point $\sigma = 0$ (radius L)
- 4D: define $Q \equiv -\dot{\Phi} + \frac{3}{2}\dot{\sigma}$
  - full equations:
$$\dot{Q} = -\frac{3}{2}\dot{\sigma}^2$$
$$\ddot{\sigma} + 2Q\dot{\sigma} = -\frac{2e^{-2\sigma}}{L^2}\left(1 - e^{-4\sigma}\right)$$
  - classical motion of a particle at coordinate σ in a potential
$$V = V_0 - \frac{e^{-2\sigma}}{L^2}\left(1 - \frac{e^{-4\sigma}}{3}\right)$$
    with a friction force $-2Q\dot{\sigma}$ due to the dilaton motion (like the inflaton [lectures by Lazarides and Sasaki])

4D dynamics: steady-state universe $\equiv$ linear dilaton solution

$$\sigma = 0,\ \Phi = \Phi_0 - \bar{Q}t$$

space $\equiv$ IR fixed point of the 3D sigma model

4D dynamics: full regime $\to$ friction $\to$ the kinetic term $\ddot{\sigma}$ becomes subdominant $\to$ first-order 3D equation with

$$t = -2Q \log \mu$$

4D genuine time evolution $\simeq$ RG i.e. Ricci flow in 3D

Evolution towards the RG regime

- At early times the 4D genuine time evolution is
  - oscillatory if $Q_0 < \sqrt{8}/L$ (large initial curvature)
  - monotonic if $Q_0 > \sqrt{8}/L$ (small initial curvature)
- At late times (3D Ricci-flow regime) $\rightarrow$ monotonic

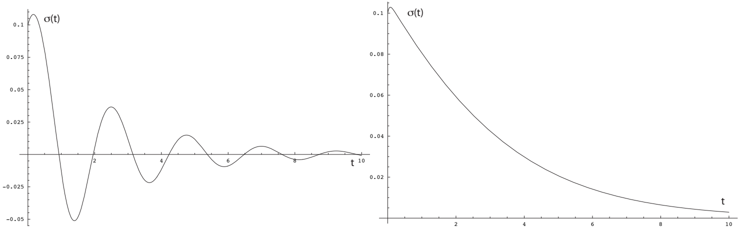

Figure: Typical behaviour for $\sigma(t)$ in the non-linear system

General result *[Schmidhuber, Tseytlin, 1994]*

- RG fixed points in $D \equiv$ steady-state universes in $D+1$
- In the vicinity of the fixed points: evolution $\equiv$ generalised Ricci flow
- Far or between fixed points: dynamics reminiscent of Zamolodchikov's c-theorem

Origin: dilaton motion (necessary for CFT) $\longrightarrow$ friction

Limitations

- Usual approximations
- Case of marginal lines in 3D sigma models

Highlights

Framework

Gravitational instantons: positive action conjecture, tools to handle non-perturbative transitions in quantum gravity, … *[70s]*

- Non-singular solutions of Euclidean Einstein's equations
- Hard task in general – easier with some specific ansatz
  - 4D geometry foliated with 3D leaves: $\mathcal{M}_4 = \mathcal{T} \times \mathcal{M}_3$
  - homegeneous spatial sections $\mathcal{M}_3$ – $\exists G_{\text{hom}}$ of Bianchi type
  - self-duality in vacuum $\longrightarrow$ first-order equations – consistent if G_{hom} algebra is unimodular

Connection with Ricci flows of homogeneous 3D spaces *[Cvetič, Gibbons, Lü, Pope, 2001; Bourliot, Estes, Petropoulos, Spindel 2009]*

- The gravitational instantons on $\mathcal{M}_4$ are in one-to-one correspondence with the flat $SU(2)$ connections on $\mathcal{M}_3$
- In all unimodular cases Euclidean-time evolution of the leaves is a geometric flow on $\mathcal{M}_3$ driven by Ricci plus a flat $SU(2)$ gauge connection

Again the dynamics of Ricci flows seems to capture – exactly – higher-dimensional gravitational set-ups

Solving 4D Euclidean Einstein equations

Usual set-up: orthonormal frame $\mathrm{d}s^2 = \delta_{ab}\theta^a\theta^b$ *invariant under local* $SO(4)$ *transformations, connection one-form* $\omega^a{}_b$*, curvature two-form* $\mathcal{R}^a{}_b$

