Quantum Chromodynamics: basic features of the pertubative approach

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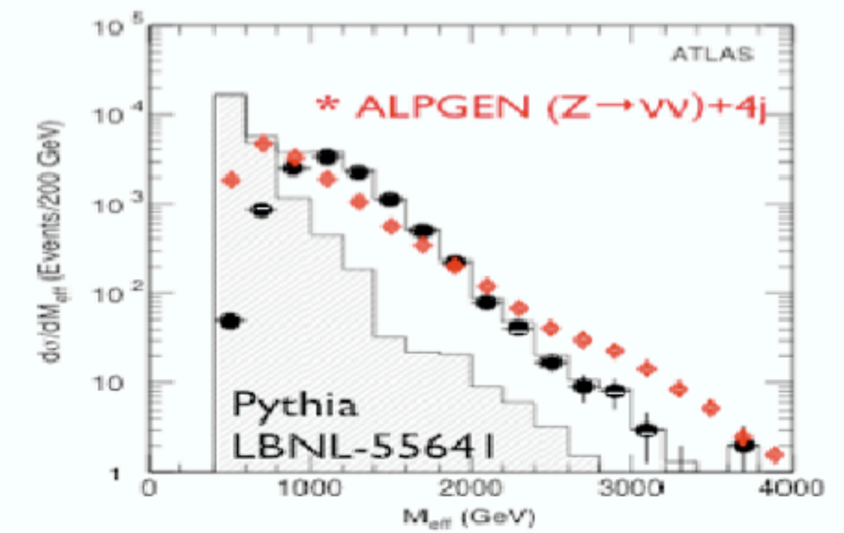
New phenomena at TeV energies

- LEP, SLC, Tevatron, muon experiments, B-meson factories, ...
  - *Discovered all Standard Model particles except the Higgs boson!*
  - *Few inconclusive* ($< 3\sigma$) *deviations:* $(g-2)_\mu, \ldots$
  - *A host of precision measurements, pointing to a light Higgs boson!*
- The SM is not the whole story:
  - *Massive neutrinos.*
  - *Dark matter + dark energy.*
  - *Gravity?*
- New theoretical possibilities:
  - *Supersymmetry, Extra dimensions, New Strong Sector.*

Physics at hadron colliders: TEVATRON and LHC

- Probing directly the physics of electroweak symmetry breaking
  - Tevatron is measuring cross-sections for EWK gauge-boson pair production.
  - After LEP in 2001, a collider places new limits on the production cross-section.
- LHC will be capable of both discoveries and precision studies.
- Perturbative QCD is the main theoretical tool for any type of studies which are interesting in typical new physics searches.

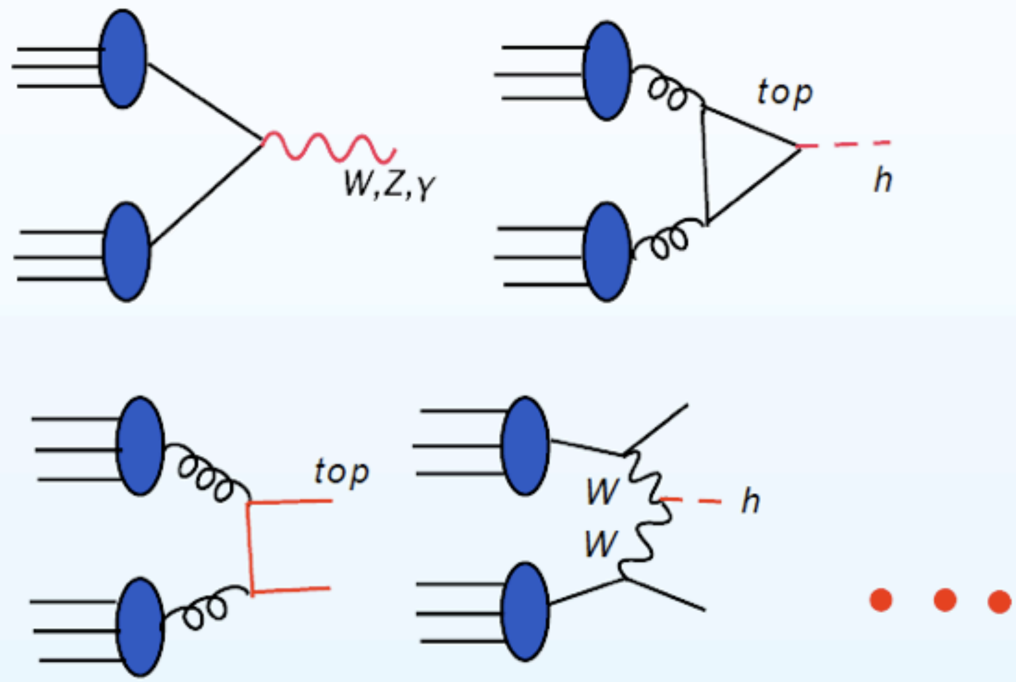
Complicated signals of new physics



Mangano

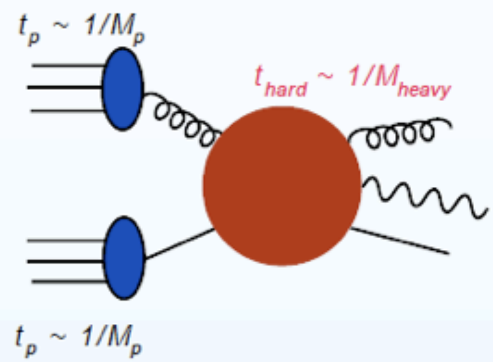
- Supersymmetry signal: Jets + missing energy.
- Two approaches to describe the background.
  - Shower Monte-Carlo!
  - Exact matrix-elements at leading order in α_s!
- Two different aproximations: we should learn how to use them, and their limitations.
- We should discover methods how to improve them!

Processes at the LHC



- A vast experimental program: pointless to give a list of "interesting" processes.
- We hope to discover many new BSM processes. But even within the SM, there is a lot to do!

Hard scattering processes in hadronic collisions



- When a high transverse momentum jet or heavy object is produced, this occurs in a very short time:

$$\tau_{hard} \sim \frac{1}{m_{heavy}}$$

- Interactions inside the protons are too slow

$$\tau_{protons} \sim \frac{1}{M_{proton}} >> \tau_{hard}$$

to change the proton content during the hard scattering.

Factorization theorem

- The protons are beams of quarks and gluons $p_1 = x_1 P_1, p_2 = x_2 P_2$.
- $f_i(x)$: Momentum distributions of quarks and gluons in the proton do not change during the scattering.

$$\sigma = \sum_{ij} \int f_i^{proton}(x_1)\, f_j^{proton}(x_2)\, \sigma_{ij}(x_1, x_2)$$

- Cross-section for the hard scattering of quarks and gluons factorizes. At large scales the strong coupling is small; we can use perturbation theory

$$\sigma_{ij} = a_s^N \left(\text{Leading Order} + \alpha_s \text{ Next LO} + \alpha_s^2 \text{ NNLO } + \ldots\right)$$

Does this really hold beyond the LO? YES

Collins, Soper, Sterman, . . .

- Perturbation theory includes all types of interactions:

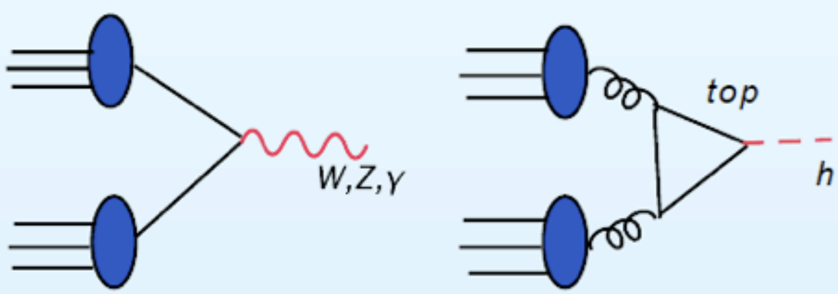
$$\int \frac{d^4k}{k^2 \dots}$$

Infrared: $k \sim 0$, *hard:* $k \sim M$, and *ultraviolet:* $k \sim \infty$!

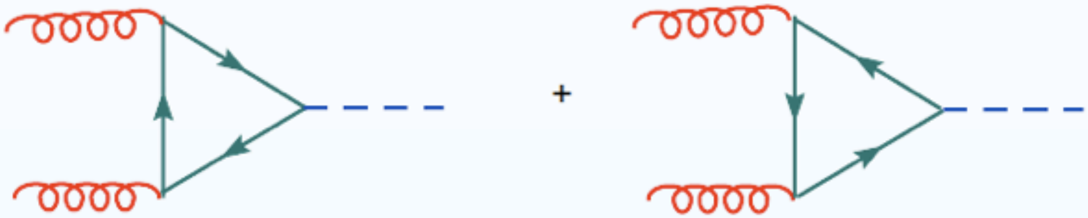
- The contributions of the ultraviolet interactions renormalize the physical values of the Lagrangian parameters, e.g. α_s.
- Some infrared interactions do not contribute (cancelations) in interesting physical observables.
- The remaining infrared interactions change the parton distribution functions.

