

The Spin Foam Lectures

2: Skein algebra

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$U_q \mathfrak{sl}_2$

Algebra with generators K_+, K_-, K_0

$$[K_0, K_+] = 2K_+$$

$$[K_0, K_-] = -2K_-$$

$$[K_+, K_-] = \frac{e^{hK_0} - e^{-hK_0}}{e^h - e^{-h}} \quad h \in \mathbb{C}$$

Two cases

1. $h = \frac{i\pi}{r}$, $r \in \mathbb{Z}$, so $q = e^h = e^{i\pi/r}$
2. $h = 0$ ($r = \infty$) Lie algebra \mathfrak{sl}_2

$U_q\mathfrak{sl}_2$ Coproduct

$$\Delta K_0 = K_0 \otimes 1 + 1 \otimes K_0$$

$$\Delta K_+ = K_+ \otimes e^{hK_0} + 1 \otimes K_+$$

$$\Delta K_- = K_- \otimes 1 + e^{-hK_0} \otimes K_-$$

The coproduct determines the tensor product of reps

$$\xi \otimes \eta \mapsto \Delta(K)\xi \otimes \eta$$

Two-dimensional representation " $\frac{1}{2}$ "

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad K_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The tensor

$$\epsilon = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - A^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

is invariant, if $A^2 = e^h$. Define this to be

$$\frac{1}{2} \quad \text{U} \quad \frac{1}{2}$$

$\frac{1}{2}$ is self-dual.

Unknot

$$\frac{1}{2} \bigcirc = -A^2 - A^{-2}$$

From now on, a line without a number is spin $\frac{1}{2}$.

Crossing

Define the crossing intertwiner

$$\text{Crossing} = A^{-1} \left(\text{Cup} + A \text{Cap} \right)$$

This satisfies Reidmeister moves II and III

$$\text{Cup} = \text{Cup} \quad \text{II}$$

$$\text{Crossing} = \text{Crossing} \quad \text{III}$$

Crossings can always be removed.

Other irreducibles

$$j \subset \frac{1}{2} \otimes \frac{1}{2} \otimes \dots \otimes \frac{1}{2}, \quad 2j \text{ copies}$$

If $\sigma \in S_{2j}$, symmetric group, then $\hat{\sigma}$ is a +ve braid:

$$\sigma = \text{X} \quad \hat{\sigma} = \text{X} \quad (\#cr \text{ minimal})$$

The projector onto the spin j irreducible

$$\text{[Diagram: a vertical line with a box labeled } j \text{]} = \text{[Diagram: } 2j \text{ vertical lines with a box across them]} = c \sum_{\sigma \in S_{2j}} (A^{-3})^{\#cr} \text{[Diagram: } 2j \text{ vertical lines with a box labeled } \hat{\sigma} \text{ across them]}$$

with

$$c^{-1} = \sum_{\sigma \in S_{2j}} (A^{-4})^{\#cr} .$$

Example: spin 1

$$\begin{aligned}
 \text{Diagram: a box with a vertical line through it, labeled 1} &= (1 + A^{-4})^{-1} \left(\text{Diagram: a vertical line with a cross} + A^3 \text{Diagram: a vertical line} \right) \\
 &= \text{Diagram: a vertical line} + \frac{A^{-2}}{1 + A^{-4}} \text{Diagram: a cup and a cap}
 \end{aligned}$$

