Renormalizable nc QFT

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(joint work with Raimar Wulkenhaar, arxiv:0909.1389)

Introduction

- Improve QFT use NCG and RG flow
- Scalar field on Moyal space, modified action 2004
- Taming the Landau ghost
  M Disertori, R Gurau, J Magnen, V Rivasseau, $=$ Paris group, 2006
  used Ward identity and Schwinger-Dyson equ for singular part
- Progress in solving the model
  - Ward identity
  - 2 pt Schwinger-Dyson equation
  - Integral equation for ren 2 pt fct
  - Perturbative solution
  - 4 pt Schwinger-Dyson equ
- Conclusions

Introduction

- Minkowski-Euclidean QFT. Require:
- Covariance, spectrum condition, locality, vacuum,...
  Algebraic, constructive, perturbative RG flow approaches
- regularization - renormalization
  UV, IR, zero convergence radius
  Landau ghost, trivial Higgs model?
- add ”Gravity” or deform Space-Time many approaches
  limited ”wedge” localization

Project

merge general relativity with quantum field theory through noncommutative geometry

Manifold: nc algebra, differential calculus, projective modules,...

Φ^4_Θ

ϕ^4 on $[x_\mu, x_\nu] = i\Theta_{\mu\nu}$,
IR/UV mixing: nonrenenormalizable
Theorem: Action Modified action

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi + \lambda\phi\star\phi\star\phi\star\phi\Big)(x)$$

for $\tilde{x}_\mu := (\theta^{-1})_{\mu\nu}x^\nu$
is perturbativly renormalizable to all orders in λ,

- cyclic order of momenta leads to ribbon graphs
- 3 proofs: Polchinski, multiscale analysis (Paris)
- Action has Langmann-Szabo position-momentum duality $S[\phi; \mu_0, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]$
- Curvature of cutoff-algebra explains potential term (Buric-Wohlgenannt)

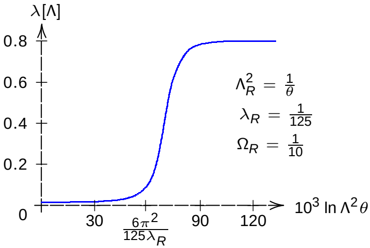
β function

$$\beta_\lambda = \frac{\lambda_{\text{phys}}^2}{48\pi^2} \frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3} + \mathcal{O}(\lambda_{\text{phys}}^3)$$

- flow bounded, L. ghost killed!
  Due to wave fct. renormalization
- $\Omega = 1$ betafunction vanishes
  up to all loops!
  Paris group

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in comm. case)



- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

Matrix model

- Action in matrix base at $\Omega = 1$
- Action functionals for *bare* mass μ_{bare}
- Wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$.

Fix $\theta = 4$, $\phi_{mn} = \overline{\phi_{nm}}$ real:

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\,,$$

$$H_{mn} = Z\left(\mu^2_{bare} + |m| + |n|\right)\,, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\,,$$

- Λ is cut-off. μ_{bare}, Z divergent
- No infinite renormalisation of coupling constant

$m, n, \ldots$ belong to $\mathbb{N}^2$, $|m| := m_1 + m_2$.

Ward identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, infinitesimally $\phi_{mn} \mapsto \phi_{mn} + \mathrm{i}\sum_{k\in\mathbb{N}^2_\Lambda}(B_{mk}\phi_{kn} - \phi_{mk}B_{kn})$
- not a symmetry of the action, but invariance of measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives

$$\begin{aligned} 0 &= \frac{\delta W}{\mathrm{i}\delta B_{ab}} = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\,\Big(-\frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}}(\mathrm{tr}(\phi J))\Big)e^{-S+\mathrm{tr}(\phi J)} \\ &= \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi \sum_n \Big((H_{nb} - H_{an})\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na})\Big)e^{-S+\mathrm{tr}(\phi J)} \end{aligned}$$

