Contracted and expanded quantum algebraic structures

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Motivations

• Resolve a quite old misconception regarding the expansion of E_2 to sl_2 (*Gilmore '74*). This is explained via the representation theory of quadratic quantum algebras arising in quantum integrable systems (quantum spin chains).

• Moreover using the quadratic (boundary) algebras we are able to obtain centrally extended algebras. More precisely, we exploit the symmetry breaking mechanism due to the presence of suitable boundaries to extract the centrally extended algebras. Note, centrally extended algebras arise within AdS/CFT (*Beisert '08*).

• This process provides the most natural and straightforward means to obtain higher Casimir operators associated to any Lie or deformed Lie algebra. Focus here on particular prototype examples gl_3, $U_q(gl_3) \to E_2^c$ and $U_q(E_2^c)$

A simple observation

Consider the E_2 algebra (also from sl_2 via Inonü-Wigner contraction) defined by generators $J,\ P^{\pm}$ and exchange relations

$$\Big[J,\ P^{\pm}\Big] = \pm P^{\pm}, \qquad \Big[P^{+},\ P^{-}\Big] = 0$$

with quadratic Casimir: $C = P^{+}P^{-}$.

Define now $Y^{\pm} = JP^{\pm}$ then $Y^{\pm},\ J$ satisfy (*Gilmore '74*):

$$\Big[J,\ Y^{\pm}\Big] = \pm Y^{\pm}, \qquad \Big[Y^{+},\ Y^{-}\Big] = -2JP^{+}P^{-}$$

Already a 'hint' of sl_2

Define $\tilde{Y}^{\pm} = \frac{Y^{\pm}}{\sqrt{P^{+}P^{-}}}$ then 'recover' the sl_2 exchange relations

$$\left[J,\ \tilde{Y}^{\pm}\right] = \pm\tilde{Y}^{\pm}, \qquad \left[\tilde{Y}^{+},\ \tilde{Y}^{-}\right] = -2J$$

'Naively' one may say the E_2 is expanded to sl_2, but not true!

Check compatibility of representations: only the unit works suggesting that something is wrong. Moreover, check co products of sl_2 again inconsistencies arise! Search for a broader (quadratic) algebra $\tilde{Y}^{\pm},\ J$ part of this. Quantum algebras arise in quantum integrable systems.

Quantum algebras: a brief review

Introduce the fundamental object in quantum integrability, YBE (*Baxter '72*)

$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

The equation acts on $V \otimes V \otimes V$, R acts on $V \otimes V$ and $R_{12} = R \otimes \mathbb{I}$, $R_{23} = \mathbb{I} \otimes R$. Consider a tensor product sequence then the notation:

$$A_n = \mathbb{I} \otimes \ldots \mathbb{I} \otimes \underbrace{A}_{n} \otimes \mathbb{I} \ldots \mathbb{I}$$

The R-matrix physically describes the scattering of low lying excitations arising in quantum integrable systems.

An example, solution of the Yangian $\mathcal{Y}(gl_n)$ (*Yang '67*)

$$R(\lambda) = \mathbb{I} + \frac{i}{\lambda}\mathcal{P}$$

$\mathcal{P}(a \otimes b) = (b \otimes a)$. The gl_n case:

$$\mathcal{P} = \sum_{i.j=1}^{n} e_{ij} \otimes e_{ji}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Both spaces represented by the fundamental rep of gl_n. For sl_2 in particular:

$$\mathcal{P} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} = \frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z + \mathbb{I})$$

$\sigma^{x,y,z}$ the usual 2×2 Pauli matrices.

Given a solution of the YBE (*Faddeev, Reschetikhin, Takhtajan 80's*) $\longrightarrow$ defining relations of quantum algebras. Consider $L \in \text{End}(V) \otimes \mathcal{A}$ (quantum Lax operator):

$$R_{12}(\lambda_1 - \lambda_2)\ L_{1j}(\lambda_1)\ L_{2j}(\lambda_2) = L_{2j}(\lambda_2)\ L_{1j}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2)$$

Traditionally the indices 1, 2 denote the 'auxiliary' space the index i the 'quantum'.

