SUPERFIELDS
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FIRST ORDER FLOWS FOR N=2 EXTREMAL BLACK HOLES AND DUALITY INVARIANTS

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CORFU' 2009 : SCHOOL AND WORKSHOP ON COSMOLOGY AND STRINGS
THEORY, COSMOLOGY, PHENOMENOLOGY

8 September 2009

AC.
Dall'Agata
Ferrara
Yeranyan

MOTIVATIONS

- BLACK HOLES IN EINSTEIN GRAVITY: $\begin{cases} \text{Schwarzschild} & M \\ \text{Kerr} & J \\ \text{R. N.} & (p,q) \end{cases}$

- STRING / M-THEORY $\Rightarrow$ EINSTEIN – MAXWELL + SCALARS

$$d: \ \{ g_{\mu\nu} , F^{\Lambda}_{\mu\nu} , \varphi^i \} \qquad G/H \qquad N \text{ SUSY}$$

(ungauged) SG $\begin{cases} N = 1 \\ N = 2 \ \Rightarrow \text{SPECIAL GEOMETRY} \qquad d = 4,5 \\ \vdots \\ N = 8 \ \Rightarrow \text{HIGH CONSTRAINTS , U.V properties} \end{cases}$

SOLUTIONS IN CLASSICAL LIMIT: p-branes, domain walls, ...

- B.H. $\begin{cases} \text{THERMODYNAMICAL PROPERTIES } T, S \Leftrightarrow \text{GEOMETRY} \quad S = \dfrac{A}{4} \\ \\ \text{DYNAMICAL PROPERTIES : ATTRACTOR MECHANISM} \end{cases}$

EXTREMAL BH : $M = Z ; T = 0 \qquad V_{BH}(\varphi^i_H ; p, q)$

- WIDE RANGE OF APPLICATIONS $\begin{cases} \text{QUANTUM CORRECTIONS} \\ \text{ENTROPY FORMALISM} \\ \text{SPLIT ATTRACTORS , ...} \end{cases}$

MAIN POINTS

- EXTREMAL BH SOLUTIONS OF EXTENDED SG HAVE

  ATTRACTOR BEHAVIOUR $\begin{cases} \text{BPS} \quad N=8 \begin{cases} 1/8 \\ 1/4 \\ 1/2 \end{cases} \\ \text{non-BPS} \end{cases}$

  Ferrara
  Kallosh
  Strominger
  Trivedi
  Jena
  Goldstein
  Mandal
  Larsen
  Gimon
  Simon
  Hotta
  Kubota

- SUSY FEATURES OF SOLUTIONS ARE ENCODED IN

  U-DUALITY INVARIANTS $\begin{cases} \text{CHARACTERIZATION} \quad \text{BPS/non BPS} \\ \text{CLASSIFICATION} \quad \text{ORBITS OF } Q=(p^\Lambda, q_\Lambda) \end{cases}$

- SOLUTIONS COME FROM 1ST ORDER FLOW EQUATIONS

  ($\Longrightarrow$ SOLVE IN TERMS OF HARMONIC FUNCTIONS)

- NON-BPS "FAKE" SUPERPOTENTIAL DRIVING 1ST ORDER FLOWS

  IS ALSO GIVEN BY DUALITY INVARIANTS

$\boxed{\begin{array}{c}\text{AC. Dall'Agata, Ferrara, Yeranyan} \\ \text{0908.1110}\end{array}}$

