Open M2-branes – ABJM & BLG with Boundary

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Outline

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- Define Chern-Simons theory with boundary by adding new boundary degrees of freedom.
- Define ABJM theory with boundary by adding new boundary degrees of freedom – Interpret as effective action for multiple self-dual strings in M5.

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- Define ABJM theory with boundary by adding new boundary degrees of freedom – Interpret as effective action for multiple self-dual strings in M5.
- Derive D3 NC Geometry from matrix string action – Requiring modified Nahm eqn. as b.c. with B-field.
- Derive M5 Quantum Geometry from BLG action – Requiring modified Basu-Harvey eqn. as b.c. with C-field.

Motivation

In string theory, open strings play a fundamental role, particularly in defining D-branes and describing field theories. Specifically, a fundamental string with a boundary end on D-branes. Quantising the string with boundary give the DBI action. If we include a constant NSNS B-field, quantisation of open strings gives non-commutative field theories on D-branes.

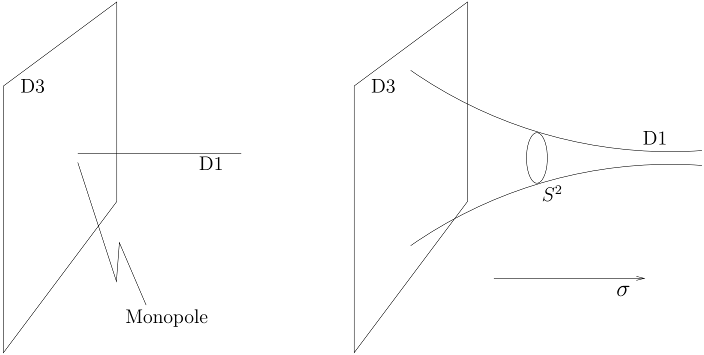
String/branes ending on other branes give a geometric interpretation of field theory states. E.g. for a D3-brane:

$$\begin{array}{rcl} \text{Particle States} & \leftrightarrow & \text{Endpoints of strings} \\ \text{Electric} & \leftrightarrow & F1 \\ \text{Magnetic} & \leftrightarrow & D1 \end{array}$$

Nahm Equation

The ADHMN construction give BPS monopole solutions by solving the Nahm equation

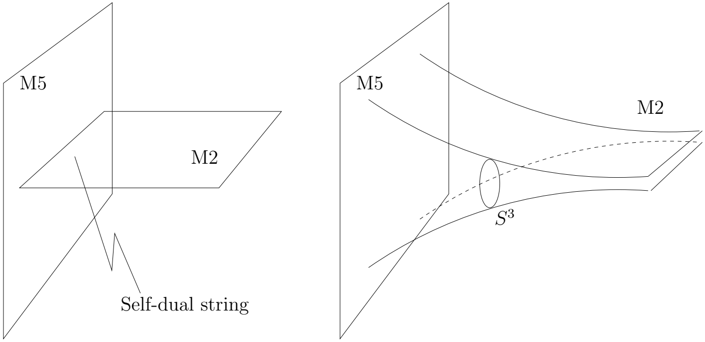
$$\partial_\sigma \phi^i \sim \epsilon^{ijk} \left[\phi^j , \phi^k \right]$$



Basu-Harvey Equation

- M2 on M5 describes $D = 6$ self-dual string solitons.
- Quantisation of M2 with boundary gives M5 WV action?
- ADHMN construction should generalise, with Nahm equation being replaced by the Basu-Harvey equation:

$$\partial_\sigma \phi^i \sim \epsilon^{ijkl} \left[\phi^j, \phi^k, \phi^l\right]$$



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- Expected to describe WV theory of multiple M2-branes.
- Symmetries and Basu-Harvey equation manifest.
- Only one suitable 3-algebra – 2 M2 on C^4/Z_2.

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ABJM (Aharony, Bergman, Jafferis & Maldecena) constructed a $D = 2 + 1$ $\mathcal{N} = 6$ SCFT which is a Chern-Simons + Matter theory, which does not require a 3-algebra.