In these lectures

- QCD is an enormous field of study. I can only talk about very few things.
- I will not repeat the QCD course that you may take from a Master's curriculum.
- Learn from a "simple" example:
  - Production of electroweak gauge bosons (Drell-Yan): A classical process which will be extremely well studied at the LHC. Luminosity, pdf's, M_w, $sin^2\theta$, new physics, ...
  - Production of a Higgs boson: *the best bet for a new discovery in particle physics at the LHC!*



Leading Order cross-section for $pp \to h$



- Fold partonic cross-sections with gluon densities:

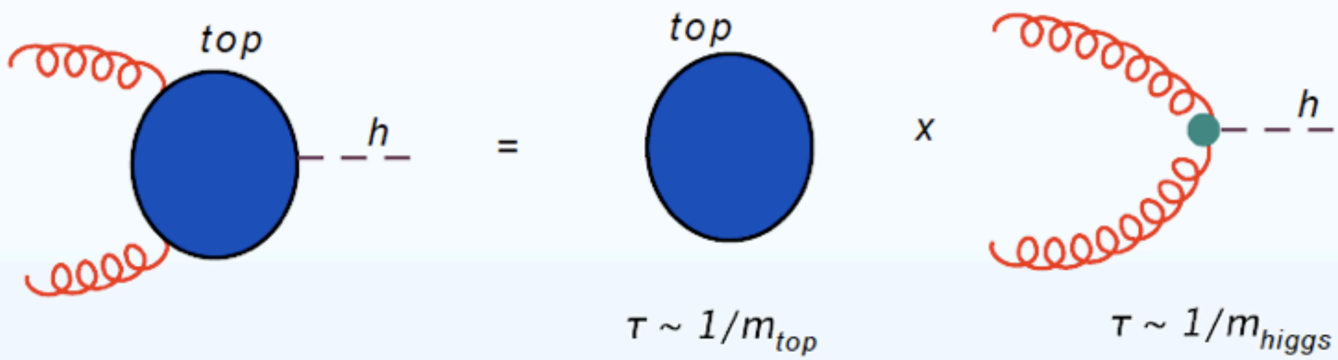
$$\sigma^{LO}_{gg\to h} = \frac{\alpha_s^2 G_F}{128\sqrt{2}\pi} |\mathcal{G}(\tau)|^2 \int d^4p_h \delta^+(p_h^2 - m_h^2)\delta^4(p_1 + p_2 - p_h)$$

$$\sigma^{LO}_{pp\to h} = \frac{\alpha_s^2 G_F}{128\sqrt{2}\pi} |\mathcal{G}(\tau)|^2 \int dx_1 dx_2 g(x_1) g(x_2) \delta(1-z),$$

$$\tau = \frac{m_h^2}{m_{top}^2}, \quad z = \frac{m_h^2}{\hat{s}}, \quad \hat{s} = x_1 x_2 s.$$

Heavy top-quark limit

Wilczek; Voloshin; Shifman; Chetyrkin, Kniehl, Steinhauser

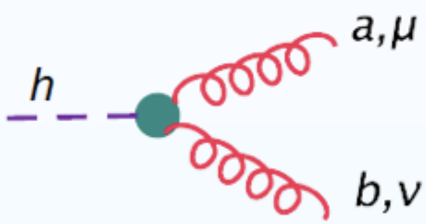


- If $m_{top} >> m_h$, interactions inside the top-loop happen very fast; insensitive to variations of the external gluon fields which interact much slower. Factorization:

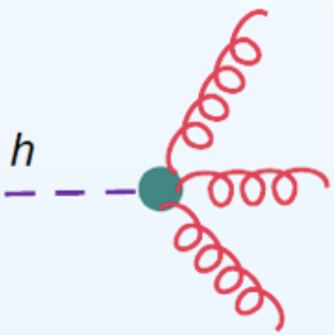
$$\mathcal{L}_{hgg} = C(m_t)\frac{h}{v}\left[-\frac{Z}{4}G^{a}_{\mu\nu}G^{\mu\nu;a}\right],$$

$$C(m_t) = \frac{-1}{3\pi}\left[1 + \frac{11}{4}\frac{\alpha_s}{\pi} + \ldots\right]$$

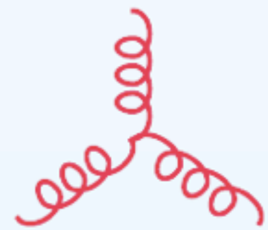
Higgs-gluon interactions for large m_t

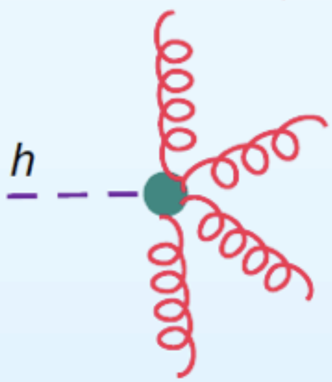


$$= \quad C/v\, Z \;\left[\, g^{\mu\nu}\, p_1 \cdot p_2 - p_1^{\nu}\, p_2^{\mu} \,\right]\; \delta^{ab}$$



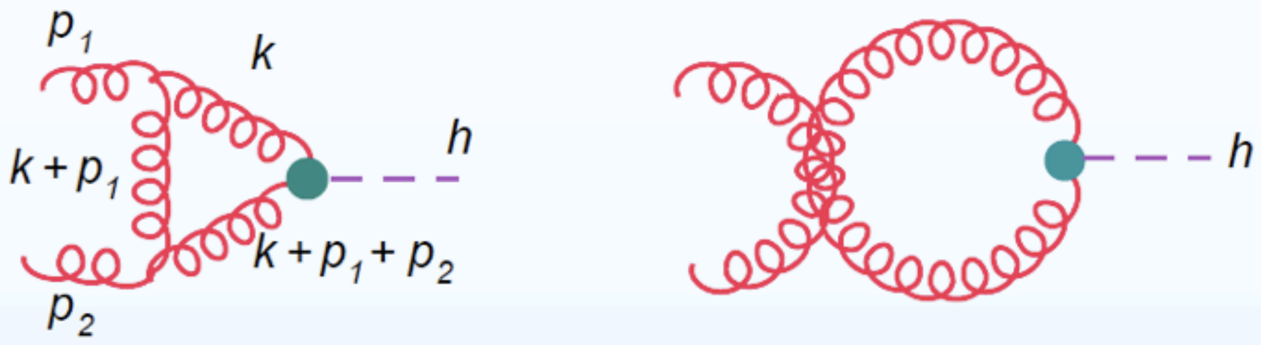
$= \quad C/v\ Z$





$= \quad C/v\ Z$

NLO virtual corrections



- The one-loop amplitude is:

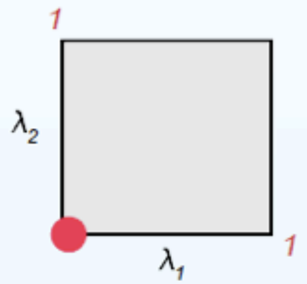
$$\mathcal{M}_1^{gg\to h} = C/vZg_s^2 N\delta^{ab}\epsilon_1^\mu \epsilon_2^\nu \int \frac{d^{4-2\epsilon}k}{i\pi^{2-\epsilon}} \frac{\{k^\mu, k^\nu, p_1, p_2\}}{k^2\,(k+p_1)^2\,(k+p_1+p_2)^2}$$

- Introduce three Feynman parameters, and integrate the loop momentum.

$$\mathcal{M}_1^{gg\to h} = C/vZg_s^2 \left(\frac{\mu^2}{-s}\right)^\epsilon \int_0^1 dx_1 dx_2 dx_3 \delta(x_1+x_2+x_3-1)\frac{f(x_1,x_2,x_3)}{(x_1x_2)^{1+\epsilon}}$$

Expansion in the regulator ϵ

- Map to a square: $x_1 = \lambda_1$, $x_2 = \lambda_2(1-\lambda_1)$, $x_3 = (1-\lambda_1)(1-\lambda_2)$



$$\mathcal{M}_1^{gg\to h} = \ldots \int_0^1 d\lambda_1 d\lambda_2 (1-\lambda_1)^{-\epsilon} \frac{f(\lambda_1, \lambda_2(1-\lambda_1), (1-\lambda_1)(1-\lambda_2))}{(\lambda_1\lambda_2)^{1+\epsilon}}$$

- The singularities are factorizable. So I can apply:

$$\lambda^{-1-\epsilon} = -\frac{\delta(\lambda)}{\epsilon} + \sum_{n=0}^{\infty}\left[\frac{\ln^n \lambda}{\lambda}\right]_+ \frac{(-\epsilon)^n}{n!}$$

Divergences of the loop amplitude

- The result is an expansion in ϵ

$$\mathcal{M}_1^{gg\to h} = C/vZg_s^2 N\delta^{ab}\left(\frac{\mathcal{A}_2}{\epsilon^2} + \frac{\mathcal{A}_1}{\epsilon} + \mathcal{F}inite + \mathcal{O}\left(\epsilon\right)\right)$$

- We should renormalize!

$$\mathcal{M} = C/vZ\left[M_{bare}^{(0)} + \frac{a_s^{bare}}{\pi}M_{bare}^{(1)} + \ldots\right]$$

  - Coupling constant renormalization:

$$\alpha_s^{bare} = \alpha_s(\mu)(4\pi)^{-\epsilon}e^{\gamma\epsilon}\left[1 - \frac{\alpha_s(\mu)}{\pi}\frac{\beta_0}{\epsilon} + \ldots\right]$$

  - Composite operator renormalization:

$$Z = 1 - \frac{\alpha_s(\mu^2)}{\pi}\frac{\beta_0}{\epsilon} + \ldots$$

One-loop renormalization

- Re-arrangement of the series:

$$\mathcal{M} = C/v \left[M^{(0)} + \frac{a_s(\mu)}{\pi} M^{(1)} + \ldots \right]$$

with

$$M^{(1)} = \left[-\frac{N}{2\epsilon^2} - \frac{\beta_0}{\epsilon} + \frac{7\pi^2 N}{24} + \mathcal{O}(\epsilon) \right] \left(\frac{\mu^2}{-s} \right)^{\epsilon} M^{(0)}$$

- Divergences do not dissappear after only ultraviolet renormalization!
- We have left over infrared singularities. Perturbation theory is, in this case, meaningless.