Example the $\mathcal{Y}(gl_n)$ L-matrix

$$L(\lambda) = \mathbb{I} + \frac{i}{\lambda}\mathbb{P}, \qquad \mathbb{P} = \sum_{i,j} e_{ij} \otimes \mathbb{P}_{ij}$$

$\mathbb{P}_{ij}$ elements of gl_n satisfying:

$$\Big[\mathbb{P}_{ij},\ \mathbb{P}_{kl}\Big] = \delta_{il}\mathbb{P}_{kj} - \delta_{jk}\mathbb{P}_{il}\ , \qquad i = 1, 2, \ldots, n$$

This allows the construction of tensorial representations of the FRT algebra (QISM, *St. Petersburg group 80's*):

$$T_0(\lambda) = L_{0N}(\lambda - \theta_N)\ L_{0N-1}(\lambda - \theta_{N-1}) \ldots L_{02}(\lambda - \theta_2)\ L_{01}(\lambda - \theta_1)$$

$T(\lambda) \in \mathrm{End}(\mathbb{V}) \otimes \mathcal{A}^{\otimes N}$. θ_i are free complex parameters. Using FRT one may show that

$$\Big[tr_0 T_0(\lambda),\ tr_0 T_0(\mu)\Big] = 0\ ,$$

Integrability condition

$t(\lambda) = trT(\lambda) \in \mathcal{A}^{\otimes N}$ system. $\mathcal{A} \hookrightarrow V$ the tensorial representation acquires the meaning of the monodromy matrix of a quantum spin chain and and trT, the transfer matrix. building periodic spin chains!

Example: XXX Hamiltonian

$L \hookrightarrow R$ one obtains local Hamiltonian from the transfer matrix (Heisenberg model solved (*Bethe '31*))

$$H \propto \frac{dt(\lambda)}{d\lambda}|_{\lambda=0} \propto -\frac{1}{2}\sum_{i=1}^{N}\Big(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z\Big)$$

periodic b.c. $\vec{\sigma}_1 = \vec{\sigma}_{N+1}$

Fundamental question: how integrable boundary conditions are incorporated in this context? Answer next!

The reflection algebra

Reflection equation (*Cherednik, Sklyanin 80's*)

$$R_{12}(\lambda_1-\lambda_2)\ \mathbb{K}_1(\lambda_1)\ R_{21}(\lambda_1+\lambda_2)\ \mathbb{K}_2(\lambda_2) =$$
$$\mathbb{K}_2(\lambda_2)\ R_{12}(\lambda_1+\lambda_2)\ \mathbb{K}_1(\lambda_1)\ R_{21}(\lambda_1-\lambda_2)$$

$\mathbb{K}$ reflection matrix, physically describes the interactions of the excitation with the end of the chain.

Operatorial rep of the reflection algebra (*Sklyanin '84*)

$$\mathbb{K}(\lambda) = L(\lambda-\Theta)\ K(\lambda)\ L^{-1}(-\lambda-\Theta)$$

K c-number solution of RE. $\mathbb{K} \in \mathrm{End}(V) \otimes \mathcal{R}$; $\mathcal{R}$ the reflection algebra

Tensorial rep of the reflection algebra $\longrightarrow$ open spin chain systems

$$\mathbb{T}_0(\lambda) = T_0(\lambda)\ K_0(\lambda)\ T_0^{-1}(-\lambda)$$

Define also the open transfer matrix:

$$t(\lambda) = Tr_0\{K_0^+(\lambda)\ \mathbb{T}_0(\lambda)\}$$

K^+ also a c-number solution of RE, henceforth $K^+ \propto \mathbb{I}$.

Use RE to show the integrability condition

$$\Big[t(\lambda),\ t(\mu)\Big] = 0.$$

Family of commuting operators $\longrightarrow$ integrability.

After the brief review we come back to our problem: inconsistencies from the E_2 'expansion' to sl_2. Focus on rep of a 'modified' RE (twisted Yangian)

$$\mathbb{K}(\lambda) = L(\lambda - \frac{i}{2})\ \hat{L}(\lambda + \frac{i}{2}),$$
$$\hat{L}(\lambda) = V\ L^t(-\lambda - i)\ V$$

$\mathbb{K}$ satisfies the quadratic algebra (twisted Yangian). Focus on the sl_2 case then:

$$L(\lambda - \frac{i}{2}) = \mathbb{I} + \frac{i}{\lambda}\ \mathbb{P}\ , \qquad \mathbb{P} = \begin{pmatrix} J & -J^- \\ J^+ & -J \end{pmatrix}$$
$$\hat{L}(\lambda + \frac{i}{2}) = \mathbb{I} + \frac{i}{\lambda}\ \hat{\mathbb{P}}\ , \qquad \hat{\mathbb{P}} = \begin{pmatrix} J+1 & J^- \\ -J^+ & -J+1 \end{pmatrix}$$