Generalisation

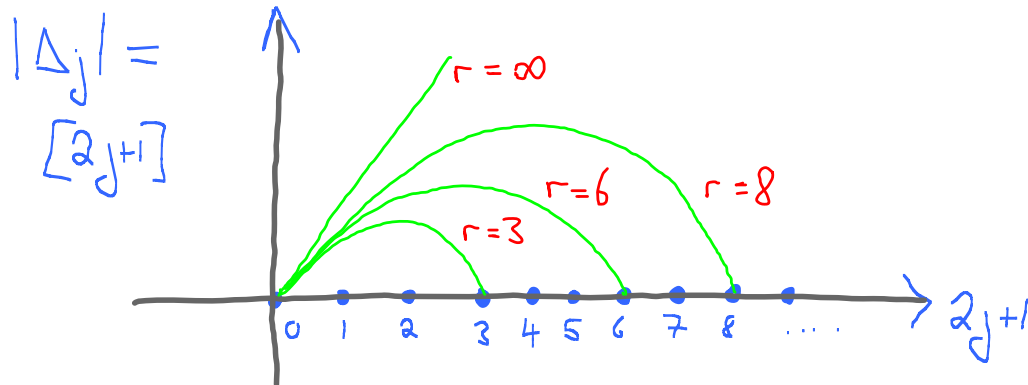
$$\text{Diagram: } j \text{ vertical lines} = \text{Diagram: a box with a vertical line through it} + \text{diagrams with } \leq 2j-2 \text{ strands in middle}$$

Quantum dimension

$$\Delta_j = \text{[Diagram of a torus with a handle]}^j$$

$$= (-1)^{2j} \frac{A^{4j+2} - A^{-4j-2}}{A^2 - A^{-2}} = (-1)^{2j} \frac{\sin \frac{\pi}{r} (2j+1)}{\sin \frac{\pi}{r}} = (-1)^{2j} [2j+1]$$

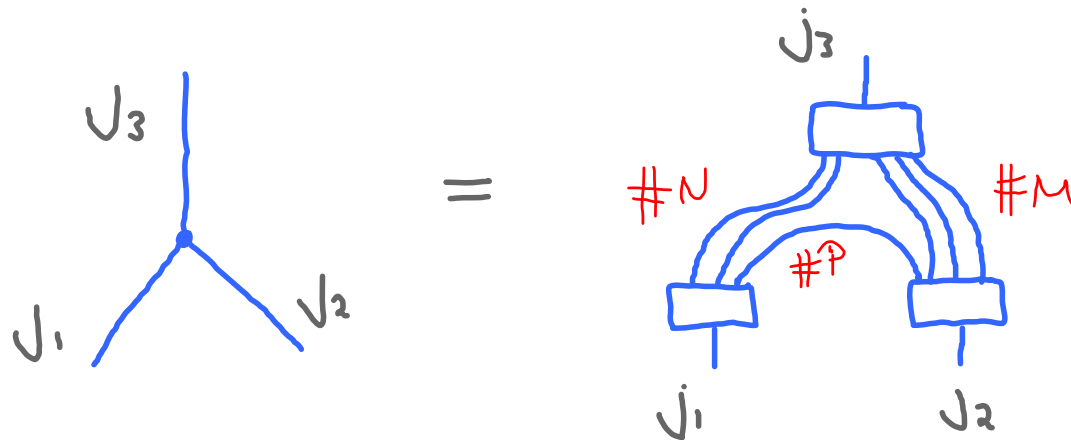
↑
quantum integer
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Intertwiners

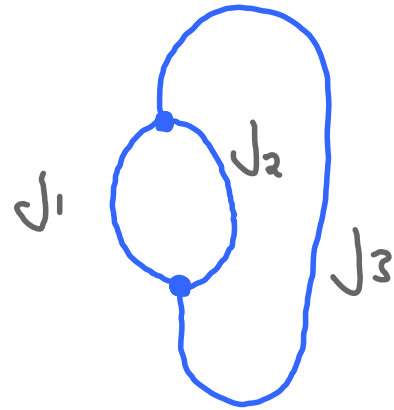
These are constructed from 

$\text{Hom}(j_1 \otimes j_2, j_3)$ has dimension 0 or 1. Canonical intertwiner



$$j_1 = \frac{1}{2}(N + P), \quad j_2 = \frac{1}{2}(M + P), \quad j_3 = \frac{1}{2}(N + M).$$

Theta



$$= \theta_{j_1 j_2 j_3} = (-1)^{j_1 + j_2 + j_3} \times \frac{[j_1 + j_2 + j_3 + 1]! [j_1 + j_2 - j_3]! [j_1 + j_3 - j_2]! [j_2 + j_3 - j_1]!}{[2j_1]! [2j_2]! [2j_3]!}$$

using $[n]! = [n][n-1] \dots [1]$.