Infrared divergences

- Ultraviolet divergences are universal; they "redefine" the fields, couplings, masses of the Lagrangian in exactly the same way for all processes.
- Infrared singularities are also universal and have a known form.
- Without doing a calculation we know that:

Giele, Glover; Kunszt, Soper; Kunszt, Signer, Troczanyi; Catani, Seymour

$$M^{(1)} = \mathbf{I}^{(1)} M^{(0)} + M^{(1)}_{finite}$$

One-loop infrared divergences

The operator $I^{(1)}$ changes the colour of the leading order amplitude and contains the universal singularities.

$$I^{(1)} = \frac{1}{2}\sum_i \left(\frac{T_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}\right)\sum_{j\neq i}\frac{T_i\cdot T_j}{T_i^2}\left(\pm\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon}$$

with

$$I^{(1)} = \frac{1}{2}\sum_i \left(\frac{T_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}\right)\sum_{j\neq i}\frac{T_i\cdot T_j}{T_i^2}\left(\pm\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon}$$

with

$$T_q^2 = C_F, \quad T_g^2 = C_A, \quad \gamma_q = \frac{3C_F}{2}, \quad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R N_f .$$

Multi-loop infrared divergences

Catani; Sterman, Tejeda-Yeomans;Aybat,Sterman,Dixon;Becher,Neubert;Gardi,Magnea

- Universality holds at higher loops:

$$M^{(2)} = I^{(1)} M^{(1)} + I^{(2)} M^{(0)} + Finite$$

with

$$I^{(2)} = -\frac{1}{2} I^{(1)}(\epsilon) \left(I^{(1)} + \frac{2\beta_0}{\epsilon} \right) + \frac{const}{\epsilon} I^{(1)}(2\epsilon) + H$$

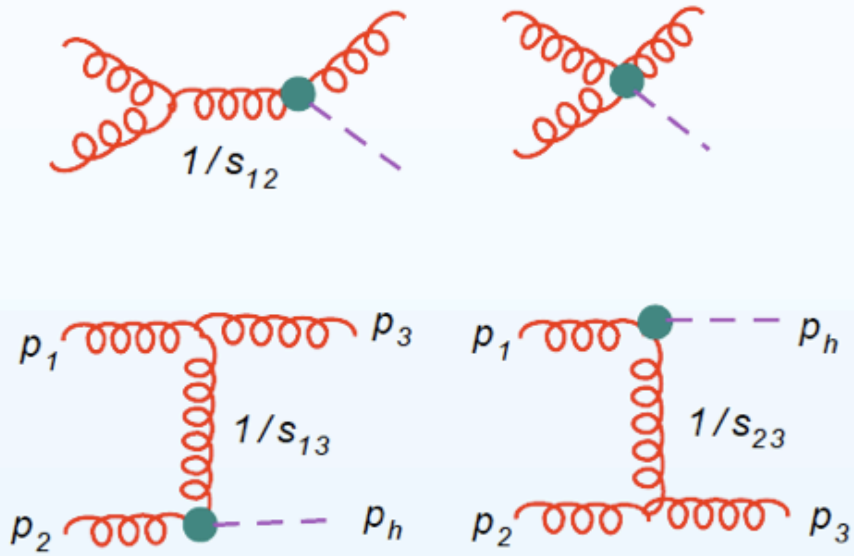
- General factorization picture:

$$\mathcal{M}_{\text{all orders}} = \left(\prod_{all\ \ legs} J^{leg}(\alpha_s, \epsilon) \right) \mathbf{Soft}(\alpha_s, \epsilon) Hard(\alpha_s)$$

Do they cancel?

- Loop divergences are universal and factorizable!
- But how do they cancel?
- The Kinoshita-Lee-Neuenberg theorem: processes which "look the same" in the infrared limit (soft and collinear radiation) must be added together!

The radiative process: $gg \to h + g$



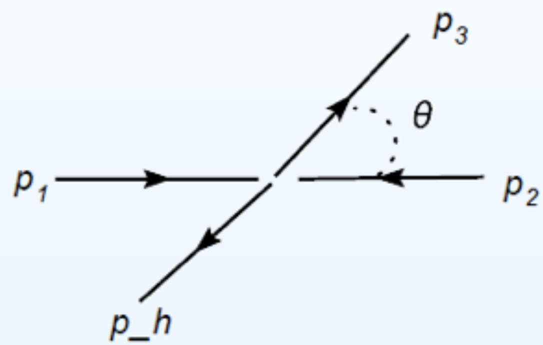
- M is singular when

$$s_{13} = (p_3 - p_1)^2 = 0 \rightsquigarrow E_3 E_1 (1 - cos\theta_{13}) = 0 \rightsquigarrow E_3 = 0 \text{ or } \theta_{13} = 0,$$

$$s_{23} = 0 \rightsquigarrow E_3 = 0 \text{ or } \theta_{13} = 0, \pi$$

Phase-space integration

$$I_{gg\to hg} = \int d^d p_h d^d p_3 \delta^+(p_h^2 - m_h^2)\delta^+(p_3^2)\delta^d(p_h + p_3 - p_1 - p_2)\frac{F(s_{13}, s_{23})Obj(s_{13}, s_{23})}{s_{13}s_{23}}$$



$$s_{13} = s(1-z)(1-\cos\theta), \quad s_{23} = s(1-z)(1+\cos\theta), \quad m_h^2 = zs, \quad E_3 = \sqrt{s}(1-z)$$

$$I_{gg\to hg} = \ldots (1-z)^{1-2\epsilon} \int_0^\pi d\theta \sin\theta^{1-2\epsilon} \frac{F(s_{13}, s_{23})Obj(s_{13}, s_{23})}{(1-z)^2(1-\cos^2\theta)}$$

Phase-space integration

Change variable: $cos\theta = 1 - 2\lambda$,

$$s_{13} = 2s(1-z)(1-\lambda), \quad s_{23} = 2s(1-z)\lambda$$

$$I \sim (1-z)^{-1-2\epsilon} \int_0^{\pi} d\lambda \left[\lambda^{-1-\epsilon}(1-\lambda)^{-\epsilon} + (1-\lambda)^{-1-\epsilon}\lambda^{-\epsilon}\right] F(s_{13}, s_{23}) Obj(s_{13}, s_{23})$$

Expand using:

$$\lambda^{-1-\epsilon} = -\frac{\delta(\lambda)}{\epsilon} + \left[\frac{1}{\lambda}\right]_+ + \ldots$$

- An infrared pole in ϵ develops only when it is exactly: $\lambda = 0, 1$ or $z = 1$!

Infrared safe observables

- The $Obj(s_{13}, s_{23})$ is 0 or 1! It defines when we include an "event" in our measurement and when we exclude it from the $gg \to hg$ process.
- We hope that by adding together the cross-section for this process and the cross-section for $gg \to h$

$$\sigma_{gg \to h} \sim \left(-\frac{N}{\epsilon^2} + \ldots \right) \delta(1 - z)$$

  we obtain a finite result.
- Our observable must include ($= 1$) the phase-space points where the matrix-elements develop poles in ϵ. In other words, we should measure only "infrared safe" quantities:

$$Obs(gg \to h + \text{soft/collinear gluon}) = Obs(gg \to h)$$

Cancelation?