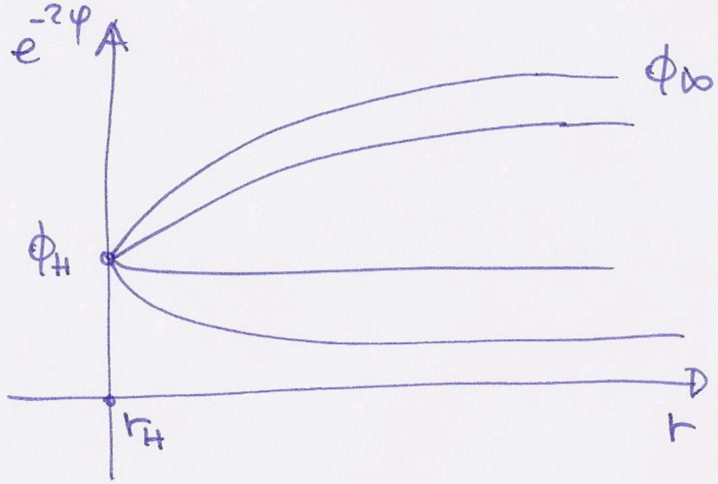
ATTRACTOR MECHANISM

Ferrara
Kallosh
Strominger

- SUSY BH with e/m CHARGES (Q,P) are SOLITONIC SOLUTIONS

1-dim QUANTUM MECH. PROBLEM : RADIAL EVOLUTION $\phi^i(r)$

$$r \to r_H \begin{cases} \phi^i(r) \to \phi^i_H(r_H) = \phi(P,q;M) \\ \dot{\phi}(r) \to 0 \end{cases}$$



scalars at horizon do not depend on

$$\lim_{r\to\infty} \phi^i(r) = \phi^i_\infty \qquad \{\phi^i_\infty\} = \text{MODULI SPACE}$$

$\Rightarrow$ NO SCALAR HAIR

- FIXED POINTS ARE EXTREMA OF EFFECTIVE POTENTIAL OF GEODESIC ACTION

$$V_{BH}(q,P;\varphi^i) = -\frac{1}{2} Q^T \mathcal{M} Q \qquad \mathcal{M}(\mathcal{N}) \ 2n \times 2n \text{ MATRIX}$$

$$Q = (P^\Lambda, q_\Lambda) \quad Sp(2n,\mathbb{R})$$

Bekenstein - Hawking

$$\boxed{S = \frac{A}{4} = \pi V^*_{BH}(\varphi_H(P,q);P,q)}$$

SET UP FOR EXTENDED SUPERGRAVITIES

- **SUSY ALGEBRA** $\{Q_{\alpha A}, Q_{\beta B}\} = \epsilon_{\alpha\beta} Z_{AB}(p, q; \varphi)$ BPS: $M \geq |Z|$

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M} Q = \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I$$

$$\begin{cases} Z_{AB} = -Z_{BA} & \text{CENTRAL CHARGES} \\ Z_I & \text{MATTER CHARGES} \end{cases}$$

A, B SU(N)

I fundam. of matter group (when present)

- **N=8** $G/H = \frac{E_{7(7)}}{SU(8)}$ U-duality group ($\mathbb{Z}$)

QUARTIC CARTAN – CREMMER – JULIA INVARIANT:

$$I_4 = \mathrm{Tr}(Z\bar{Z})^2 - \frac{1}{4}(\mathrm{Tr}\, Z\bar{Z})^2 + 4(\mathrm{Pf}\, Z + \mathrm{Pf}\, \bar{Z}) = T_{ijkl}\, q^i q^j q^k q^l$$

$$Z_{AB} \xrightarrow{SU(8)} e^{i\varphi} \begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \rho_3 & \\ & & & \rho_4 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

NORMAL FRAME

$$\boxed{\frac{\partial I_4}{\partial \varphi^i} = 0}$$

BPS: $\frac{S_{BH}}{\pi} = \sqrt{|I_4|}$ non-BPS: $\frac{S_{BH}}{\pi} = \sqrt{-I_4}$

Kallosh – Kol

Ferrara Gibbons Kallosh

BLACK HOLE SOLUTIONS: $N=2$

ACTION

$$\mathcal{L} = -\frac{R}{2} + \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma\mu\nu} + \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} (*F^{\Sigma})^{\mu\nu} + g_{i\bar{j}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}}$$

$\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ VECTOR KINETIC MATRIX $\quad g_{i\bar{j}}$: SPECIAL kähler σ-MODEL

$$\int_{S_2} F^{\Lambda} = 4\pi p^{\Lambda} \qquad \int_{S_2} G_{\Lambda} = 4\pi q_{\Lambda}$$

e/m CHARGES $\quad Q = (p^{\Lambda}, q_{\Lambda})$

BH ANSATZ

STATIC, SPHER. SYMM., CHARGED, ASYMPT. FLAT : $g_{\mu\nu}(r)$, $\phi^i(r)$

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left[\frac{c^4\,dr^2}{\sinh^4(cr)} + \frac{c^2}{\sinh^2(cr)}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$c^2 = 2ST$ EXTREMALITY PARAMETER $\quad \boxed{c=0}$

