Black Holes, Instantons and Harmonic Functions

Thomas Mohaupt

Department of Mathematical Sciences
University of Liverpool

Corfu, September 2009

Extremal Black Holes, Einstein-Maxwell

Extremal black hole solution in $D \geq 4$ dimensions

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^{\frac{2}{D-3}}(\vec{x})d\vec{x}^2$$

where $H(\vec{x})$ is a harmonic function.

- Spherical symmetry, single-centered solution

$$H(\vec{x}) = H(r) = 1 + \frac{q}{r^{D-3}}$$

- Multi-centered solution

$$H(\vec{x}) = 1 + \sum_{a=1}^{N} \frac{q_{(a)}}{|\vec{x} - \vec{x}_a|^{D-3}}$$

Extremality: Mass = total charge

$$M = \sum_{a=1}^{N} q_{(a)}$$

(I have assumed $q_{(a)} > 0$)

Extremal Black Holes, Supergravity

Field content:

- 5d supergravity multiplet $\{e_\mu{}^\alpha, \ldots, A_\mu, \ldots\}$
- 5d vector multiplets $\{A^x_\mu, \ldots, \phi^x, \ldots\}$. Label $x = 1, \ldots, n$
- Special real geometry (aka very special (real) geometry): all couplings encoded in a prepotential (cubic polynomial).

Extremal, multi-centered solutions:

$$ds^2 = -e^{2\tilde{\sigma}(\vec{x})}dt^2 + e^{-\tilde{\sigma}(\vec{x})}d\vec{x}^2$$

Metric (i.e. $\tilde{\sigma}$), scalars ϕ^x, and gauge fields A_μ, A^x_μ, can be expressed *algebraically* in terms of $n+1$ harmonic functions $H_I(\vec{x})$, $I = 0, \ldots, n$. ('Generalized stabilization equations').

Near horizon behaviour determined by the charges ('attractor mechanism').

Work Plan

- Objective: construct *multi-centered* solutions without imposing supersymmetry.
- Strategy: reduce equations of motion to decoupled harmonic equations. Use dimensional reduction over time. Do *not* impose spherical symmetry (in contrast to methods based on first order gradient flow equations).
- Use 'Einstein-Maxwell-Scalar'-type action which contains vector multiplet action as a subclass but do not assume from the start that the couplings are encoded by special real geometry. Rather insist that multi-centered solutions exist and can be expressed *algebraically* in terms of harmonic functions.

(Some) References

- Talk based on T.M. and K. Waite, arXiv:0906.3451. Related: V. Cortés and T.M. JHEP 0907 (2009) 066 (Euclidean Supergravity and para-complex geometry).
- Attractor mechanism (via Kiling spinors): S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D 52 (1995) 5412.,...
- Generalized stabilisation equations: K. Behrndt, D. Lüst and W. Sabra, Nucl. Phys. B 510 (1998) 264, G. Cardoso, B. de Wit, J. Käppeli and T.M. JHEP 0012 (2000) 019.
- Attractor mechanism w/o Killing spinors, non-BPS extremal black holes: S. Ferrara, G. Gibbons and R. Kallosh, Nucl. Phys. B 500 (1997) 75, K. Goldstein, N. Iizuka, R. Jena and S. Trivedi, Phys. Rev. D 72 (2005) 124021, ...

(Some) References (cont'd)

- First order gradient flow equations: ..., A. Ceresole and G. Dall'Agata, JHEP 0703 (2007) 110, G. Cardoso, A. Ceresole, G. Dall'Agata, J. Oberreuter and J. Perz, JHEP 10 (2007) 063, J. Perz, P. Smyth, T. Van Riet and B. Vercnocke, JHEP 0903 (2009) 150, ...
- 5d vector multiplets, special real geometry: M. Gunaydin, G. Sierra and P. Townsend, Nucl. Phys. B 242 (1984) 244, ...
- BPS black holes in 5d supergravity: A. Chamseddine and W. Sabra, Phys. Lett. B 426 (1998), Phys. Lett. B 460 (1999) 63.
- Temporal reduction: G. Neugebauer and D. Kramer, Ann. Phys. (Leipzig) 24 (1969) 253, P. Breitenlohner, D. Maison and G. Gibbons, Comm. Math. Phys. 120 (1988) 253, ...

Dimensional reduction over time

Four-dimensional Euclidean theory:

$$S = \frac{1}{2} \int d^4x \sqrt{|g_{(4)}|} \left(R_{(4)} - N_{IJ}(\sigma)(\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J) + \cdots \right)$$

Dictionary

$$ds^2_{(5)} = -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} ds^2_{(4)}$$

$$\begin{array}{ccc} \tilde{\sigma}, \phi^x & \rightarrow & \sigma^I \\ A_\mu, A^x_\mu & \rightarrow & b^I \end{array}$$

$n + 1$ gauge symmetries $\rightarrow$ $n + 1$ shift symmetries $b^I \rightarrow b^I + C^I$.