- Now I can add together:

$$\sigma^{nlo-part}_{gg\to h+X} = \sigma^{1-loop}_{gg\to h} + \sigma^{tree}_{gg\to h+g} = Finite - 2\mathcal{N}^{LO}_{gg\to h}\frac{\alpha_s}{\pi}\frac{P^{(0)}_{gg}}{\epsilon}$$

finding in our result the universal gluon splitting function:

$$P^{(0)}_{gg} = N\left(\frac{1}{z} + z(1-z) - 2 + \left[\frac{1}{1-z}\right]_+\right) + \beta_0\delta(1-z)$$

- Remember that: $\sigma^{LO}_{gg\to h} = \mathcal{N}^{LO}_{gg\to h}\delta(1-z)$:

$$\sigma^{1-loop}_{gg\to h} + \sigma^{tree}_{gg\to h+g} + 2\int\frac{dx}{x}\frac{\alpha_s(\mu)}{\pi}\frac{P^{(0)}_{gg}(x)}{\epsilon}\sigma^{LO}_{gg\to h}\left(\frac{z}{x}\right)$$

- i.e. the gluons emit collinear radiation before the hard scattering

Redefining the parton densities

- Collinear radiation from the initial state before the hard scattering, changes the energy distribution of the gluons. This effect should be included in a re-definition of the parton densities. The renormalized pdf's are:

$$\tilde{f}_i(x,\mu^2) = \sum_j \int_x^1 \frac{dy}{y}\left(\delta_{ij}\delta(1-y) - \frac{\alpha_s(\mu)}{\pi}\frac{P_{ij}^{(0)}}{\epsilon} + \dots\right) f_j\left(\frac{x}{y}\right)$$

- The finite partonic cross-section is:

$$\begin{aligned}\sigma_{ij}(x_1,x_2) &= \int_0^1 dy_1 dy_2 \left(\delta_{ik}\delta(1-y_1) - \frac{\alpha_s}{\pi}\frac{P_{ik}^{(0)}(y_1)}{\epsilon} + \dots\right)\\ &\times\left(\delta_{kj}\delta(1-y_2) - \frac{\alpha_s}{\pi}\frac{P_{kj}^{(0)}(y_2)}{\epsilon} + \dots\right)\\ &\times\hat{\sigma}_{ij}(x_1y_1,x_2y_2) \qquad (1)\end{aligned}$$

Factorization and renormalization scale

$$\sigma = \sum_{ij} \int dx_1 dx_2 \tilde{f}_i(x_1, \mu_f) \tilde{f}_j(x_2, \mu_f) \hat{\sigma}_{ij \to h+X}(x_1, x_2, \mu_f, \mu_r)$$

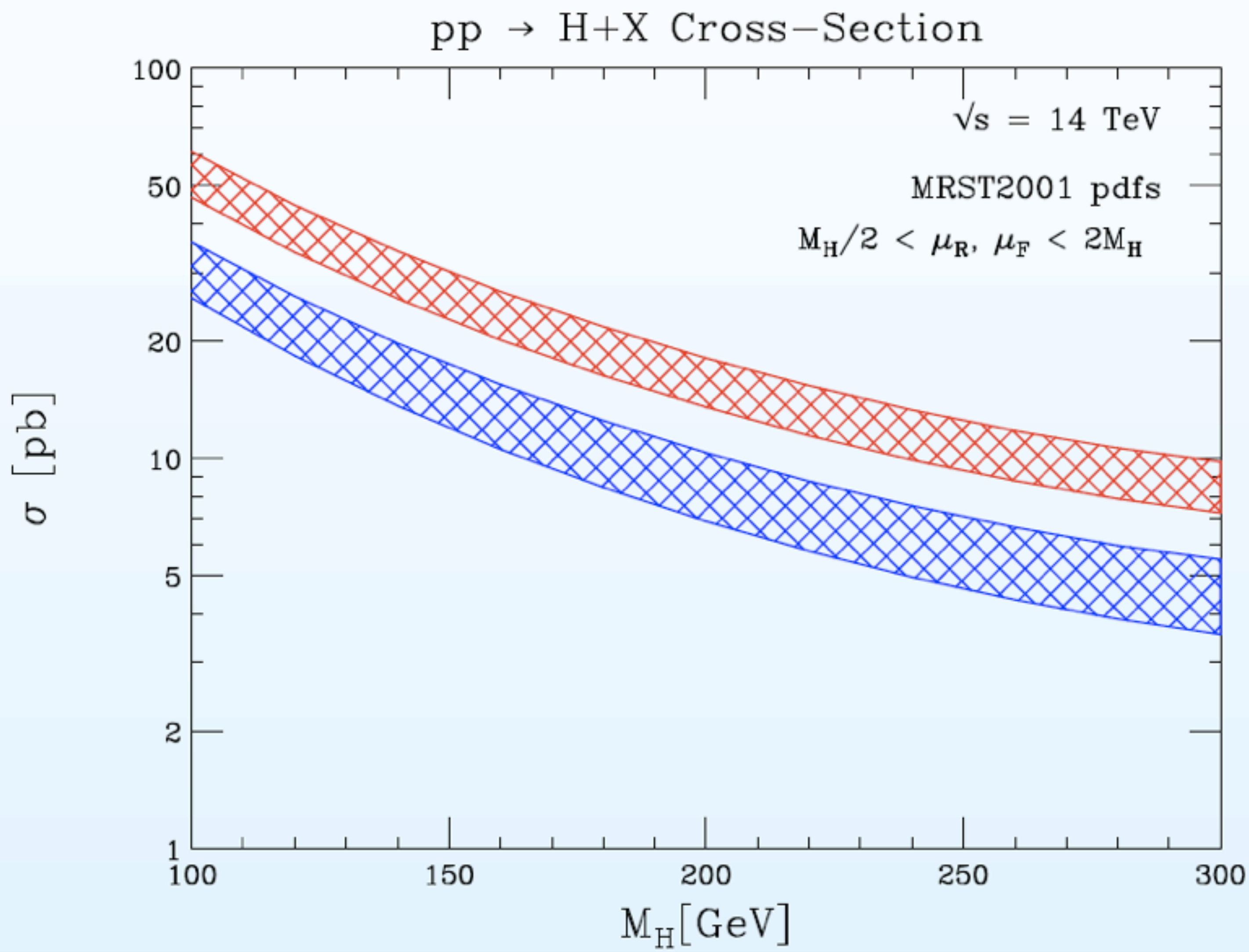
The renormalization scale enters through the running of the strong coupling:

$$\alpha_s(\mu_f) = \alpha_s(\mu_r) \left(1 - \beta_0 \frac{\alpha_s(\mu_r)}{\pi} \ln\left(\frac{\mu_f}{\mu_r}\right)\right)$$

Finally:

$$\hat{\sigma}_{gg\to h+X} = \frac{\pi}{576 v^2} \left(\frac{\alpha_s(\mu_r)}{\pi}\right)^2 \Bigg[\delta(1-z) + \frac{\alpha_s(\mu_r)}{\pi} \Big\{ 2\beta_0 \ln\left(\frac{\mu_r}{m_h}\right) \delta(1-z) - 2P_{gg}^{(0)}(z) \ln\left(\frac{\mu_f}{m_h}\right) + \delta(1-z)\left(\frac{11}{2} + \pi^2\right) + 12 \left[\frac{\ln(1-z)}{1-z}\right]_+ + \dots \Big\} \Bigg] \quad (2)$$

Higgs boson cross-section at LO and NLO



Summary

- Factorization theorem and perturbation theory are the main framework for making quantitative predictions at the LHC.
- Objervables must be infrared safe
- Infrared contributions are universal. Ultraviolet singularities renormalize α_s, m. Soft singularities cancel. Collinear singularities are factored into the parton densities.
- Cross-sections at fixed order depend on “arbitrary” renormalization and factorization scales.
- QCD effects are important in our example calculation (and not only!)

Perturbative QCD: a glimpse at the present and future

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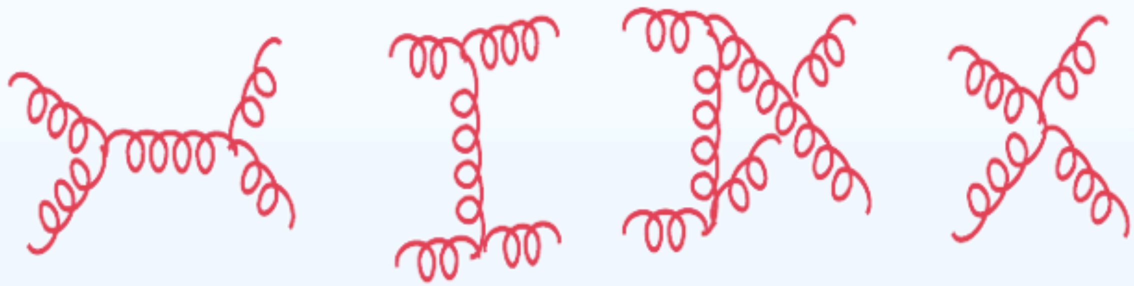
Introduction

- Perturbative QCD is a very active field in recent years.
- We have made progress in every aspect of it:
  - Leading order, Next to LO, NNLO
  - Resummation, merging fixed order calculations and parton showers.
  - All orders!
- Progress has been made with the generation of very good new ideas. Not just by turning the crank!
- Refreshing influx of ideas and people from other fields (string theory)
- A very competitive research area, with many challenges to be taken up.

Leading order perturbation theory

- It provides a rough estimate for cross-sections.
- Usually, it involves the calculation of tree diagrams:



  - Derive Feynman rules from Lagrangian.
  - Write down diagrams.
  - Perform Dirac and colour algebra.
  - Numerically integrate over the phase-space.

- A conceptually solved problem (like most in pQCD)! But in practice we need to be more clever.

