The Spin Foam Lectures
1: Introduction and Spin Networks

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Spin foam models

- Quantum gravity without matter
- Not realistic physics
- Models quantum space-time (technology, concepts)
- Observables
- Planck scale structure

Planck scale

- Planck area $= G\hbar$ is only scale
- Discrete structure at Planck scale (superpositions)
- Discreteness compatible with symmetries (c.f. angular momentum)
- General relativity in $G\hbar \to 0$ limit
- Continuum quantum picture?

3d QG: History

- Ponzano, Regge 1968 (3d gravity state sum model)
- Penrose 1970 (Spin networks)
- Witten 1989 (3d gravity functional integral)
- Turaev, Viro 1991 (3d gravity Λ ssm)

References

- Kauffman and Lins book: Temperley-Lieb recoupling theory...
- Moussouris: PhD thesis
- Major: A spin network primer
- JWB and Naish-Guzman: The Ponzano-Regge model
- JWB and Westbury: Invariants of PL 3-manifolds
- Roberts: Skein theory and TV invariants

Spin networks

Representations of a group/Hopf algebra G

$$X, Y, \ldots, X \otimes Y, \ldots$$

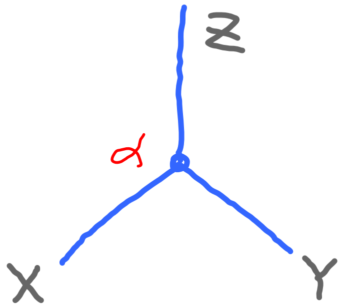
Intertwiners

$$\alpha \colon X \to Y$$

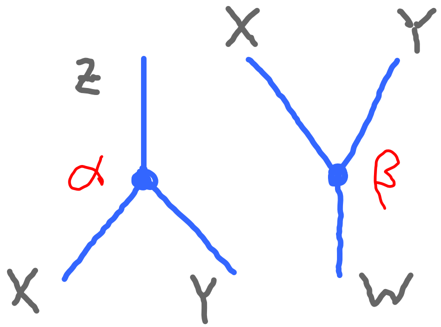
$$\alpha(gx) = g\alpha(x), \qquad g \in G, x \in X.$$