INTEGRATE OVER $\mathbb{R}_t \times S^2 \Rightarrow$ $\boxed{\text{1d ACTION for } \{U(r), z^i(r)\}}$

$$\begin{cases} \mathcal{L} = (U'(r))^2 + g_{ij} z'^i \bar{z}'^j + e^{2U} V_{BH} - c^2 \\ H = (U'(r))^2 + g_{ij} z'^i \bar{z}'^j - e^{2U} V_{BH} + c^2 = 0 \end{cases}$$

HAMILTONIAN CONSTRAINT

$\boxed{\text{EQUATIONS OF MOTION}}$

$$\begin{cases} U'' = e^{2U} V_{BH} \\ z'' + \Gamma^i_{jk} z'^j z'^k = e^{2U} g^{ij} \partial_j V_{BH} \end{cases}$$

EXTREMAL CASE $c=0$: ACTION TAKES $\boxed{\text{BPS FORM}}$

$$S = \int dr \left[(U' \pm e^U W)^2 + |z'^i \pm 2 e^U g^{ij} \partial_j W|^2 \mp 2 \frac{d}{dr}(e^U W) \right)$$

$\Rightarrow$ $\boxed{\text{FIRST ORDER FLOW EQUATIONS}}$

$$\begin{cases} U' = \pm e^{U} W \\ z'^{i} = \pm 2 e^{U} g^{ij} \partial_j W \end{cases} \qquad e^{U} W \sim M_{ADM}$$

$$V_{BH} = W^2 + 4 g^{ij} \partial_i W \partial_j W$$

$(\partial_i + \partial_i K) Z$

N=2 $$V_{BH} = |Z|^2 + 4 g^{ij} \partial_i |Z| \partial_j |Z| = |Z|^2 + |DZ|^2$$

BPS SUSY SOLUTIONS: $$\begin{cases} D_i Z = 0 \qquad Z \neq 0 \\ \dfrac{\partial V_{BH}}{\partial z^i} = 0 \qquad \text{BPS FIXED POINTS} \end{cases}$$

(EXTENDED SG : $V_{BH} = \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I$ CENTRAL CHARGES MATTER CHARGES)

DUALITY INVARIANTS FOR $N=8$

(Ferrara Kallosh
A.C. Ferrara Guecchi
AM6

SUSY ALGEBRA $\{Q_{\alpha A}, Q_{\beta B}\} = \epsilon_{\alpha\beta} Z_{AB}(p, q; \varphi)$

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M} Q = \frac{1}{2} Z_{AB} \bar{Z}^{AB} \qquad A = 1, \dots, 8$$

$G/H = \frac{E_{7(7)}}{SU(8)}$ U-duality group ($\mathbb{Z}$) spanned by 70 φ^i

$$Z_{AB} \xrightarrow{SU(8)} \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & z_3 & \\ & & & z_4 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{NORMAL FRAME}$$

QUARTIC CARTAN CREMMER-JULIA INVARIANT:

$$I_4 = \mathrm{Tr}(Z\bar{Z})^2 - \frac{1}{4}(\mathrm{Tr}\, Z\bar{Z})^2 + 4(\mathrm{Pf}\, Z + \mathrm{Pf}\, \bar{Z})$$

$$\begin{cases} \text{BPS:} & \dfrac{S_{BH}}{\pi} = \sqrt{I_4} \quad I_4 > 0 \\ \text{NON-BPS:} & \dfrac{S_{BH}}{\pi} = \sqrt{-I_4} \quad I_4 < 0 \end{cases} \qquad \boxed{\frac{\partial I_4}{\partial \varphi^i} = 0}$$

$A = Z\bar{Z}$: INVARIANTS: $\{\mathrm{tr}\, A, \ \mathrm{tr}\, A^2, \ \mathrm{tr}\, A^3, \ \mathrm{tr}\, A^4, \ \mathrm{Re}\, \mathrm{Pf}\, Z\}$