Algebraic explosion

- For example, in $gg \to N$ gluons we need to compute:

N	diagrams
2	4
4	220
6	34,300
8	10,525,900

- Feynman rules in gauge theory

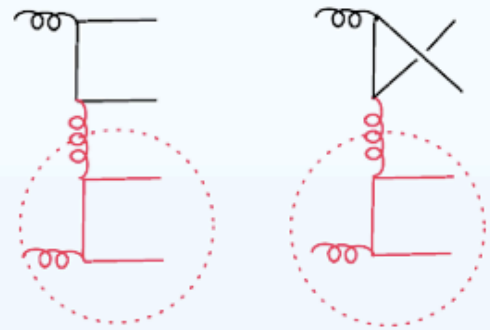
$$\mathcal{V}_{ggg} = f^{abc}\left[g_{\mu_1\mu_2}(p_1 - p_2)^{\mu_3} + g_{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} + g_{\mu_3\mu_1}(p_3 - p_1)^{\mu_2}\right]$$

- Algebra of γ matrices, colour algrebra, etc.

$$Tr(\gamma^{\mu_1}\gamma^{\mu_2}) = \text{1 term}$$
$$Tr(\gamma^{\mu_1}\cdots\gamma^{\mu_8}) = \text{105 terms}$$
$$Tr(\gamma^{\mu_1}\cdots\gamma^{\mu_{14}}) = \text{26,931 terms}$$

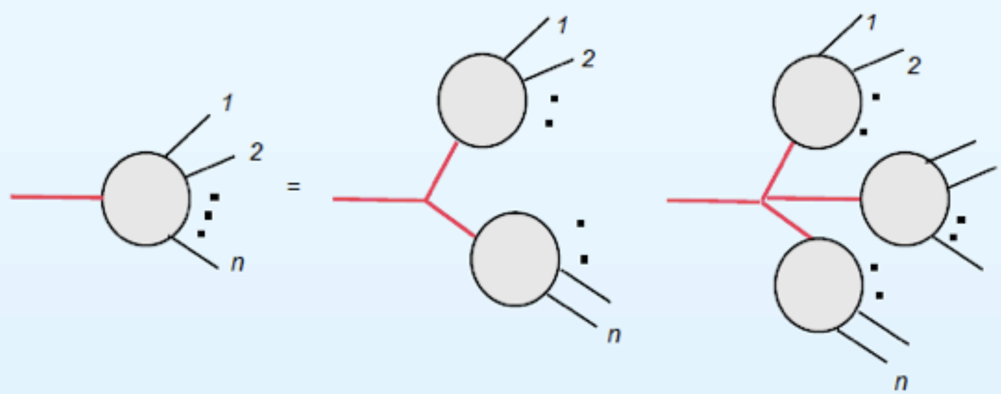
Recursion at tree-level

- Feynman diagrams contain sub-parts which we compute over and over.



- It is possible to organize the evaluation of tree amplitudes recursively

e.g. Berends, Giele



Britto-Cachazo-Feng-Witten recursions

- Amplitudes are functions of external momenta.

$$A(p_1, p_2, \ldots, p_n)$$

- For massless particles $p^\mu \to p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$; this can be written as the product of two spinors:

$$p = \lambda^a \tilde{\lambda}^{\dot{a}}$$

- Then they considered a more general object, extending two of the momenta to be complex but preserving momentum conservation:

$$p_1 = p_1 + z\lambda_1^a \tilde{\lambda}_4^{\dot{a}} \qquad p_4 = p_4 - z\tilde{\lambda}_4^{\dot{a}} \lambda_1^a$$

- Counter-intuitive extension, given that we like to think of the momenta of scattered particles as real.

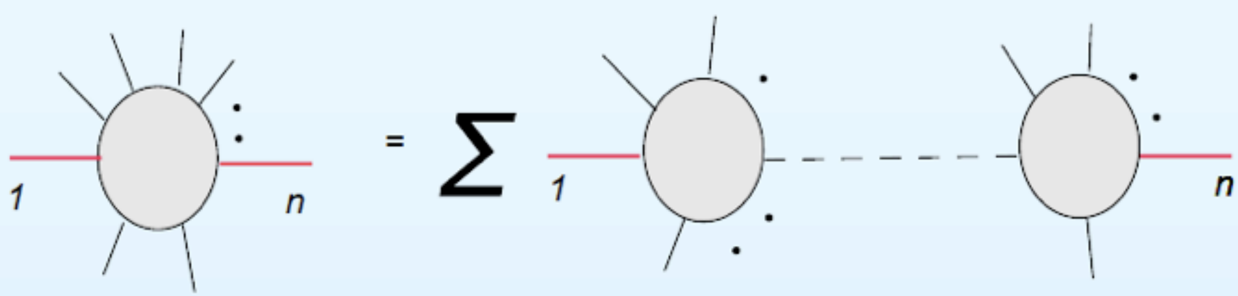
Analytic extension of tree amplitudes

- The generalized tree-amplitudes, if $(A(z \to \infty) = 0)$, have simple poles; all possible poles that can be found in propagators of Feynman diagrams.

$$A(z) = \sum_{p_{i\ldots j}} \frac{c_{ij}(z)}{p_{i\ldots j}^2 - z}$$

- The physical amplitude is:

$$A(0) = \sum_{p_{i\ldots j}} \frac{c_{ij}(0)}{p_{i\ldots j}^2}$$



Leading order scale variation for $pp \to \nu\bar{\nu} + N$jets

High $p_t > 80\,\mathrm{GeV}$, central $|\eta| < 2.5$ jets. Assume that a reasonable scale is:

$$\mu^2 = M_Z^2 + \sum_{jet} p_{t,jet}^2$$

and vary: $\mu_R = \mu_F = \mu/2 - 2\mu$

N	$\sigma(2\mu)[pb]$	$\sigma(\mu/2)[pb]$	variation
1	182	216	$\pm 9\%$
2	47.1	75.4	$\pm 24\%$
3	6.47	13.52	$\pm 35\%$
4	0.90	2.48	$\pm 47\%$

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Quantitative predictions at NLO

- Reduced senitivity in factorization and renormalization scales

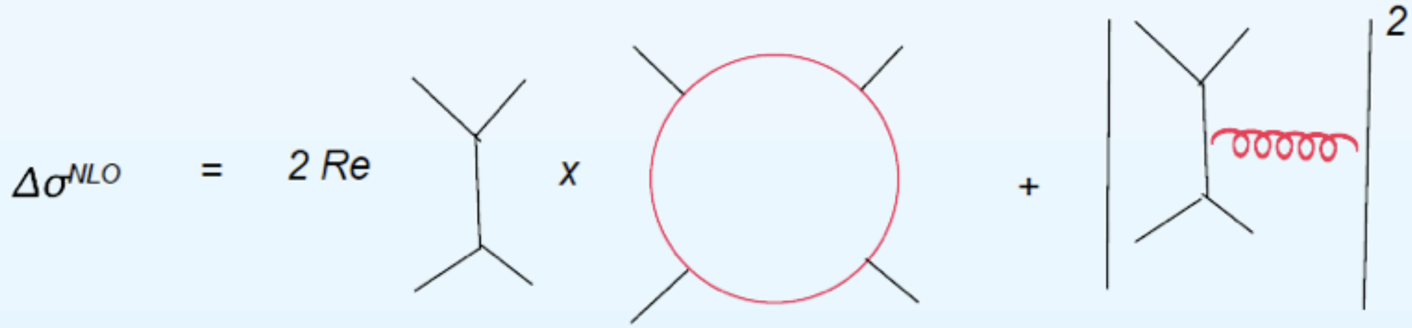
$$\frac{\partial \alpha_s}{\partial \log(\mu)} = -\beta_0 \alpha_s^2 + \mathcal{O}(\alpha_s^3)$$

$$\frac{\partial f(x,\mu)}{\partial \log(\mu)} = \alpha_s \int_z^1 \frac{dy}{y} P_{ab}(y) f(x/y,\mu) + \mathcal{O}(\alpha_s^2)$$

- New channels: For example, in Higgs production we included the processes $gg \to hg$, $qg \to hq$ and $q\bar{q} \to hg$.
- More realistic cover of the phase-space. At leading order, the Higgs boson has no transverse momentum. At NLO, $p_t \geq 0$.
- We have seen many examples where NLO corrections cannot be neglected ($gg \to h$, Drell-Yan production, squark and gluino production,W-pair production, ...)

A “professional” approach to NLO

- We followed a rather primitive method to compute the NLO corrections in the Higgs cross-section.
  - Brute force calculation of virtual corrections
  - Map phase-space into variables with range $[0, 1]$
  - Expand singularities in terms of $delta$ functions and plus distributions.
- Exploit universality of infrared singularities. We always cancel the same divergences.



- Can we arrange for an UNIVERSALLY “automated” cancelation of divergences?