$J,\ J^{\pm}$ generators of sl_2:

$$[J,\ J^{\pm}] = \pm J^{\pm}\ , \qquad [J^+,\ J^-] = -2J\ .$$

Expand $\mathbb{K}$ it in powers of $\frac{1}{\lambda}$:

$$\mathbb{K}(\lambda) = \mathbb{I} + \frac{1}{\lambda}\mathbb{K}^{(0)} + \frac{1}{\lambda^2}\mathbb{K}^{(1)} \ ,$$

where

$$\mathbb{K}^{(0)} = i \begin{pmatrix} 2J+1 & 0 \\ 0 & -2J+1 \end{pmatrix} \ ,$$
$$\mathbb{K}^{(1)} = \begin{pmatrix} -J^2 - \frac{1}{2}\{J^+,\ J^-\} - 2J & -2JJ^- \\ -2JJ^+ & -J^2 - \frac{1}{2}\{J^+,\ J^-\} + 2J \end{pmatrix} \ .$$

Taking the trace we end up:

$$\bar{t}(\lambda) = tr\{\mathbb{K}(\lambda)\} = \mathbb{I} + \frac{i}{\lambda} + \frac{t^{(1)}}{\lambda^2} \ , \qquad \text{where} \qquad t^{(1)} \propto J^2 + \frac{1}{2}\{J^+,\ J^-\} \ .$$

$t^{(1)}$ structurally n su_2 like Casimir. t is $u(1)$ symmetric.

Define $\mathbb{K}^{(1)}_{12} = -2Y^-$, $\mathbb{K}^{(1)}_{21} = -2Y^+$. After performing the Inonü–Wigner contraction

$$J^\pm \to \frac{1}{\epsilon} P^\pm, \qquad \epsilon \to 0$$

we end up

$$Y^\pm = J P^\pm \ ,$$

precisely as in Gilomre's 'expansion'.

J and $Y^\pm$ are elements of an extended algebra, i.e. the contracted sl_2 twisted Yangian, with exchange relations dictated by the quadratic equation reflection equ. This will be more transparent in the following when constructing the N-tensor representation of the twisted Yangian.

The N-particle construction

Define the N-tensor representation of the twisted Yangian

$$\bar{\mathbb{T}}_0(\lambda) = L_{0N}(\lambda - \frac{i}{2}) \dots L_{01}(\lambda - \frac{i}{2})\ \hat{L}_{01}(\lambda + \frac{i}{2}) \dots \hat{L}_{0N}(\lambda + \frac{i}{2})\ .$$

Expansion in powers of $\frac{1}{\lambda}$ leads to

$$\mathbb{T}(\lambda) = \mathbb{I} + \sum_{k=1}^{2N} \frac{\mathbb{T}^{(k-1)}}{\lambda^k}\ ,$$

with

$$\begin{aligned}
\mathbb{T}_0^{(0)} &= \sum_{i=1}^{N} \left(\mathbb{P}_{0i} + \hat{\mathbb{P}}_{0i}\right)\ , \\
\mathbb{T}_0^{(1)} &= -\Big(\sum_{i>j} \mathbb{P}_{0i}\ \mathbb{P}_{0j} + \sum_{i<j} \hat{\mathbb{P}}_{0i}\ \hat{\mathbb{P}}_{0j} + \sum_{i,j} \mathbb{P}_{0i}\ \hat{\mathbb{P}}_{0j}\Big)\ \dots
\end{aligned}$$

The first non-trivial conserved quantity

$$t^{(1)} \propto \sum_{i=1}^{N} \Big(J_i^2 + \frac{1}{2} \{ J_i^+ ,\ J_i^- \} \Big) + 4 \sum_{i<j} J_i \ J_j \ .$$

After I-W contraction

$$t^{(1)} = \sum_{i=1}^{N} P_i^+ P_i^- \ .$$

For $N = 1$ one obtains the expected E_2 Casimir

$$t^{(1)} = P^+ P^- \ .$$

The non-diagonal elements: $\bar{\mathbb{T}}_{12} = -2\mathbb{Y}^-$, $\bar{\mathbb{T}}_{21} = -2\mathbb{Y}^+$, where

$$\mathbb{Y}^\pm = \sum_{i=1}^{N} J_i P_i^\pm + 2\sum_{i<j} J_i P_j^\pm \ ,$$

The N co-product of $Y^\pm$

For $N = 2$

$$\Delta(Y^\pm) = Y^\pm \otimes \mathbb{I} + \mathbb{I} \otimes Y^\pm + 2J \otimes P^\pm$$

Non trivial co-product, another hint of extended (deformed) algebra.
$\Rightarrow Y^{(\pm)}$, J belong to a broader deformed algebra: the contracted sl_2 twisted Yangian.