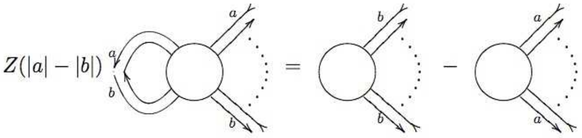
where $W[J] = \ln \mathcal{Z}[J]$ generates connected functions

trick $\phi_{mn} \mapsto \frac{\partial}{\partial J_{nm}}$

$$\begin{aligned} 0 = \Big\{ \sum_n \Big((H_{nb} - H_{an})\frac{\delta^2}{\delta J_{nb}\,\delta J_{an}} + \Big(J_{na}\frac{\delta}{\delta J_{nb}} - J_{bn}\frac{\delta}{\delta J_{an}}\Big)\Big) \\ \times \exp\Big(-V\Big(\frac{\delta}{\delta J}\Big)\Big)e^{\frac{1}{2}\sum_{p,q} J_{pq}H^{-1}_{pq}J_{qp}}\Big\}_c \end{aligned}$$

Interpretation

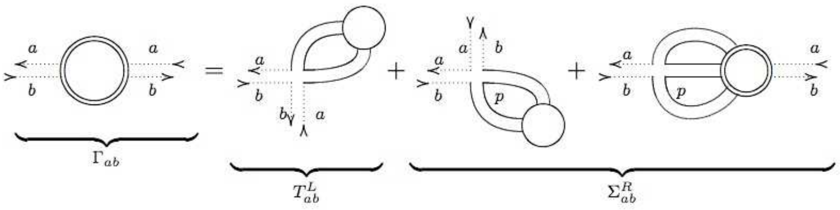
The insertion of a special vertex $V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$ into an external face of a ribbon graph is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



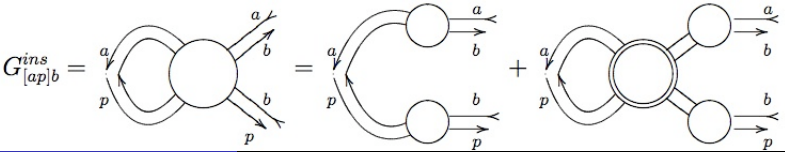
The dots stand for the remaining face indices.

$$Z(|a|-|b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$$

SD equation 2



- vertex is $Z^2\lambda$, connected two-point function is G_{ab}: first graph equals $Z^2\lambda\sum_q G_{aq}$
- open p-face in Σ^R and compare with insertion into connected two-point function; insert either into 1P reducible line or into 1PI function:



Amputate upper G_{ab} two-point function, sum over p, multiply by vertex $Z^2\lambda$, obtain: Σ^R_{ab}:

$$\Sigma^R_{ab} = Z^2\lambda \sum_p (G_{ab})^{-1} G^{ins}_{[ap]b} = -Z\lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} .$$

case $a = b = 0$ and $Z = 1$ treated.
Use $G^{-1}_{ab} = H_{ab} - \Gamma_{ab}$ and $T^L_{ab} = Z^2\lambda \sum_q G_{aq}$
gives for 2 point function:

$$Z^2\lambda \sum_q G_{aq} - Z\lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} = H_{ab} - G^{-1}_{ab} .$$

Symmetry $\Gamma_{ab} = \Gamma_{ba}$ is not manifest!

Renormalize

express SD equation in terms of the 1PI function Γ_{ab}
perform renormalisation for the 1PI part

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \Big) .$$

Taylor expand:

$$\Gamma_{ab} = Z\mu^2_{bare} - \mu^2_{ren} + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab} ,$$
$$\Gamma^{ren}_{00} = 0 \qquad (\partial\Gamma^{ren})_{00} = 0 ,$$

$\partial\Gamma^{ren}$ is derivative in a_1, a_2, b_1, b_2. Implies

$$G^{-1}_{ab} = |a| + |b| + \mu^2_{ren} - \Gamma^{ren}_{ab} .$$

... the resulting equation is

$$(Z-1)(|a|+|b|) + \Gamma^{ren}_{ab} = \int_0^\Lambda |p|\, d|p| \Big(\frac{Z}{|b|+|p|+\mu^2 - \Gamma^{ren}_{bp}} + \frac{Z^2}{|a|+|p|+\mu^2-\Gamma^{ren}_{ap}}$$
$$- \frac{Z^2+Z}{|p|+\mu^2 - \Gamma ren_{0p}} \qquad - \frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}} \frac{\Gamma^{ren}_{bp} - \Gamma^{ren}_{ab}}{(|p|-|a|)} + \frac{Z}{|p|+\mu^2-\Gamma^{ren}_{0p}} \frac{\Gamma^{ren}_{0p}}{|p|} \Big) ,$$

change variables,....