Infrared subtraction method at NLO

Giele, Glover; Giele Glover, Kosower; Kunst, Soper; Frixione, Kunszt, Signer; Catani, Seymour

$$\Delta\sigma_{NLO} = \int dPS_m (2Tree_m Loop_m) Obs_m + \int dPS_{m+1} |Tree_{m+1}|^2 Obs_{m+1}$$

- The single infrared limit (one soft or two collinear partons) of tree amplitudes is universal "antennae" functions:

$$|Tree_{m+1}|^2 \to \text{infrared limit} \to |Tree_m|^2 \times Antenna$$

- We can rearrange:

$$\Delta\sigma_{NLO} = \int dPS_{m+1} \left[|Tree_{m+1}|^2 \mathbf{Obs_{m+1}} + |Tree_m|^2 \times Antenna \mathbf{Obs_m} \right]$$
$$+ \int dPS_m (2Tree_m Loop_m) Obs_m + \int dPS_m |Tree_m|^2 \times Obs_m \int PS_{1\to 2} Antenna$$

Loop integral relations

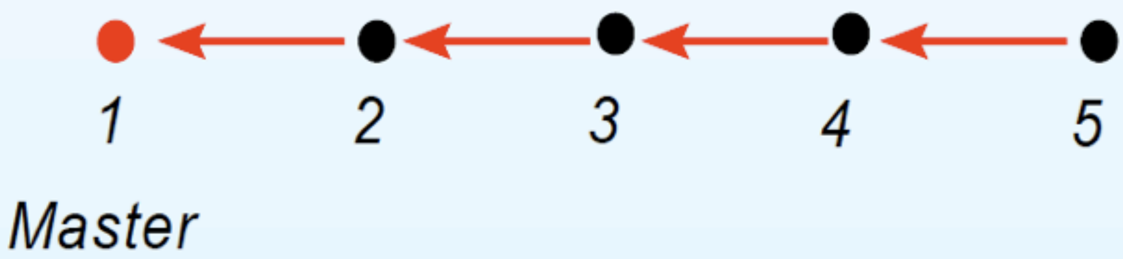
- Loop integrals are not independent:

$$\int d^dk \frac{\partial}{\partial k_\mu} \frac{k_\mu}{k^2 - M^2} = 0$$

Chetyrkin, Tkachov

$$M^2 \int d^dk \frac{1}{(k^2 - M^2)^2} + \left(\frac{d}{2} - 1\right) \int d^dk \frac{1}{(k^2 - M^2)^1} = 0$$

- We need to compute less!



Loop integral relations

- Tensor integrals are not independent (Pasarino, Veltman):

$$I\left[k^{\mu}\right]=\int d^{d}k\frac{k_{\mu}}{\left[k^{2}-M_{0}^{2}\right]\left[(k+q_{1})^{2}-M_{0}^{2}\right]\left[(k+q_{2})^{2}-M_{0}^{2}\right]\cdots}=\sum_{i}A_{i}q_{i}^{\mu}.$$

- Related to scalar integrals

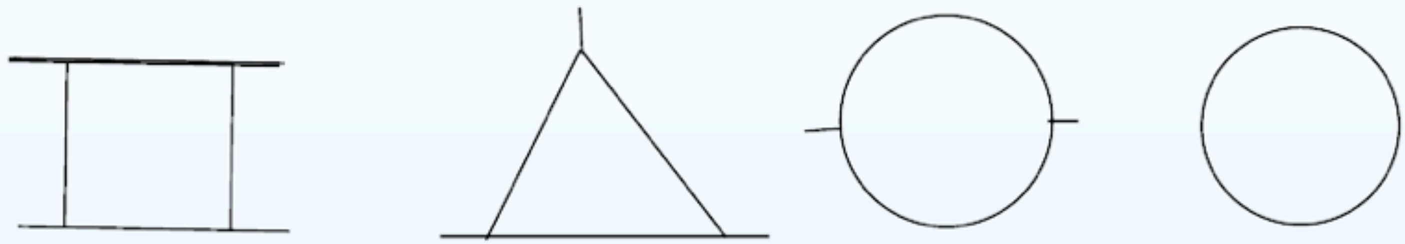
$$I\left[k\cdot q_{j}\right]=\sum_{i}\left(q_{i}\cdot q_{j}\right)A_{i}\rightsquigarrow\vec{A}=\left[q_{i}\cdot q_{j}\right]^{-1}I\left[k\cdot\vec{q}\right],$$

- Numerator scalar products cancel denominators:

$$k\cdot q_{j}=\frac{1}{2}\left\{\left[(k+q_{j})^{2}-M_{j}^{2}\right]-\left[(k)^{2}-M_{0}^{2}\right]-q_{j}^{2}+M_{j}^{2}-M_{0}^{2}\right\}$$

One-loop master integrals

- All one-loop integrals are reduced to a few known master integrals:



- known analytically since long time ago
- Gauge theory introduces a lot of algebra, but no new integrations!
- Integrals are the same as in scalar field theories

Loop and Phase-space duality

- Phase-space integrals over tree amplitudes look very different than loop integrals.

$$\text{Virtual} \to \int d^dk \frac{i}{k^2 - m^2} \qquad \text{Real} \to \int d^dk \delta^+(k^2 - m^2)$$

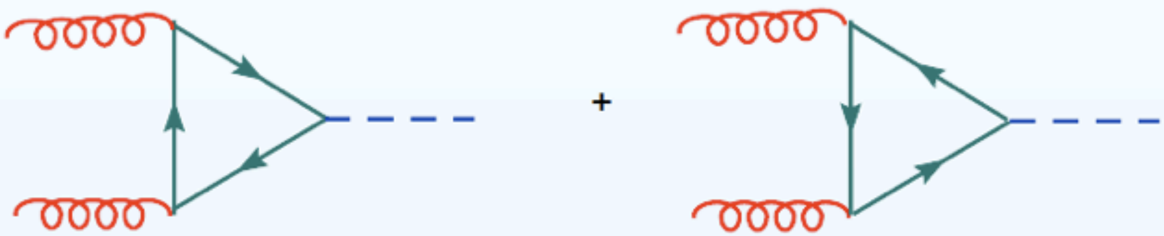
  - *Particles in loops can propagate unrestricted*
  - *Real particles must be on-shell*
- On-shell conditions ($\delta(k^2 - m^2)$) are equivalent to propagators:

$$2\pi\delta(x) = \frac{i}{x - i0} - \frac{i}{x + i0}$$

  - *Phase-Space/Loop reductions to master integrals are dual.*
  - *Phase-space/Loop master integrals satisfy dual differential equations (CA, Melnikov).*

A simple example using such ideas

- Consider our favorite process $gg \to h$ at leading order, without resorting to an effective theory approach. The amplitude is:



$$i\mathcal{M} = 2g_s^2 \frac{m_t}{v} tr\left\{t^a t^b\right\} \int \frac{d^d k}{(2\pi)^d} \frac{\mathrm{Num}(k)}{\left((k+p_1)^2 - m_t^2\right)\left(k^2 - m_t^2\right)\left((k-p_2)^2 - m_t^2\right)},$$

- an one-loop integral with a numerator

$$\mathrm{Num}(k) = tr\left\{(\not k + \not p_1 + m_t)\not \epsilon_1(\not k + m_t)\not \epsilon_2(\not k - \not p_2 + m_t)\right\}$$

Reduction to Master Integrals

- A Passarino-Veltman reduction yields,

$$\int d^d k \frac{Num(k)}{D_1 D_2 D_3} = C_{123} \int d^d k \frac{1}{D_1 D_2 D_3} + C_{12} \int d^d k \frac{1}{D_1 D_2} + C_{13} \int d^d k \frac{1}{D_1 D_3} + C_{23} \int d^d k \frac{1}{D_2 D_3} + C_1 \int d^d k \frac{1}{D_1} + C_2 \int d^d k \frac{1}{D_2} + C_3 \int d^d k \frac{1}{D_3}$$

- Use phase-space/loop duality $(1/D_i \to \delta(D_i))$:

$$\int d^d k \frac{Num(k)\delta(D_1)}{D_2 D_3} = C_{123} \int d^d k \frac{\delta(D_1)}{D_2 D_3} + C_{12} \int d^d k \frac{\delta(D_1)}{D_2} + C_{13} \int d^d k \frac{\delta(D_1)}{D_3} + C_1 \int d^d k \delta(D_1)$$

- Coefficients of master integrals are identical if we “cut a propagator”. Cuts make reductions easier.

Reduction to Master Integrals

- A Passarino-Veltman reduction yields,

$$
\begin{aligned}
\int d^d k \frac{Num(k)}{D_1 D_2 D_3} \quad = \quad & C_{123} \int d^d k \frac{1}{D_1 D_2 D_3} \\
& + C_{12} \int d^d k \frac{1}{D_1 D_2} + C_{13} \int d^d k \frac{1}{D_1 D_3} + C_{23} \int d^d k \frac{1}{D_2 D_3} \\
& + C_1 \int d^d k \frac{1}{D_1} + C_2 \int d^d k \frac{1}{D_2} + C_3 \int d^d k \frac{1}{D_3}
\end{aligned}
$$

- Use phase-space/loop duality (again):

$$
\begin{aligned}
\int d^d k \frac{Num(k) \delta\left(D_1\right) \delta\left(D_2\right)}{D_3} \quad = \quad & C_{123} \int d^d k \frac{\delta\left(D_1\right) \delta\left(D_2\right)}{D_3} \\
& + C_{12} \int d^d k \delta\left(D_1\right) \delta\left(D_2\right)
\end{aligned}
$$

- Even simpler, but we have now lost information about the coefficient of tadpoles.