The co-products do *not* satisfy sl_2 algebra. One "particle" expansion stops at order $\frac{1}{\lambda^2}$ so the relevant exchange relations emanating from FRT are truncated. Considering the N "particle" representation the expansion involves higher orders, and thus the associated exchange relations become more involved.

The E_2^c extended algebra

AIM: obtain the centrally extended E_2 algebra via a boundary breaking symmetry mechanism from $sl_2 \otimes u(1)$

Review the gl_n algebra:

$$[J_{ij},\ J_{kl}] = \delta_{il} J_{kj} - \delta_{jk} J_{il} \ , \qquad i = 1, 2, \ldots, n \ .$$

generated by

$$J^{+(i)} = J^{i+1\ i} \ , \qquad J^{-(i)} = J^{i\ i+1} \ , \qquad e^{(i)} = J^{ii} \ .$$

Define $s^{(k)} = e^{(k)} - e^{(k+1)}$, then

$$[J^{+(k)},\ J^{-(l)}] = \delta_{kl} s^{(k)} \ , \qquad [s^{(k)},\ J^{\pm(l)}] = \pm(2\delta_{kl} - \delta_{k\ l+1} - \delta_{k\ l-1}) J^{\pm(l)}$$

$\sum_{i=1}^{n} e^{(i)}$ belongs to the center of the algebra.

Recall the generic L operator: $L(\lambda) = \mathbb{I} + \frac{i}{\lambda}\mathbb{P}$.

Focus on gl_3,

$$\mathbb{P} = \begin{pmatrix} e^{(1)} & J^{-(1)} & \Lambda^{+} \\ J^{+(1)} & e^{(2)} & J^{-(2)} \\ \Lambda^{-} & J^{+(2)} & e^{(3)} \end{pmatrix} , \quad \text{where} \qquad \Lambda^{\pm} = \pm[J^{\pm(1)},\ J^{\pm(2)}] \ .$$

We choose as K (*de Vega, Gonzalez-Ruiz '93*)

$$K(\lambda) = k = \text{diag}(1,\ 1,\ -1)$$

and expand the tensor rep of the RA $\mathbb{T}(\lambda)$

$$\mathbb{T}_0(\lambda) = L_{0N}(\lambda) \ldots L_{01}(\lambda)\ k\ L_{01}^{-1}(-\lambda) \ldots L_{0N}^{-1}(-\lambda)\ ,$$

The first couple terms of the expansion:

$$\mathbb{T}^{(0)} = i\sum_{i=1}^{N}\Big(\mathbb{P}_{0i}k + k\mathbb{P}_{0i}\Big) \ ,$$

$$\mathbb{T}^{(1)} = -\sum_{i>j}\mathbb{P}_{0i}\ \mathbb{P}_{0j}\ k - k\ \sum_{i<j}\mathbb{P}_{0i}\ \mathbb{P}_{0j} - \sum_{i,j=1}^{N}\mathbb{P}_{0i}\ k\ \mathbb{P}_{0j} - k\sum_{i=1}^{N}\mathbb{P}_{0i}^{2}\ , \ldots$$

Let us first set $k = \mathbb{I}$, transfer matrix is gl_3 symmetric (*Doikou, Nepomechie '98*). Trace of $\mathbb{T}^{(k)} \rightarrow$ Casimir operators, e.g. $N = 1,\ k = 1$:

$$C = tr\{\mathbb{P}^2\} = (e^{(1)})^2 + (e^{(2)})^2 + (e^{(3)})^2 + J^{-(1)}J^{+(1)} + J^{-(2)}J^{+(2)} + \Lambda^-\Lambda^+ + J^{+(1)}J^{-(1)} + J^{+(2)}J^{-(2)} + \Lambda^+\Lambda^- \ .$$

GENERIC statement for higher rank algebras.

Come back to the situation where $k = \text{diag}(1,\ 1,\ -1)$, breaks the symmetry $gl_3 \longrightarrow sl_2 \otimes u(1)$. In general, suitable k, $gl_n \longrightarrow gl_l \otimes gl_{n-l}$ (*Doikou, Nepomechie '98*).