General remarks

Solutions to sigma model EOM $=$

$$\text{Harmonic Map : space-(time)} \rightarrow \text{scalar manifold}$$

- Single centered extremal solutions $\simeq$ Harmonic map onto null geodesic curves in the scalar manifold.
- General extremal solutions $\simeq$ Harmonic map onto totally isotropic, totally geodesic submanifold.

Terminology: Finite action Euclidean solution $=$ Instanton or (-1)-brane. (In fact, solutions will resemble D-instanton and other instanton solutions of supergravity/string theory.)

Multi-centered solutions for pedestrians

Compare Kaluza-Klein ansatz to extremal black hole ansatz

$$ds^2_{(5)} = -e^{2\tilde{\sigma}(\vec{x})} dt^2 + e^{-\tilde{\sigma}(\vec{x})} ds^2_{(4)}$$

Extremal black holes have flat $ds^2_{(4)} = \delta_{mn} dx^m dx^n$. 4d scalar EOM (assuming flat 4d metric):

$$\begin{aligned} \partial_m (N_{IJ}(\sigma) \partial^m \sigma^J) - \frac{1}{2} \partial_I N_{JK}(\sigma) (\partial_m \sigma^J \partial^m \sigma^K - \partial_m b^J \partial^m b^K) &= 0 \\ \partial_m (N_{IJ}(\sigma) \partial^m b^J) &= 0 \end{aligned}$$

Second equation $=$ conservation of charges q_I (4d: axionic or instanton charge, 5d: electric charge).
Equations simplify if we set $\partial_m \sigma^I = \pm \partial_m b^I$.

Extremal instanton solutions

Extremal instantons: $\partial_m \sigma^I = \pm \partial_m b^I$.

- Geometrical interpretation: solutions flows along null directions in scalar manifold (totally isotropic submanifold).
- Physical interpretation: equivalent to $T_{mn} = 0$, which is needed to solve the Einstein equation with $g^{(4)}_{mn} = \delta_{mn}$.
- Solution saturate a Bogomol'nyi type bound and lifts to an extremal black hole solution.

Remark: generalized relation $\partial_m \sigma^I = R^I{}_J \partial_m b^J$ can be imposed if the scalar metric has additional isometries.

Harmonic functions

Remaining equations

$$\partial_m (N_{IJ}(\sigma) \partial^m \sigma^J) = 0$$

reduce to harmonic equations $\Delta \sigma_I = 0$ if we can find dual coordinates σ_I such that

$$N_{IJ}(\sigma) \partial_m \sigma^J = \partial_m \sigma_I \ .$$

Without assumptions on the solution (i.e. no spherical symmetry), this requires the integrability condition

$$\partial_{[I} N_{J]K} = 0 \Leftrightarrow \partial_I N_{JK} \text{completely symmetric} \Leftrightarrow \Gamma_{JK|I} \text{completely symmetric}$$

which is solved locally by

$$N_{IJ}(\sigma) = \frac{\partial^2 \mathcal{V}(\sigma)}{\partial \sigma^I \partial \sigma^J}$$

i.e. the scalar metric $N_{IJ}(\sigma)$ must be Hessian, with Hesse potential $\mathcal{V}(\sigma)$.

Harmonic functions (cont'd)

Geometrical interpretation: σ_I are *affine* coordinates on a *flat* totally geodesic, totally isotropic submanifold. (Flatness $\Rightarrow$ harmonic map equation becomes linear.)
Note that

$$\sigma_I = \frac{\partial \mathcal{V}}{\partial \sigma^I}$$

so that the solution (i.e. σ^I) can be expressed algebraically in terms of harmomic functions:

$$\frac{\partial \mathcal{V}}{\partial \sigma^I} = H_I(\vec{x}) \ .$$

Explicit expressions for the σ^I can only be obtained if the Hesse potential $\mathcal{V}(\sigma)$ is sufficiently simple.

Lifting to 5d

We have assumed that the 4d Euclidean action

$$\int d^4x \sqrt{|g_{(4)}|} \left(\frac{1}{2} R_{(4)} - \frac{1}{2} N_{IJ}(\sigma)(\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J) + \cdots \right)$$

came from a 5d Einstein-Maxwell type action

$$\int d^5x \sqrt{|g_{(5)}|} \left(\frac{1}{2} R_{(5)} - \frac{1}{2} N_{xy}(\phi) \partial_m \phi^x \partial^m \phi^y - \frac{1}{4} N_{IJ}(\phi) F^I_{\mu\nu} F^{J|\mu\nu} + \cdots \right)$$

4d scalars $\sigma^I \leftrightarrow$ 5d scalars ϕ^x plus KK scalar $\tilde{\sigma}$.
A consistent lift is obtained for logarithmic Hesse potentials

$$\mathcal{V}(\sigma) = -\frac{1}{p} \log \hat{\mathcal{V}}(\sigma) ,$$

where the prepotenial $\hat{\mathcal{V}}(\sigma)$ is a homogeneous function of degree p.
The case $p = 3$ corresponds to 5d supergravity and special real geometry.