- Riemann tensor: $\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$
- (Anti-)self-dual-curvature solutions:
  - $\mathcal{R}^a{}_b = \pm\tilde{\mathcal{R}}^a{}_b$ & Cyclic id. $\Rightarrow$ vacuum eqs.
  - $\mathcal{R}^a{}_b = \pm\tilde{\mathcal{R}}^a{}_b \Leftrightarrow \omega^a{}_b = \pm\tilde{\omega}^a{}_b$ up to local $SO(4)$ rotations $(\tilde{\omega}^a{}_b = \frac{1}{2}\epsilon^a{}_{bc}{}^d\omega^c{}_d) \Rightarrow$ several branches of 1st order equations

Curvature and connection $\in \mathbf{6}$ *(antisymmetric) of* $SO(4)$ *– reducible as* $(\mathbf{3}, \mathbf{3})$ *under the decomposition* $SO(4) \cong SU(2)_{\mathrm{sd}} \otimes SU(2)_{\mathrm{asd}}$

In the geometries $\mathcal{M}_4 = \mathcal{T} \times \mathcal{M}_3$ the decomposition is simple

- Curvature two-form
  - $\mathcal{S}_i = \frac{1}{2}\left(\mathcal{R}_{0i} + \frac{1}{2}\epsilon_{ijk}\mathcal{R}^{jk}\right)$
  - $\mathcal{A}_i = \frac{1}{2}\left(\mathcal{R}_{0i} - \frac{1}{2}\epsilon_{ijk}\mathcal{R}^{jk}\right)$
- Connection one-form
  - $\Sigma_i = \frac{1}{2}\left(\omega_{0i} + \frac{1}{2}\epsilon_{ijk}\omega^{jk}\right)$
  - $A_i = \frac{1}{2}\left(\omega_{0i} - \frac{1}{2}\epsilon_{ijk}\omega^{jk}\right)$
- Relations
  - $\mathcal{S}_i = \mathrm{d}\Sigma_i - \epsilon_{ijk}\Sigma^j \wedge \Sigma^k$
  - $\mathcal{A}_i = \mathrm{d}A_i + \epsilon_{ijk}A^j \wedge A^k$

$\{\mathcal{S}_i, \Sigma_i\}$ *vectors of* $SU(2)_{\mathrm{sd}}$ *and singlets of* $SU(2)_{\mathrm{asd}}$ *and vice-versa for* $\{\mathcal{A}_i, A_i\}$

Self-dual gravitational instantons

$$\mathcal{A}_i = 0 \leftrightarrow \text{flat } SU(2) \text{ connections on } \mathcal{M}_3$$

Further assumption $\mathcal{M}_3$ homogeneous of Bianchi type locally G_{hom}
flat $SU(2)$ connections on $\mathcal{M}_3 \leftrightarrow$ homomorphisms $G_{\text{hom}} \rightarrow SU(2)$

Example: Bianchi IX $G_{\text{hom}} \equiv SU(2)$

1. Trivial homomorphism: $G_{\text{hom}} \rightarrow$ identity of $SU(2)$, $A_i = 0\ \forall i$
   Eguchi–Hanson branch e.g.
2. Isomorphism (rank-3): $A_i = {}^{\sigma^i}/_2\ \forall i$ Taub–NUT branch e.g.

Note: non-singular real self-dual gravitational instantons exist only for unimodular – class A – Bianchi groups and for Bianchi III

The view from the leaf: geometric flows

4D self-dual gravitational instantons foliated in time with Bianchi-A homogeneous spaces

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \sum_i \left(\gamma_i \sigma^i\right)^2$$

- $\gamma_i(T)$: 1st order differential equations – branches: choice of flat $SU(2)$ anti-self-dual Levi–Civita connection A_i ($\mathcal{A}_i = 0$)
- Example: Bianchi IX

$$\begin{cases} 2\dfrac{\dot{\gamma}_1}{\gamma_1} = \left(\gamma_2 - \gamma_3\right)^2 - \gamma_1^2 + 2\tilde{\lambda}\gamma_2\gamma_3 \text{ and cyclic} \\ A_i = \left(1 - \tilde{\lambda}\right)\dfrac{\sigma^i}{2} \end{cases}$$

  - $\tilde{\lambda} = 0$: Taub–NUT – Darboux–Halphen system
  - $\tilde{\lambda} = 1$: Eguchi–Hanson – Lagrange system (Euler top)

The evolution of the leaves in Euclidean time is covariant – resembles a flow due to the privileged rôle of time in $\mathcal{M}_4 = \mathcal{T} \times \mathcal{M}_3$ – what kind of flow?