Focus on $N = 1$ situation, after expanding

$$\mathbb{T}^{(0)} = 2i \begin{pmatrix} e^{(1)} & J^{-(1)} & 0 \\ J^{+(1)} & e^{(2)} & 0 \\ 0 & 0 & -e^{(3)} \end{pmatrix} \ldots$$

$sl_2 \otimes u(1)$ algebra.

After taking the trace:

$$t^{(0)} \propto c - 2e^{(3)},$$
$$t^{(1)} \propto (e^{(1)})^2 + (e^{(2)})^2 - (e^{(3)})^2 + J^{-(1)}J^{+(1)} + J^{+(1)}J^{-(1)}\ ,$$

$c = e^{(1)} + e^{(2)}$ central element of sl_2. $t^{(1)}$ is quadratic Casimir of $sl_2 \otimes u(1)$.

For notational convenience, we set

$$s^{(1)} \equiv 2J \ , \qquad e^{(3)} \equiv 2\tilde{J} \ , \qquad J^{-(1)} \equiv -J^{-} \ , \qquad J^{+(1)} \equiv J^{+} \ .$$

Then one finds

$$t^{(0)} \propto \tilde{J}, \qquad t^{(1)} \propto J^2 - \frac{1}{2}\{J^+,\ J^-\} - \tilde{J}^2 - c\tilde{J}$$

$sl_2 \otimes u(1)$ Casimir.

The charges may be written as

$$I^{(0)} = \tilde{J} \ , \qquad I^{(1)} = J^2 - \frac{1}{2}\{J^+,\ J^-\} - \tilde{J}^2$$

The contraction

Saletan type contraction (*Sfetsos '94*):

$$J^{\pm} = \frac{1}{\sqrt{2\epsilon}} P^{\pm} \ , \qquad J = \frac{1}{2}\left(T + \frac{F}{\epsilon}\right) \ , \qquad \tilde{J} = -\frac{F}{2\epsilon} \ , \qquad \epsilon \to 0 \ .$$

Then one obtains the E_2^c algebra:

$$[P^+, \ P^-] = -2F \ , \qquad [T, \ P^{\pm}] = \pm P^{\pm} \ ,$$

F *exact* central element of the algebra. After contracting and keeping the leading order contribution in $\frac{1}{\epsilon}$:

$$I^{(0)} = F \ , \qquad I^{(1)} = TF - \frac{1}{2}\{P^+, \ P^-\} \ .$$

The $U_q(E_2^c)$ algebra

the $U_q(sl_n)$ with the Chevalley–Serre generators e_i , f_i , $q^{\pm\frac{s_i}{2}}$, $i = 1,2,\dots,n-1$, obey ($Jimbo$ '86)

$$\left[q^{\pm\frac{s_i}{2}},\ q^{\pm\frac{s_j}{2}}\right] = 0 \qquad q^{\frac{s_i}{2}}\ e_j = q^{\frac{1}{2}a_{ij}} e_j\ q^{\frac{s_i}{2}} \qquad q^{\frac{s_i}{2}}\ f_j = q^{-\frac{1}{2}a_{ij}} f_j\ q^{\frac{s_i}{2}} \ ,$$
$$\left[e_i,\ f_j\right] = \delta_{ij}\frac{q^{s_i} - q^{-s_i}}{q - q^{-1}} \ , \qquad i,j = 1,2,\dots,n-1 \ .$$

a_{ij} elements of the Cartan matrix of sl_n.

Let $q^{\pm s_i} = q^{\pm(\epsilon_i - \epsilon_{i+1})}$. The $U_q(gl_n)$ algebra is derived by adding to $U_q(sl_n)$ the elements $q^{\pm\epsilon_i}$ $i = 1,\dots,n$: $q^{\sum_{i=1}^n \epsilon_i}$ belongs to the center.