Diagrams

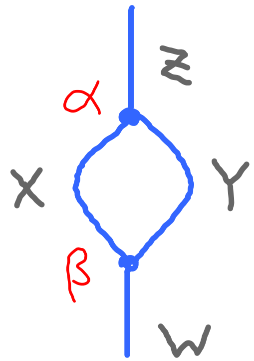
$\alpha\colon X \otimes Y \to Z$



$\alpha \otimes \beta$



$\alpha\beta$

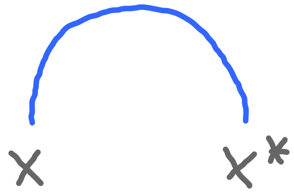


Equivalence of diagrams... see later

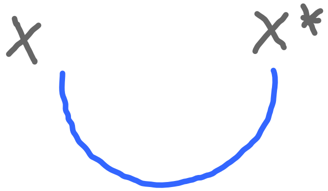
Duals

For any X, a dual representation X^*, and maps

$X \otimes X^* \to \mathbb{C}$



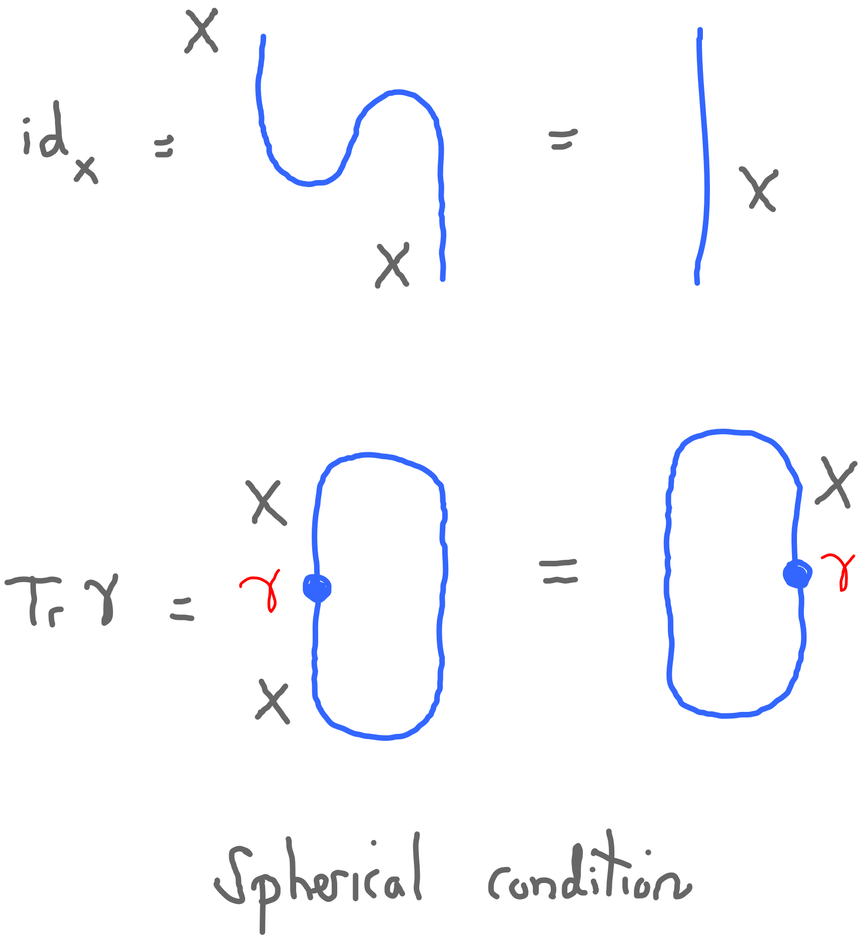
$\mathbb{C} \to X \otimes X^*$

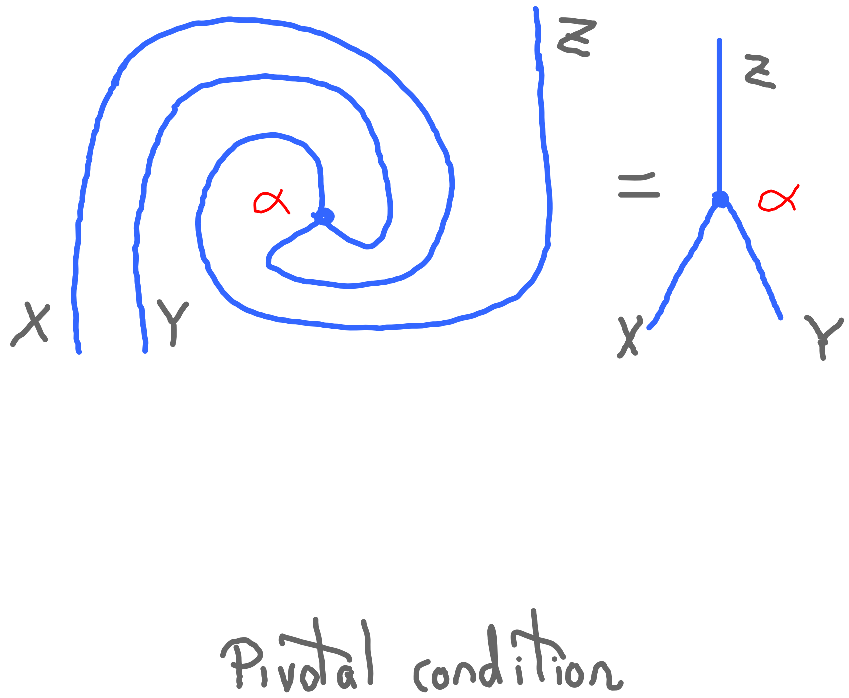


Always, $X^{**} = X^*$. For some representations, $X = X^*$.

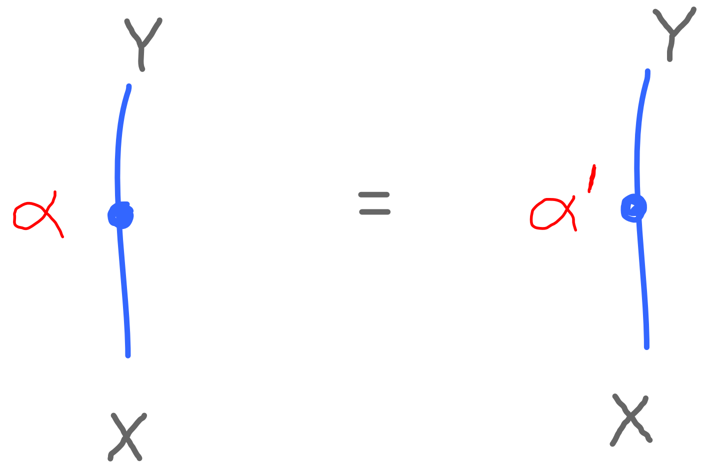
Coherence conditions

Symmetries: diffeomorphisms of S^2. Examples:





Equivalence of diagrams



if $\mathrm{Tr}(\beta\alpha) = \mathrm{Tr}(\beta\alpha')$ for all $\beta\colon Y \to X$.