$$|a| =: \mu^2 \frac{\alpha}{1-\alpha}\;, \quad |b| =: \mu^2 \frac{\beta}{1-\beta}\;, \quad |p| =: \mu^2 \frac{\rho}{1-\rho}\;, \quad |p|\,d|p| = \mu^4 \frac{\rho\, d\rho}{(1-\rho)^3}$$
$$\Gamma_{ab} =: \mu^2 \frac{\Gamma_{\alpha\beta}}{(1-\alpha)(1-\beta)}\;, \quad \Lambda =: \mu^2 \frac{\xi}{1-\xi}$$

...get expression for ren.constant,..take cutoff limit:

Theorem

The renormalised planar two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory (with continuous indices) satisfies the integral equation

$$\begin{aligned} G_{\alpha\beta} = 1 + \lambda\Big(& \frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \\ & + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}) - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) \\ & + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y}\Big)\;, \end{aligned}$$

expansion

- Integral equation for Γ_{ab} is non-perturbatively defined. Resisted an exact treatment.
- We look for an iterative solution $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G^{(n)}_{\alpha\beta}$.
- This involves iterated integrals labelled by rooted trees.

Up to $\mathcal{O}(\lambda^3)$ we need

$$I_{\alpha} := \int_0^1 dx \, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\;,$$

$$I_{\underset{\bullet}{\alpha}} := \int_0^1 dx \, \frac{\alpha \, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2$$

$$I_{\underset{\bullet\;\bullet}{\alpha}} = \int_0^1 dx \, \frac{\alpha \, I_x \cdot I_x}{1-\alpha x} = -2\,\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big)$$

$$I_{\underset{\underset{\bullet}{\bullet}}{\alpha}} = \int_0^1 dx \, \frac{\alpha \, I_{\underset{\bullet}{x}}}{1-\alpha x} = -2\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big) - 2\mathrm{Li}_3(\alpha) - \ln(1-\alpha)\zeta(2)$$

$$+\ln(1-\alpha)\mathrm{Li}_2(\alpha) + \frac{1}{6}\big(\ln(1-\alpha)\big)^3$$

In terms of I_t and $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$:

$$
\begin{aligned}
G_{\alpha\beta} &= 1 + \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\
&\quad + \lambda^2\Big\{A\big(\beta I_{\underset{\bullet}{\beta}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
&\qquad + B\big(\alpha I_{\underset{\bullet}{\alpha}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big) \\
&\qquad + AB\big((I_{\underset{\bullet}{\alpha}} - \alpha) + (I_{\underset{\bullet}{\beta}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2) + 1)\big)\Big\} \\
&\quad + \lambda^3\Big\{A\mathcal{W}_\beta + \alpha AB\big(-\mathcal{U}_\beta + I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta\big) + \alpha A^2 B\mathcal{V}_\beta \\
&\qquad + B\mathcal{W}_\alpha + \beta BA\big(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_{\underset{\bullet}{\beta}} I_\alpha\big) + \beta B^2 A\mathcal{V}_\alpha \\
&\qquad + AB\big(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta (I_\alpha)^2 - I_\alpha (I_\beta)^2 - 6 I_\alpha I_\beta\big) \\
&\qquad + AB^2\big((1-\alpha)(I_{\underset{\bullet}{\alpha}} - \alpha) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\beta}} I_\alpha + I_\beta (I_\alpha)^2\big) \\
&\qquad + BA^2\big((1-\beta)(I_{\underset{\bullet}{\beta}} - \beta) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta + I_\alpha (I_\beta)^2\big)\Big\} + \mathcal{O}(\lambda^4)
\end{aligned}
$$

$$\mathcal{T}_\beta := \beta I_{\underset{\bullet}{\beta}} - \beta I_\beta + (I_\beta - \beta)\ ,$$

Similar for $\mathcal{U}_\beta, \mathcal{V}_\beta, \mathcal{W}_\beta$

Remark: $\frac{I_\beta - \beta}{\beta} = \int_0^1 dx\ \frac{\beta x}{1 - \beta x}$

Observations

Polylogarithms and multiple zeta values appear in singular part of individual graphs of e.g. ϕ^4-theory (Broadhurst-Kreimer)
We encounter them for regular part of all graphs together

Conjecture

- $G_{\alpha\beta}$ takes values in a polynom ring with generators $A, B, \alpha, \beta, \{I_t\}$, where t is a rooted tree with root label α or β
- at order n the degree of A, B is $\leq n$,
  the degree of α, β is $\leq n$,
  the number of vertices in the forest is $\leq n$.