$U_q(gl_n)$ equipped with $\Delta : U_q(gl_n) \to U_q(gl_n) \otimes U_q(gl_n)$,

$$\Delta(y) = q^{-\frac{s_i}{2}} \otimes y + y \otimes q^{\frac{s_i}{2}} \ , \qquad y \in \{e_i,\ f_i\} \ , \qquad \Delta(q^{\pm\frac{\epsilon_i}{2}}) = q^{\pm\frac{\epsilon_i}{2}} \otimes q^{\pm\frac{\epsilon_i}{2}} \ .$$

$\mathbb{U}_q(\widehat{gl_n})$ R-matrix (*Jimbo '87*)

$$R(\lambda) = a(\lambda) \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + b(\lambda) \sum_{i \neq j=1}^{n} e_{ii} \otimes e_{jj} + c \sum_{i \neq j=1}^{n} e^{-sgn(i-j)\lambda} e_{ij} \otimes e_{ji} \ ,$$

$R \in \mathrm{End}((\mathbb{C}^n)^{\otimes 2})$ and

$$a(\lambda) = \sinh \mu(\lambda + i) \ , \qquad b(\lambda) = \sinh \mu\lambda \ , \qquad c = \sinh i\mu \ , \qquad q = e^{i\mu} \ .$$

The associated L operator

$$L(\lambda) = e^{\mu\lambda} L^+ - e^{-\mu\lambda} L^- \ ,$$

$$L^+ = \sum_{i \leqslant j} \hat{e}_{ij} \otimes t_{ij} \ , \qquad L^- = \sum_{i \geqslant j} \hat{e}_{ij} \otimes t_{ij}^- \ .$$

The most natural and simplest way to obtain Casimir operators of q deformed algebras.

Choose $K = \text{diag}(e^{\mu\lambda},\ e^{\mu\lambda},\ e^{-\mu\lambda})$ then the asymptotics of the RA (in general $U_q(gl_n \rightarrow U_q(gl_l) \otimes U_q(gl_{n-l}))$ (*Doikou, Nepomechie '98*))

$$\mathbb{K}^+ = L^+\ K^+\ \hat{L}^+ =\sim \mathbb{K}^+ = \begin{pmatrix} q^{2\epsilon_1} + t_{12}\hat{t}_{21} & t_{12}q^{\epsilon_2} & 0 \\ q^{\epsilon_2}\hat{t}_{21} & q^{2\epsilon_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$t_{ii} = q^{\epsilon_i} , \qquad t_{12} = (q-q^{-1})q^{-1/2}q^{\frac{\epsilon_1+\epsilon_2}{2}} f_1 , \qquad \hat{t}_{21} \equiv (q-q^{-1})q^{-1/2}q^{\frac{\epsilon_1+\epsilon_2}{2}} e_1 .$$

Also implement:

$$e_1 \equiv J^+ , \qquad f_1 \equiv -J^- , \qquad \epsilon_1 - \epsilon_2 \equiv 2J , \qquad \epsilon_1 + \epsilon_2 = -2\tilde{J} .$$

$\tilde{J}$ is central element of $U_q(sl_2)$. The asymptotics $\longrightarrow$ quadratic Casimir

$$t^+ \propto q^{-2\tilde{J}}\Big(q^{2J+1} + q^{-2J-1} - (q-q^{-1})^2 J^- J^+\Big) ,$$

The contraction

Saletan-type contraction after $q = e^{\epsilon\eta} \longrightarrow U_q(E_2^c)$ algebra

$$[T,\ P^{\pm}] = \pm 2P^{\pm}\ , \qquad [P^{+},\ P^{-}] = -2F_\eta\ , \qquad F_\eta = \frac{\sinh(\eta F)}{\eta}\ ,$$

F_η *exact* central element of the algebra. The associated co-products

$$\begin{aligned}
&\Delta(P^{\pm}) = e^{-\frac{\eta F}{2}} \otimes P^{\pm} + P^{\pm} \otimes q^{\frac{\eta F}{2}},\\
&\Delta(e^{\pm\eta F}) = e^{\pm\eta F} \otimes e^{\pm\eta F},\\
&\Delta(T) = \mathbb{I} \otimes T + T \otimes \mathbb{I}\ .
\end{aligned}$$

The non-trivial quadratic Casimir

$$C = e^{\eta F}\left(2\cosh(\eta F) + 2\eta^2\epsilon(TF_\eta - \frac{1}{2}\{P^{+},\ P^{-}\})\right)\ .$$

Comments

• Extension to the generic case especially associated to super algebras. Consider the boundary symmetry breaking $\mathrm{G} \otimes \mathrm{H}$ where G and H are generic algebras ($\mathrm{H} \subset \mathrm{G}$)

• This is also useful in deriving the universal R matrix associated to the the Yangian of E_2, E_2^c and $U_q(E_2^c)$.

• Can we use this generic context in order to uncover the full underlying algebraic structure within AdS/CFT, that is to extract by the methodology proposed the associated centrally extended algebra $gl(2|2)$?