- N M2 on C^4/Z_k for $U(N) \times U(N)$ & CS level $(k, -k)$.
- $\mathcal{N} = 8$ expected but not manifest for $k = 1, 2$.
- Basu-Harvey eqn. & Fuzzy S^3? – multiple D2-brane theory?

Chern-Simons Action

Consider a Chern-Simons theory with gauge group G on a 3-dimensional manifold M

$$S_{CS} = \frac{k}{4\pi}\int_M \omega_3(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}(AdA + \frac{2}{3}A^3),$$

where A^3 denotes $A \wedge A \wedge A$ etc and $A = dx^\mu A_\mu$ is a Lie algebra valued one-form. When M is closed, the theory is gauge invariant and topological.

Under an infinitesimal gauge transformation

$$\delta_\alpha A = d\alpha + [A, \alpha], \quad \delta_\alpha F = [F, \alpha],$$

the Chern-Simons form transforms as

$$\delta_\alpha \omega_3(A) = d\omega_2^1(A; \alpha).$$

where

$$\omega_2^1(A; \alpha) = \mathrm{Tr}(\alpha dA).$$

Chern-Simons with Boundary

Since $\delta_\alpha \omega_3(A)$ is exact, in the presence of a boundary ∂M, the gauge variation of S_{CS} gives a boundary term. So it seems that a boundary breaks the gauge symmetry.
To study the theory we must first fully define it when there is a boundary. There are two ways to do this:

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- Specify boundary conditions for A.
- Introduce new boundary degrees of freedom.

Imposing boundary conditions

Consider an arbitrary infinitesimal variation of A, the variation of the Chern-Simons action gives

$$\delta S_{CS} = \frac{k}{2\pi} \int_M \mathrm{Tr}(\delta A\, F) + \frac{k}{4\pi} \int_{\partial M} \mathrm{Tr}\,(\delta A\, A).$$

The bulk term gives the equation of motion $F = 0$.
For the boundary term to vanish we must impose a boundary condition on A and only allow variations which preserve this boundary condition.
Different inequivalent classes define different theories.

- $A_0 = 0$
- $A_0 \pm A_1 = 0$
- $A_1 = 0$

Boundary Chern-Simons $\rightarrow$ WZW

Let's consider explicitly a boundary at $x^2 = 0$ and choose the boundary condition $A_0 = 0$. Then

$$S_{CS} = \frac{k}{2\pi} \int_M \mathrm{Tr}(\epsilon^{ij} F_{ij} A_0 - \frac{1}{2}\epsilon^{ij} A_i \dot{A}_j),$$

after integration by parts, using the boundary condition.
We see that A_0 is a Lagrange multiplier, imposing the constraint $F_{12} = 0$. Therefore

$$A_i = U^{-1}\partial_i U \;\; \text{for} \;\; i = 1, 2$$

for $U \in G$. Substituting back into the above action we find

$$S = -\frac{k}{8\pi} \int_{\partial M} \mathrm{Tr}\left(U^{-1}\partial_0 U U^{-1}\partial_1 U\right) - \frac{k}{12\pi} \int_M \mathrm{Tr}\left(U^{-1} dU\right)^3 .$$

Note that a "chiral kinetic term" is obtained for U.

Similar results for other boundary conditions, but kinetic term gets modified.

- $A_0 = 0$ or $A_1 = 0$ give chiral kinetic term, but with opposite signs
- $A_0 \pm A_1 = 0$ give conventional kinetic term, again with opposite signs.

So, choosing appropriate boundary conditions for A, we arrive at the well-known WZW action

$$S_{WZW}[U] = -\frac{k}{8\pi}\int_{\partial M} \mathrm{Tr}(U^{-1}\partial_\mu U)^2 - \frac{k}{12\pi}\int_M \mathrm{Tr}\left(U^{-1}dU\right)^3 ,$$

This action describes the dynamics for the field U living on the boundary ∂M.