Admissibility conditions

Conditions for $\theta_{j_1 j_2 j_3} \neq 0$, i.e., $\dim \text{Hom} = 1$.

$$\#\text{strings} = j_1 + j_2 + j_3 \in \mathbb{Z} \quad (1)$$

$$M = j_3 + j_2 - j_1 \geq 0 \quad (2)$$

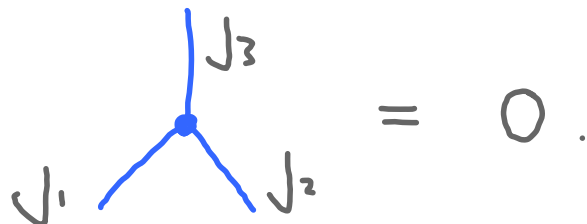
$$N = j_1 + j_3 - j_2 \geq 0 \quad (3)$$

$$P = j_1 + j_2 - j_3 \geq 0 \quad (4)$$

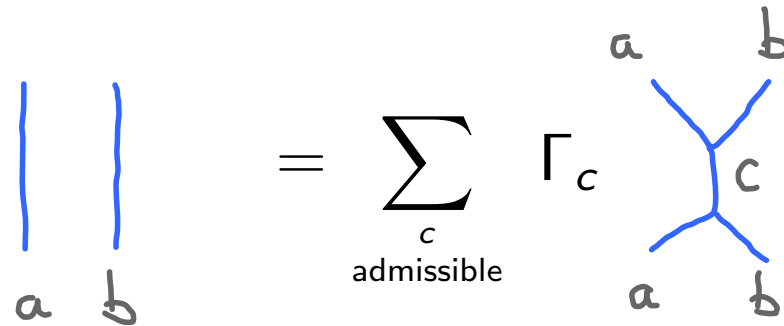
$$j_1 + j_2 + j_3 \leq r - 2 \quad (5)$$

$$2j_1, 2j_2, 2j_3 \leq j_1 + j_2 + j_3 \leq r - 2 \quad (6)$$

If not admissible,


$$= 0.$$

Semisimplicity



The diagram shows an equality between two expressions. On the left, there are two vertical blue lines. The left line is labeled 'a' at its base, and the right line is labeled 'b' at its base. This is followed by an equals sign. To the right of the equals sign is a summation symbol with a subscript 'c' and the word 'admissible' below it. To the right of the summation is a blue diagram of a crossing. The top-left strand is labeled 'a', the top-right strand is labeled 'b', the bottom-left strand is labeled 'a', and the bottom-right strand is labeled 'b'. The crossing is labeled with the Greek letter Γ_c to its left and the letter 'c' to its right.

Proof: use "generalisation".

Coefficients: $\Gamma_c = \Delta_c / \Theta_{abc}$.

Exercises - lecture 2

1. Show that the representation of $U_q\mathfrak{sl}_2$ on \mathbb{C}^2 given in the lecture satisfies the relations for the algebra.
2. Show that if u and d are basis vectors in \mathbb{C}^2 , then

$$\epsilon = Au \otimes d - Bd \otimes u$$

is an invariant tensor, deriving the relation between A , B and $q = e^h$.

3. Prove the Reidemeister II and III moves for $U_q\mathfrak{sl}_2$ spin $1/2$. Calculate the crossing intertwiner in the classical cases $A = \pm 1$. In which case does it simply permute the two factors?

Exercises - lecture 2

- 4 Show that the admissibility conditions for $\text{Hom}(j_1 \otimes j_2, j_3)$ are related to the inequalities on the edge-lengths $j_1 + 1/2, j_2 + 1/2, j_3 + 1/2$ of a non-degenerate triangle on a sphere. How is the radius of the sphere related to r ?