AM BIS

ATTRACTORS & DUALITY

- KALLOSH KOL (1996) : AREA OF HORIZON FOR $N=8$ EXTREMAL BH IS PROPORTIONAL TO $\sqrt{I_4}$; 1/8 PRESERVED SUSY
  $A=0$ FOR 1/4 AND 1/2

- FERRARA - MALDACENA (1998) : DIFFERENT SUSY FEATURES ARE DISTINGUISHED BY G-INVARIANT CONDITIONS ON CHARGES

- FERRARA - GÜNAYDIN : (1998) : FOR FIXED VALUES OF I_4 INVARIANT IN $d=4$ AND I_3 INVARIANT IN $d=5$ CHARGE VECTORS $Q=(p^\Lambda, q_\Lambda)$ FOR SUPERGRAVITIES ON SYMMETRIC SPACES DESCRIBE ORBITS WHOSE NATURE IS STRICTLY RELATED TO SUSY PROPERTIES OF FIXED POINTS

- FERRARA BELLUCCI, GUNAYDIN, MARRANI, ...

AM 8

ATTRACTOR EQUATIONS : $N=2$ $d=4$

$$Z_{AB} = \epsilon_{AB} Z$$

$\{X^\Lambda, F_\Lambda\}$ $Sp(2n_V+2)$ Kähler Hodge manifold

$$F_\Lambda = \partial_\Lambda F(X)$$

$$Z = e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda)$$

$$K(z,\bar z) = -\log i(\bar X^\Lambda F_\Lambda - X^\Lambda \bar F_\Lambda) \qquad g_{i\bar j} = \partial_i \partial_{\bar j} K(z,\bar z)$$

$$V_{BH}(z,\bar z; p,q) = |Z| + g^{i\bar j} D_i Z D_{\bar j} \bar Z$$

$$\boxed{\begin{aligned} I_1 &= |Z|^2 + |DZ|^2 \\ I_2 &= |Z^2| - |DZ|^2 \end{aligned}}$$

ATTRACTOR EQUATION :

$$\frac{\partial V}{\partial z^i} = 0 \implies 2|Z| D_i Z + i C_{ijk} g^{j\bar j} g^{k\bar k} D_{\bar j} \bar Z D_{\bar k} \bar Z = 0$$

BPS $\quad D_i Z = 0 \quad \partial_i \partial_j V = 0 \quad \partial_i \partial_{\bar j} V = 2 g_{i\bar j} V$ ATTRACTIVE

non-BPS $\quad D_i Z \neq 0 \quad \partial_i V = 0 \begin{cases} Z = 0 \\ Z \neq 0 \end{cases}$ possible SADDLE POINTS

$\implies$ FLAT DIRECTIONS

QUESTION: HOW ABOUT non-BPS?

- A.M.
- C-FUNCTION
- SOLUTIONS
- ORBITS

Ferrara Gibbons
Kallosh - Goldstein -
Iizuka - Jena - Trivedi
Larsen, Sinica, Simon
...

ANSWER: A.C. + Dall'Agata 0702088

A) LOOK FOR A "FAKE SUPERPOTENTIAL" $W \neq |Z|$

? $W(z,\bar{z})$: a) $V_{BH} = W^2 + 4 g^{ij} \partial_i W \partial_j W$ SAME EFFECTIVE POTENTIAL

b) $\partial_i W = 0$ GIVES non-BPS CRITICAL POINTS

B) Andrianopoli, D'Auria, Orazi, Trigiante 0706.0712

PROPOSE THAT $\boxed{W = \sum_I \alpha^I I_I}$ FOR EXTENDED SUSY

LINEAR COMBINATION OF U-DUALITY INVARIANTS

0905.3938 W = HAMILTON'S PRINCIPAL FUNCTION FOR non-BPS FLOWS

$\boxed{\text{GUIDELINE}}$

A.C. + Dall'Agata 0702088

$$x^A = \{u, z^i, \bar{z}^{\bar{i}}\} \qquad V(u, z, \bar{z}) = e^{2u} V_{BH}(z, \bar{z})$$