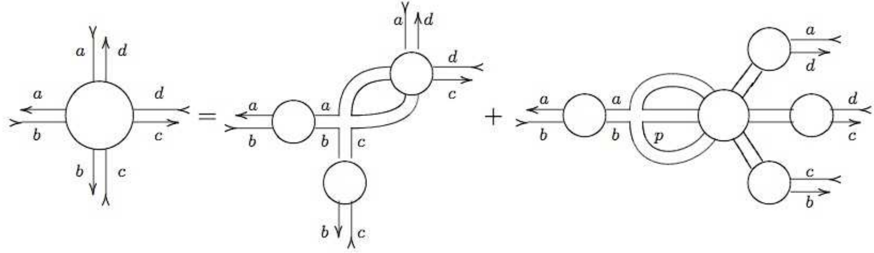
If true:

- There are $n!$ forests of rooted trees with n vertices at order n
- estimate: $|G^{(n)}_{\alpha\beta}| \leq n!(C_{\alpha\beta})^n$

may lead to Borel summability?.

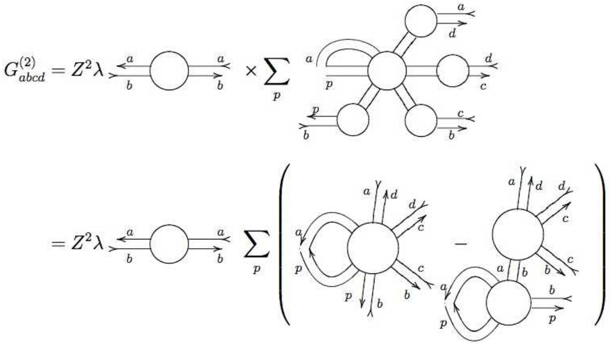
Schwinger-Dyson equ 4 pt fct

Follow *a*-face, there is a vertex at which *ab*-line starts:



1. First graph: Index c and a are opposite
   It equals $Z^2 \lambda G_{ab} G_{bc} G^{ins}_{[ac]d}$
2. Second graph: Summation index p and a are opposite. We open the p-face to get an insertion.

   This is not into full connected four-point function, which would contain an *ab*-line not present in the graph.



second graph equals

$$= Z^2\lambda\Big(\sum_p G_{ab} G^{ins}_{[ap]bcd} - G^{ins}_{[ap]b} G_{abcd}\Big)$$

1PI four-point function

$$\Gamma^{ren}_{abcd} = Z\lambda\Big\{\frac{G^{-1}_{ad} - G^{-1}_{cd}}{|a| - |c|} + \sum_p \frac{G_{pb}}{|a| - |p|}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\Big\}$$

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory satisfies

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho\, d\rho\, \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{G_{\alpha\delta}+\lambda\Big((\mathcal{M}_\beta-\mathcal{L}_\beta-\mathcal{Y})G_{\alpha\delta}+\int_0^1 d\rho\, \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho\, d\rho\, \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)}$$

Corollary

$\Gamma_{\alpha\beta\gamma\delta} = 0$ is not a solution!
We have a non-trivial (interacting) QFT in four dimensions!

Conclusions

- Studied model at $\Omega = 1$
- RG flows save
- Used Ward identity and Schwinger-Dyson equation
- ren. 2 point fct fulfills nonlinear integral equ
- ren. 4 pt fct linear inhom. integral equ
  perturbative solution:

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2 \Big(\frac{(1-\gamma)(I_\alpha - \alpha) - (1-\alpha)(I_\gamma - \gamma)}{\alpha - \gamma} + \frac{(1-\delta)(I_\beta - \beta) - (1-\beta)(I_\delta - \delta)}{\beta - \delta} \Big) + \mathcal{O}(\lambda^3)$$

- is nontrivial and cyclic in the four indices
- nontrivial Φ^4 model ?