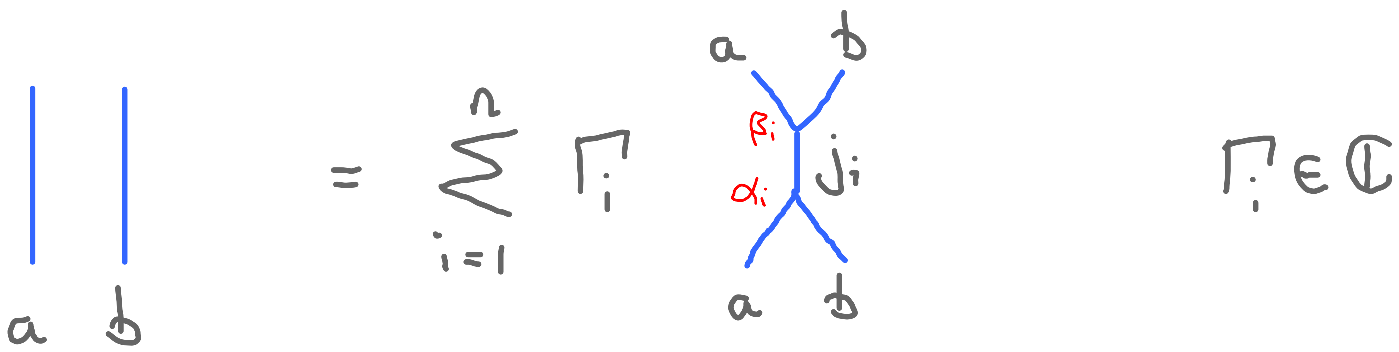
- Diagrams are equivalence classes of intertwiners
- A diagram is 0 if any closed diagram containing it is 0.
- Equivalence clear if only closed diagrams used

Semisimple condition

There is a list of irreducible representations $j_1, j_2, \ldots$
For any X,

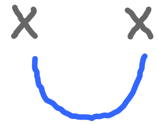
$$X \cong \bigoplus_{i=1}^{n} j_i$$

Example: for $X = a \otimes b$, and $a, b \in \mathrm{Irrep}$,

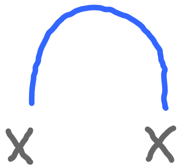


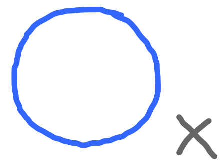
- Quantum group: semsimple after equivalence

Exercises

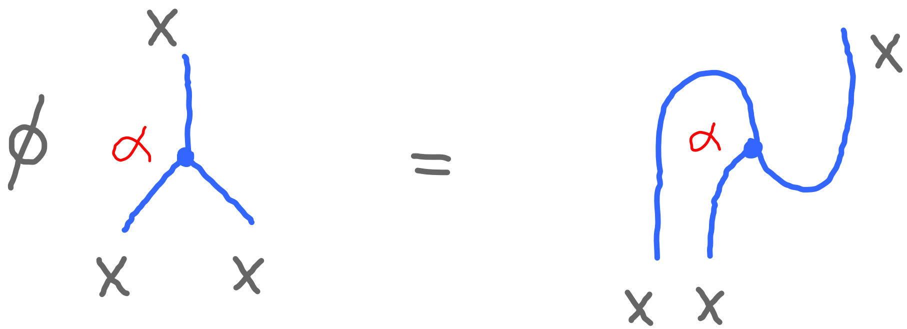
1. If $X = X^* = \mathbb{C}^2$, with basis u and d. Suppose 

is $1 \to Au \otimes d - A^{-1} d \otimes u$, for a constant $A \in \mathbb{C}$.

   Calculate 

and the number 

2. Suppose $X = X^*$. Denote the space of intertwiners between $X \otimes X$ and X by $\mathrm{Hom}(X \otimes X, X)$. Define a linear map $\phi\colon \mathrm{Hom}(X \otimes X, X) \to \mathrm{Hom}(X \otimes X, X)$ by



   Why is ϕ invertible? What are the possible eigenvalues of ϕ?

Exercises

3. Suppose $X = \mathbb{C}^2$, and the set of all intertwiners $X \to X$ is

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$$

and the trace is the matrix trace. Which diagrams $X \to X$ are equivalent to zero?

4. Prove that

$$\mathrm{Tr}(\mathrm{id}_X) = \mathrm{Tr}(\mathrm{id}_{X^*}).$$

(This number is called the *quantum dimension* of X.)