DEFINE BH POTENTIAL AS $V(x^A) = g^{AB} \partial_A \hat{W}(x) \partial_B \hat{W}(x)$

$$g_{AB} = \left(\begin{array}{c|c} 1 & \\ \hline & g_{ij} \end{array}\right)$$

$\Rightarrow$ THE POTENTIAL IS INVARIANT UNDER ROTATIONS $R(z, \bar{z})$:

$$\partial_A W(x) = R_A{}^B(z, \bar{z}) \partial_B \tilde{W}(x)$$

$$\boxed{R^T g R = g}$$

THEY MUST PRESERVE THE NORM OF $\partial_A W(x)$

QUESTION: WHAT IS A COMPLETE SET OF INVARIANTS FOR $N=2$?

Cerchiai Marrani Ferrara Zumino
0902.3973

ANSWER: $Sp(2n+2;\mathbb{R})$ INVARIANTS ARE:

$$\begin{cases} i_1 = Z\bar{Z} \\ i_2 = g^{i\bar{j}} Z_i \bar{Z}_{\bar{j}} \\ i_3 = \frac{1}{6}\left[Z N_3(\bar{Z}) + \bar{Z} \bar{N}_3(Z) \right] \\ i_4 = \frac{i}{6}\left[Z N_3(\bar{Z}) - \bar{Z} \bar{N}_3(Z) \right] \\ i_5 = g^{i\bar{j}} C_{ijk} C_{\bar{i}\bar{j}\bar{k}} \bar{Z}^{\bar{i}} \bar{Z}^{\bar{j}} \bar{Z}^{\bar{k}} \end{cases}$$

$Z_i = D_i Z$

CUBIC NORM:

$$N_3(\bar{Z}) = C_{ijk} \bar{Z}^i \bar{Z}^j \bar{Z}^k$$

$$\bar{N}_3(Z) = C_{\bar{i}\bar{j}\bar{k}} Z^{\bar{i}} Z^{\bar{j}} Z^{\bar{k}}$$

WITH 1 RELATION $\quad I_4 = (i_1 - i_2)^2 + 4 i_4 - i_5$

FOR SYMMETRIC SPECIAL GEOMETRIES $\quad \partial_i I_4 = 0$

$\Rightarrow$ **CLAIM** : $W(z,\bar{z}) = W(i_1, \dots, i_5)$

SYMMETRIC GEOMETRIES

$n=1$ Acer. + Dall'Agata

n: Gnecchi Marrani Ferrara

Andrianopoli D'Auria Trigiante

EXAMPLE I

QUADRATIC SERIES: $F(X) = \frac{i}{2}\left[(X^0)^2 - \sum_{i=1}^{n}(X^i)^2\right]$

MODULI SPACE: $\frac{SU(1,n)}{SU(n)\times U(1)}$ $\quad C_{ijk}=0 \quad \Rightarrow i_3=i_4=i_5=0$

ATTRACTOR EQ. $\partial_i V_{BH}=0 \Rightarrow 2\bar{Z}D_iZ=0$

non-BPS fixed point: $DZ\neq 0\,,\ Z=0$

$W=W(i_1,i_2)$: DEMONSTRATE USING SPECIAL GEOMETRY:

$$\Longrightarrow \begin{cases} W_{BPS} = |Z| = \sqrt{i_1} & DZ=0 \quad DW\neq 0 \\ W_{non-BPS} = \sqrt{i_2} & DW=0 \quad DZ\neq 0 \end{cases}$$

$$= \sqrt{g^{i\bar{j}}D_iZD_{\bar{j}}\bar{Z}}$$

$$V_{BH} = |Z|^2 + g^{i\bar{j}}D_iZ\bar{D}_{\bar{j}}\bar{Z} = W^2 + 4g^{i\bar{j}}\partial_iW\partial_{\bar{j}}W$$