Reduction to Master Integrals

- A Passarino-Veltman reduction yields,

$$\int d^dk \frac{Num(k)}{D_1 D_2 D_3} = C_{123} \int d^dk \frac{1}{D_1 D_2 D_3}$$
$$+C_{12} \int d^dk \frac{1}{D_1 D_2} + C_{13} \int d^dk \frac{1}{D_1 D_3} + C_{23} \int d^dk \frac{1}{D_2 D_3}$$
$$+C_1 \int d^dk \frac{1}{D_1} + C_2 \int d^dk \frac{1}{D_2} + C_3 \int d^dk \frac{1}{D_3}$$

- Use phase-space/loop duality (and again):

$$\int d^dk Num(k) \delta(D_1) \delta(D_2) \delta(D_3) = C_{123} \int d^dk \delta(D_1) \delta(D_2) \delta(D_3)$$

- Computing C_{123} is now very simple..., but a “triple cut” tells nothing(?) about the coefficients of bubbles and tadpoles.

Computing C_{123} from a triple cut

- In this case, computing just C_{123} may be meaningful since this is a leading order UV finite amplitude, while bubbles and tadpoles are divergent in $D = 4$.
- Consider for simplicity that the two gluons have the same ("$+$" or "$-$") helicity.
- A general solution for the on-shell conditions,

$$D_1 = D_2 = D_3 = 0 \qquad \rightsquigarrow k^2 - m_t^2 = k \cdot p_1 = k \cdot p_2 = 0$$

- is given by,

$$k^\mu = \lambda_1 \epsilon_1^\mu + \lambda_2 \epsilon_2^\mu, \qquad \lambda_1 \lambda_2 = \frac{m_t^2}{2\epsilon_1 \cdot \epsilon_2}$$

(we used/chose: $\epsilon_i \cdot p_i = 0, \epsilon_i \cdot p_j = 0, \epsilon_i^2 = 0$)

Computing C_{123} from a triple cut

- The numerator in the amplitude trivializes when we take a simple cut,

$$\mathrm{Num}(k)|_{D_1=D_2=D_3=0} = \mathrm{Num}(\lambda_1\epsilon_1+\lambda_2\epsilon_2) = 2\epsilon_1\cdot\epsilon_2\left(4m_t^2 - m_h^2\right).$$

- this is the coefficient of the triple-cut master integral
- from loop/phase-space duality, this is also the coefficient of the triangle (loop) master integral.

$$C_{123} = 2\epsilon_1\cdot\epsilon_2\left(4m_t^2 - m_h^2\right).$$

- We are now ready to gain a deeper insight on master integral coefficients which can be obtained by taking cuts of a loop amplitude.

Master-integral coefficients $= \prod$ Tree-Amplitudes

- Our main result from the previous slide is,

$$C_{123} = \mathrm{Num}(k)|_{D_1=D_2=D_3=0}$$
$$= tr\{[\not{k}_1 + m]\ \not{\epsilon}_1\,[\not{k}_2 + m_t]\,[\not{k}_3 + m_t]\ \not{\epsilon}_2\}|_{k_1^2=k_2^2=k_3^2=m_t^2}$$

- On-shell conditions $\rightsquigarrow$ Dirac equation

$$C_{123} = \sum_{lmn} [u(k_1)_l \bar{u}(k_1)_l]\ \not{\epsilon}_1 [u(k_2)_m \bar{u}(k_2)_m][u(k_3)_n \bar{u}(k_3)_n]\ \not{\epsilon}_2$$

$$= \sum_{lmn} [\bar{u}(k_1)_l\ \not{\epsilon}_1 u(k_2)_m][\bar{u}(k_2)_m u(k_3)_n][\bar{u}(k_3)_n\ \not{\epsilon}_2 u(k_1)_l]$$

From $D = 4$ to $D = 4 - 2\epsilon$ dimensions

- Tadpole and Bubble master integrals are UV divergent

$$\int \frac{d^{4-2\epsilon}k}{i\pi^{2-\epsilon}} \frac{1}{(k^2 - m_t^2)\left[(k+q)^2 - m_t^2\right]} = \frac{1}{\epsilon} - \gamma + \int_0^1 dx \log\left[m_t^2 - x(1-x)q^2\right] + \mathcal{O}(\epsilon)$$

- Our calculation in $D = 4$ dimensions misses master integral coefficients of order ϵ, in $D = 4 - 2\epsilon$ dimensions. These are “rational terms”, or non “cut-constructible” terms.

$$\epsilon \times \mathrm{Bubble} = \epsilon \times \left(\frac{1}{\epsilon} + \mathrm{Finite}\right) = 1 + \mathcal{O}(\epsilon).$$

- We indeed missed such a coefficient in our example.
- We now understand how to reconstruct them from cuts of the amplitudes with two calculations, in different values of the dimension or of a mass regulator *(ask me at coffee break; Ossola, Pittau,Papadopoulos; Ellis,Giele,Kunszt,Melnikov)*

Is this systematic?

- Obtaining one coefficient ($e.g. C_{123}$) from cuts easily is nice but not enough
- Loop/Phase-space duality manifest after integration. Knowing one master coefficienct does not obviously help to find the rest.
- "Unitarity" methods were introduced in the 90s for modern perturbative calculations *(Bern, Dixon, Dunbar, Kosower)*, but
  - supplied with constraints from universal structure of the UV and IR divergences and collinear limits
  - no systematic method

One-loop amplitudes from trees... and masters!!!

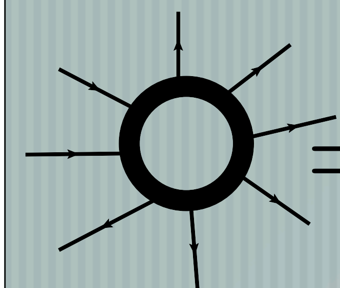
Trees in Gauge theory

Loop Master Integrals in scalar field theory

ONE-LOOP INTEGRAND

Ossola, Papadopoulos, Pittau 2006

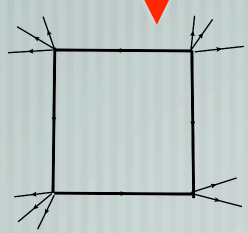
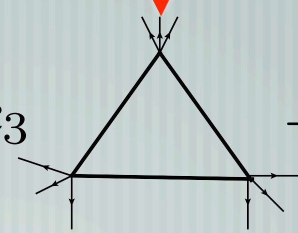
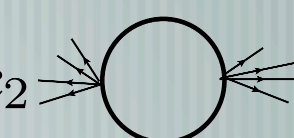
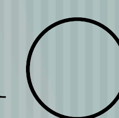
(building on del Aguila, Pittau, 2004)



$$= \int \frac{d^d k}{(2\pi)^d} \Big[c_4\, f_4(\vec{k}) + c_3\, f_3(\vec{k}) + c_2\, f_2(\vec{k}) + c_1\, f_1(\vec{k})$$

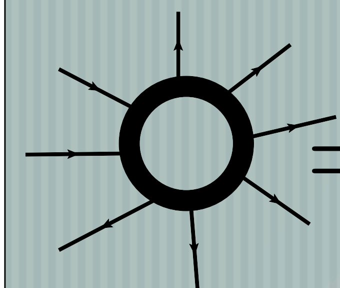
$$+ \tilde{c}_4\, \tilde{f}_4(\vec{k}) + \tilde{c}_3\, \tilde{f}_3(\vec{k}) + \tilde{c}_2\, \tilde{f}_2(\vec{k}) + \tilde{c}_1\, \tilde{f}_1(\vec{k}) \Big]$$

After Integration:

$= c_4$  $+c_3$  $+c_2$  $+c_1$ 

ONE-LOOP INTEGRAND

Ossola, Papadopoulos, Pittau 2006



$$= \int \frac{d^d k}{(2\pi)^d} \Big[c_4\, f_4(\vec{k}) + c_3\, f_3(\vec{k}) + c_2\, f_2(\vec{k}) + c_1\, f_1(\vec{k})$$

$$+ \tilde{c}_4\, \tilde{f}_4(\vec{k}) + \tilde{c}_3\, \tilde{f}_3(\vec{k}) + \tilde{c}_2\, \tilde{f}_2(\vec{k}) + \tilde{c}_1\, \tilde{f}_1(\vec{k}) \Big]$$

$\tilde{f}_i(\vec{k}), f_i(\vec{k})$: Known rational functions of the loop momentum

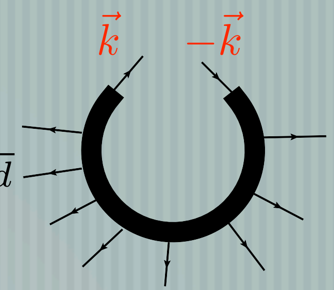
$\tilde{c}_i, c_i$: coefficients can be determined algebraically computing the integrand at a sufficient number of values for $\vec{k}$