Consider a pure 3D set-up: same Bianchi-class $\mathcal{M}_3$s – parameter t

- Metric: $\mathrm{d}\tilde{s}^2 = \sum_i \gamma_i(t) \left(\sigma^i\right)^2$
- $SU(2)$ gauge connection:
  - $\tilde{A} = \tilde{A}_i \sqrt{\gamma_i} \sigma^i = \tilde{A}_i^j T_j \sqrt{\gamma_i} \sigma^i$
  - $\tilde{F} = \mathrm{d}\tilde{A} + [\tilde{A}, \tilde{A}]$

Define a generalised Ricci flow

$$\begin{cases} \dfrac{\partial \tilde{g}_{ij}}{\partial t} = -\tilde{R}_{ij} - \dfrac{1}{2}\mathrm{tr}\left(\tilde{A}_i \tilde{A}_j\right) \\ \dfrac{\partial \tilde{A}_i}{\partial t} = 0, \quad \tilde{F} = 0 \end{cases}$$

- Dynamics in parametric time $\equiv$ coordinate-time evolution in the corresponding gravitational instanton – for each branch defined by a flat connection
- Example: Bianchi IX $\rightarrow \tilde{A}_i = -\frac{\tilde{\lambda} T_i \sigma^i}{\sqrt{\gamma_i}}$
  - $\tilde{\lambda} = 0$: pure Ricci flow (Darboux–Halphen) – Taub–NUT
  - $\tilde{\lambda} = 1$: Ricci plus gauge (Lagrange) – Eguchi–Hanson

Comments

The dynamics of self-dual gravitational instantons in vacuum is a geometric flow driven by Ricci plus flat $SU(2)$ gauge connection

- $\tilde{A}$ is a background $SU(2)$ gauge field inherited from the anti-self-dual part of the 4D Levi–Civita connection
- The geometric flow *is not* gauge invariant – not supposed to be
- The gauge field
  - *does not* flow ($\dot{\tilde{A}} = 0$)
  - its strength is set to $\tilde{F} = 0$

Towards a Ricci plus Yang–Mills flow with $\tilde{F} \neq 0$ & $\dot{\tilde{A}} \neq 0$

- We must relax the self-duality condition: $\mathcal{A}_i \neq 0$
- A similar condition is needed to provide 1st order equations
  - introduce a 4D cosmological constant Λ_c
  - impose self-duality on the Weyl tensor to solve $R_{ab} = \Lambda_c g_{ab}$

In the presence of cosmological constant 4D gravitational instantons with homogeneous spatial sections are equivalent to geometric flows of 3D leaves with a background $SU(2)$ gauge field $\tilde{A}$ s.t. $\dot{\tilde{A}}$ & $\tilde{F} \neq 0$

Example in Bianchi IX: the Fubini–Study solution (metric on $\mathbb{C}P_2$)

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3D geometric flows in homogeneous spaces versus self-duality in Einstein dynamics in 4D space–times with time foliation and homogeneous spatial sections

- Role of 4D: $SO(4) \cong SU(2) \times SU(2) \Rightarrow$ reduction is sd $\oplus$ asd
- Role of the 3D homogeneity: $G_{\text{hom}} \to SU(2) \Rightarrow$ gauge choice
- Role of the self-duality: effectively reduces the system from 4D to 3D – *off-shell though: "g_{ij} runs"*

Hard to generalise in arbitrary dimension (except perhaps is some hyper-Kähler or quaternionic spaces)

Horava–Lifshitz gravity: similar properties $\longrightarrow$ similar conclusions

[Horava 2008-09]

- Similar spaces: foliation $\mathcal{M}_4 = \mathcal{T} \times \mathcal{M}_3$
- Major difference: explicit breaking of the diffeomorphism invariance – previously this breaking was spontaneous
- Similar constraint: detailed balance – previously self-duality
- Similar effect in 4D $\longrightarrow$ 3D: the dynamics is governed by combined Ricci and Cotton flows
- Consequences:
  - valid in $D+1 \longrightarrow D$ dimensions with other flows
  - important issue of degrees of freedom and the appearance of Einstein gravity in the IR
  - richer landscape of fixed points – not always isotropic [Bakas, Bourliot, Lüst, Petropoulos 2009]

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Conclusion

Ricci and more general geometric flows appear in a wide palette of physical problems in diverse dimensions

Here: selection of topics related to gravity and time evolution with some fundamental questions: (i) Relation time evolution/RG flow/Liouville field? (ii) Relation energy evolution/holography?