EXAMPLE II

CUBIC SERIES $\quad F(X) = C_{IJK} \dfrac{X^I X^J X^K}{X^0} \Longrightarrow \dfrac{(X^1)^3}{X^0} \qquad \dfrac{X^1}{X^0} = t$

limit of STU model $\qquad K = -\log\left[-i(t-\bar{t})^3\right]$

A.C. Dall'Agata: charges $Q = (p_0, q^1)$

$$W_{BPS} = \left| e^{K/2}\left(t q_1 + p^0 t^3\right)\right|$$

$$W_{non\text{-}BPS} = \left| e^{K/2}\left(t q_1 + p^0 t^2 \bar{t}\right)\right|$$

$\Longrightarrow$ FIELD DEPENDENT TRANSFORMATION $\quad W_{BPS} \longrightarrow W_{non\text{-}BPS}$

moreover $\quad W \neq |DZ|$

THERE ARE 3 INDEPENDENT INVARIANTS $\quad \{i_1, i_2, i_3\}$

BECAUSE FOR 1 MODULUS $\quad i_4 = i_4(i_1, i_2, i_3)$ and $i_5 = i_5(i_2)$

$$I_4 = I_4(i_1, i_2, i_3)$$

FIND $\boxed{\text{W FOR GENERIC CHARGES}}$ BY

1) Take W for STU model (Bellucci Ferrara Marrani Yeranyan 0807.3503) in limit $S=T=U=t$

2) Compute it in simple symplectic frame $Q=(p^0, q_0)$ and then boost it to generic charges

$$\Rightarrow \boxed{W^2 = \frac{i_1+i_2}{4} + \frac{3}{8}\left\{\left[\left(i_1-\frac{i_2}{3}\right)^3-(i_1+i_2)I_4+4i_3\sqrt{-I_4}\right]^{1/3}+\left[\left(i_1-\frac{i_2}{3}\right)^3-(i_1+i_2)I_4-4i_3\sqrt{-I_4}\right]^{1/3}\right\}}$$

FIXED POINT $\quad i_2 = 3i_1 = \frac{3}{4}\sqrt{-I_4}\ ,\ i_3=0 \Rightarrow W^2 = \sqrt{-I_4} \equiv S$

$\Rightarrow$ NON-POLYNOMIAL EXPRESSION WITH $\sqrt[3]{}$!

FULL NON-BPS SOLUTION

GIVEN W WE CAN SOLVE FLOW EQS IN TERMS OF HARMONIC FUNCTIONS

$$\begin{cases} u' = e^{u} W \\ z'^{i} = -2 e^{u} g^{i\bar{j}} \partial_{\bar{j}} W \end{cases} \qquad \begin{aligned} t &= x + iy \\ a &= e^{-u} \end{aligned}$$

$$\Rightarrow \begin{cases} e^{-4u} = \mathcal{H}_1^3 \mathcal{H}_0 - b^2 \\ x = \dfrac{b\sqrt{-I_4}}{2(p^1)^2 (\mathcal{H}_1)^2} \\ y = \dfrac{e^{-2u}\sqrt{-I_4}}{2(p^1)^2 \mathcal{H}_1^2} \end{cases}$$

$$\mathcal{H}_0 = \frac{(-I_4)^{1/4} H_0}{\sqrt{2}\, q_0}$$

$$\mathcal{H}_1 = -\frac{(-I_4)^{1/4} H^1}{\sqrt{2}\, p^1}$$

$$H_0 = h_0 - \sqrt{2}\, q_0 \tau$$

$$H^1 = h^1 + \sqrt{2}\, p^1 \tau$$

OUTLOOK

- ONE SHOULD NOW FIND W FOR n MODULI (i_5 CONTRIBUTES)
- PREVIOUS ATTEMPTS FAILED BECAUSE (SYMMETRIC GEOMETRIES) $\left\{ \begin{array}{l} W \text{ IS NOT LINEAR} \\ \{i_1, \dots, i_5\} \end{array} \right.$
- RESULT AGREE WITH RECENT PAPER Bossard Pioline Michel (TIME REDUCTION)
- WORK ON STU IS IN PROGRESS .
- FIND TRANSFORMATION $Z \longrightarrow W$: ?
- USE EMBEDDING IN $N = 8$, $d = 4, 5$ A.C. Ferrara Gnecchi
- QUANTUM CORRECTIONS, GAUGED SG, ...

THANK YOU!

SEE YOU CORFU' 2010...