ONE-LOOP INTEGRAND

Ossola, Papadopoulos, Pittau 2006

$$\int \frac{d^dk}{(2\pi)^d} \Big[c_4\, f_4(\vec{k}) + c_3\, f_3(\vec{k}) + c_2\, f_2(\vec{k}) + c_1\, f_1(\vec{k})$$

$$+ \tilde{c}_4\, \tilde{f}_4(\vec{k}) + \tilde{c}_3\, \tilde{f}_3(\vec{k}) + \tilde{c}_2\, \tilde{f}_2(\vec{k}) + \tilde{c}_1\, \tilde{f}_1(\vec{k}) \Big] = \int \frac{d^dk}{(2\pi)^d}$$



☑ Integrand is "easy", essentially a tree amplitude

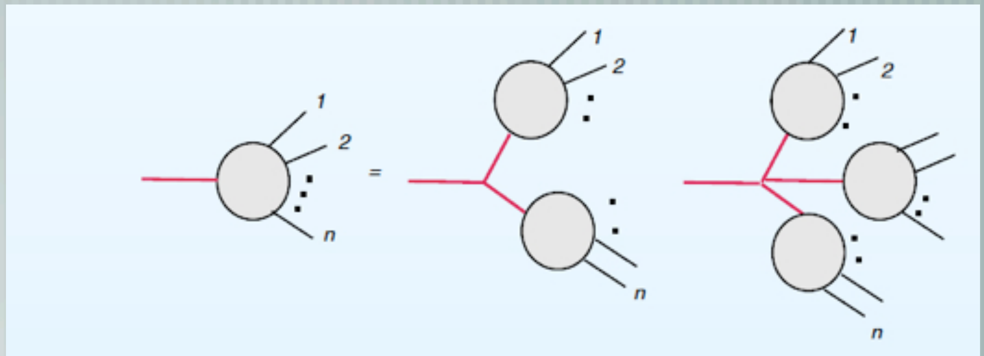
Evaluate integrand at loop momenta values such as loop particles are set ON SHELL

☑ ON-SHELL: determines coefficients successively

Coefficients as tree products

Ellis, Giele, Kunszt *2007*

- ON-SHELL loop propagators = Product of tree amplitudes
- Evaluation of trees with powerful recursive methods



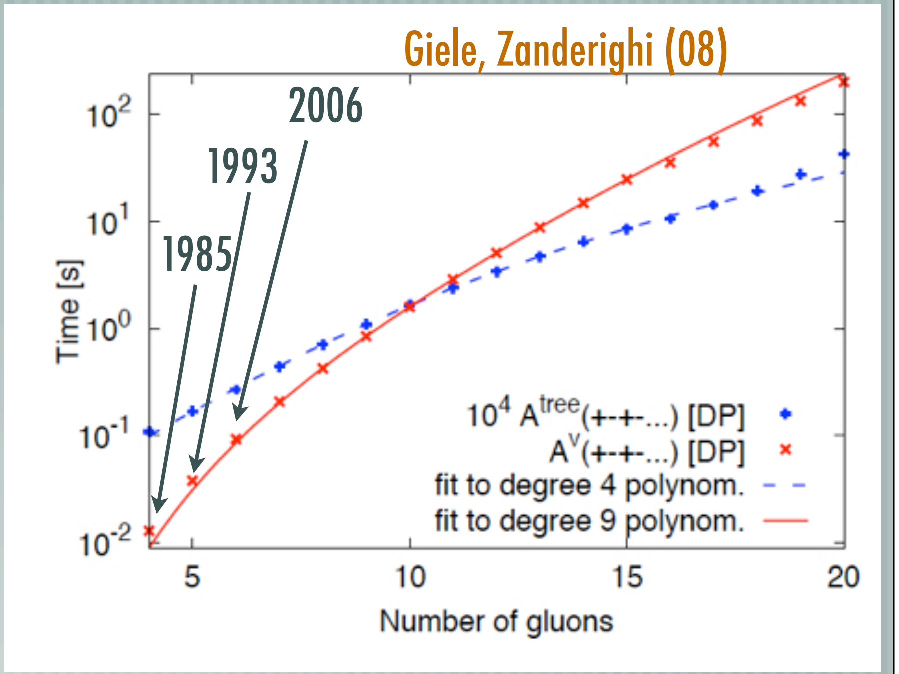
e.g. Berends-Giele, Britto-Cachazo-Feng-Witten, etc

Conflict of dimensions

- Loop Integrations in D dimensions, Tree amplitudes in four dimensions. Mismatch, i.e. missing terms from amplitude evaluation. Requires a second calculation.
  - Specialized tree-like recursions in D=4 for the missing terms
    Berger, Bern, Dixon, Forde, Kosower 2006
  - Elegant/general solution: Amplitude in a general dimension from results in D=5 and D=6. Ellis, Giele, Kunszt, Melnikov 2008
  - Specialized Feynman rules for missing terms:
    2009
    Draggiotis, Garzelli, Papadopoulos, Pittau

Breathtaking developments

- One-loop amplitudes with 22 gluons Giele, Zanderighi (08); Lazopoulos (08); Giele,Winter (09)



- Numerical evaluation of all 2 to 4 amplitudes in the Les-Houches 2007 wish-list

van Hameren, Papadopoulos, Pittau (09)

$$q\bar{q}, gg \to t\bar{t}b\bar{b}, b\bar{b}b\bar{b}, W^+W^-b\bar{b}, t\bar{t}gg$$

$$q\bar{q}' \to Wggg, Zggg$$

W+3 jets: NLO cross-section

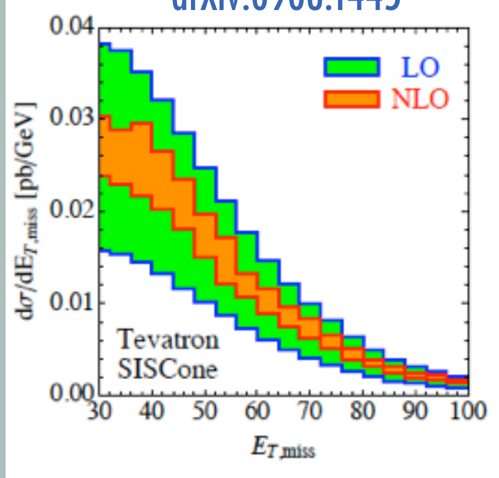
Large Nc approximation

Ellis,Giele,Kunszt,Melnikov,Zanderighi; Berger,Bern,Dixon,Cordero,Forde, Gleisberg,Ita,Kosower,Maitre

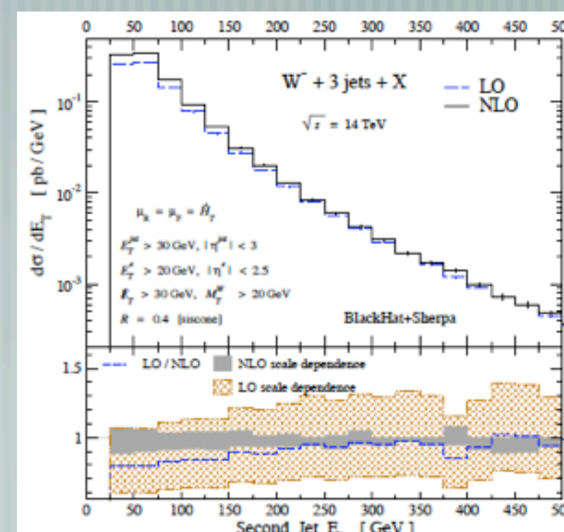
NEW: complete NLO

Berger,Bern,Dixon,Cordero, Forde,Gleisberg,Ita,Kosower, Maitre (arXiv:0907.1984)

arXiv:0906.1445



Start of a new era, with precise theoretical predictions for multi-particle production at the LHC



arXiv:0907.1984

NLO calculations @ LHC

- What can we hope for?
- We cannot do better than tree calculations..., i.e. processes with 7 or 8 particles in the final state.
- All 2 to 4 processes with both Feynman diagrammatic and unitarity methods
- 2 to 5 and perhaps 2 to 6 processes with unitarity methods

FUTURE? (or loud wishful thinking)

- Loop amplitudes can be viewed as complex integrals. Result is determined by residues, which (as in the one-loop case) maybe given from amplitudes with less loops
- Cross-order relations maybe present, similar to the cross-order or resummation formulae for infrared divergences (arising from the same poles)
- Very far from uncovering cross-order relations in QCD, but...

N=4 Super Yang-Mills

$$M_4^{(2)}(\epsilon) = \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_4^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4$$

CA,Bern,Dixon,Kosower

$$M_4^{(3)}(\epsilon) = -\frac{1}{3}\left(M_4^{(1)}(\epsilon)\right)^3 + M_4^{(1)}(\epsilon)M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_4^{(1)}(3\epsilon) + C.$$

Bern,Dixon,Smirnov

$$M_n(\epsilon) = \exp\left(\sum_{l=0}^{\infty} a^l \left[f^{(l)}(\epsilon)M_n^{(1)}(l\epsilon) + h^{(l)}\right]\right)$$

Bern,Dixon,Smirnov

Last words

- Perturbative QCD is the main theoretical tool for particle physics explorations at colliders
- We can now make very powerful computations, after understanding better recursion and the structure of one-loop amplitludes
- We have only scratched the surface.... your ideas and curiosity can take us further, “solving” (why not?) the perturbative series.