5d Solutions

The 5d theory is more conveniently described using ‘homogeneous coordinates’

$$\int d^5x \sqrt{|g_{(5)}|} \left(\frac{1}{2} R_{(5)} - \frac{1}{2} N_{IJ}(h) \partial_m h^I \partial^m h^J - \frac{1}{4} N_{IJ}(h) F^I_{\mu\nu} F^{J|\mu\nu} + \cdots \right)_{\hat{\mathcal{V}}=1}$$

where

$$\phi^x \; x = 1, \ldots, n \leftrightarrow h^I \;, I = 0, \ldots, n+1 \;, \text{subject to } \; \hat{\mathcal{V}}(h) = 1 \;.$$

Relation between 5d and 4d scalars:

$$\sigma^I = e^{\tilde{\sigma}} h^I \;. \quad \text{Note: } \; \hat{\mathcal{V}}(h) = 1 \Rightarrow \hat{\mathcal{V}}(\sigma) = e^{p\tilde{\sigma}} \;.$$

Express solution (scalars and metric) in terms of harmonic functions

$$e^{-\tilde{\sigma}} \frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} = H_I(\vec{x})$$

Attractor behaviour and entropy

Asymptotics at a center located at $r = 0$ (display single-centered case for convenience):

$$H_I \approx \frac{q_I}{r^2}\,, \quad e^{-\tilde{\sigma}} \approx \frac{Z_*}{r^2}$$

determined by ‘attractor equations’

$$Z_* \left. \frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} \right|_* = q_I$$

Z_* generalizes the central charge:

$$Z_* = \frac{1}{p} q_I h^I_*$$

and determines the entropy:

$$S_{BH} = \frac{A}{4} = \frac{\pi^2}{2} Z_*^{3/2}\,.$$

Wormholes

The hypersurface $t = \text{const.}$

$$ds^2_{(5)}\Big|_{dt=0} = e^{-\tilde{\sigma}}\delta_{mn}dx^m dx^n$$

is a semi-infinite wormhole with neck of size $\propto A$. Charges can be tuned to obtain degenerate solutions with $A = 0$.

ADM mass:

$$\begin{aligned} M_{ADM} &= -\frac{3}{2} \oint d^3\Sigma^m \partial_m e^{-\tilde{\sigma}} = -\frac{3}{2} \oint d^3\Sigma^m \partial_m \hat{\mathcal{V}}(\sigma)^{1/p} \\ &= \text{Const.}\, Z_\infty = \text{Const.}\, q_I h^I_\infty \end{aligned}$$

BPS-like formula!
Displayed single-centered case, but is additive in centers.

Instanton action

Naively, the instanton action is zero

$$S_{\rm inst} = \frac{1}{2}\int d^4x \left(N_{IJ}(\sigma)(\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J) \right)_{\partial\sigma^I = \pm \partial b^I} = 0$$

Addition of boundary term can be motivated (i) via (positive definite) dual action $\tilde{S}(\sigma^I, B^I_{mn})$ and (ii) instanton amplitudes:

$$S[\sigma^I, b^I] = S[\sigma^I, b^I]_{\rm bulk} + \oint d\Sigma^m b^I N_{IJ}(\sigma)\partial_m b^J$$

$$\Rightarrow S_{\rm inst} = \text{Const. } \sigma^I_\infty q_I$$

Another 'BPS-type' formula!

Black hole mass and instanton action

Observation:

$$M_{ADM} = S_{\mathrm{inst}} \; .$$

Might have been expected: 0-brane tension = mass $\rightarrow$ (-1)-brane tension = action. (We have set the volume of the internal dimension to unity.) However, as we have seen the instanton action is a bit subtle.

This observation provides additional motivation for adding boundary term to the sigma model action. Positivity of the instanton action and saturation of a Bogomol'nyi bound for extremal instanton solutions then follow from the positivity of the ADM mass.

Outlook

- Our approach (temporal reduction, but no other symmetries, harmonic maps instead of 1st order ‘Killing spinor-type’ flow equations) could be extended to other types of solutions: 4d black holes vs 3d instantons (dyonic), 5d: Taub-NUT charge, strings and rings, non-extremal solutions, . . .
- Generalized versions of special geometry, relations between ‘geometry of couplings’ and physical properties.