Comments

- No non-trivial Jacobian for change of variables $A \to U$ so actions quantum equivalent.
- Boundary conditions break gauge symmetry, but WZW action has new chiral $G \times G$ symmetry

  $$U \to \Omega(z) U \tilde{\Omega}(\bar{z})$$

  where $z = x^0 + ix^1$ and $\Omega, \tilde{\Omega} \in G$.
  This is a Kac-Moody symmetry with chiral currents

  $$J = \frac{k}{\pi} U^{-1} \partial U, \quad \bar{J} = \frac{k}{\pi} \bar{\partial} U U^{-1}$$

- Boundary breaks topological invariance of Chern-Simons, but WZW action retains boundary conformal invariance.

Boundary Degrees of Freedom

We can also introduce new physical degrees of freedom at the boundary. We can try to preserve bulk symmetries such as gauge invariance.

Consider a finite gauge transformation

$$A^g := g^{-1}Ag + g^{-1}dg.$$

Since $d\omega_3(A) = \mathrm{Tr}F^2(A)$, the integrand in

$$\delta S_{CS} = \frac{k}{4\pi}\int_M [\omega_3(A^g) - \omega_3(A)]$$

is closed and so δS_{CS} is a boundary term plus a bulk topological term – an integer multiple of 2π with given coefficient.

Boundary Action – Gauge Invariance

Let's introduce boundary degrees of freedom g with action

$$S_{Bdry} := \frac{k}{4\pi} \int_M \left[\omega_3(A^g) - \omega_3(A)\right],$$

Under a gauge transformation with parameter h,

$$A^g \longrightarrow (A^g)^h = A^{hg}.$$

Therefore the total action

$$S_T := S_{CS} + S_{Bdry} = \frac{k}{4\pi} \int_M \omega_3(A^g)$$

will be gauge invariant under the combined transformation

$$A \rightarrow A^h \ , \ \ g \rightarrow h^{-1} g.$$

Boundary Action – Boundary Conformal Invariance

The resulting action is not unique – we can add any further gauge-invariant boundary terms. Since the original action was topological, we can at least try to preserve conformal invariance on the boundary.

Now, if we set the gauge field $A = 0$ we are left with

$$S_{Bdry}[A=0] = -\frac{k}{12\pi}\int_M \mathrm{Tr}\left(g^{-1}dg\right)^3$$

which is only classically conformally invariant. However, we can introduce a boundary kinetic term for g, resulting in the well-known WZW CFT

$$S_{WZW}[g] = -\frac{k}{8\pi}\int_{\partial M}\mathrm{Tr}(g^{-1}\partial_\mu g)^2 - \frac{k}{12\pi}\int_M \mathrm{Tr}\left(g^{-1}dg\right)^3 .$$

Note, the beta-function vanishes for the above ratio of coefficients, and also for either choice of sign for the kinetic term. This means we can choose the correct sign for the kinetic term whether k is positive or negative.

Gauge and Boundary Conformal Invariance

Restoring the gauge field A we can maintain both gauge and conformal invariance by replacing $\partial_\mu \to D_\mu = \partial_\mu + A_\mu$ giving the boundary action

$$\begin{aligned} S_{Bdry} &= -\frac{k}{8\pi}\int_{\partial M} \mathrm{Tr}(g^{-1}D_\mu g)^2 + \frac{k}{4\pi}\int_M [\omega_3(A^g) - \omega_3(A)] \\ &= S_{WZW}[g] + \frac{k}{4\pi}\int_{\partial M} \partial_+ g g^{-1} A_- - \frac{k}{8\pi}\int_{\partial M} A_\mu^2, \end{aligned}$$

where $\partial_\pm := \partial_0 \pm \partial_1$. This is a (non-standard) gauged WZW action.

Finally, the total action is

$$S_T = -\frac{k}{8\pi}\int_{\partial M} \mathrm{Tr}(g^{-1}D_\mu g)^2 + \frac{k}{4\pi}\int_M \omega_3(A^g).$$

Boundary Equations of Motion

We can derive the boundary equations of motion by collecting the boundary contributions from variations of A and g – including from integration by parts in the bulk.

Varying A, we find the boundary term

$$-\frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(A_+^g \delta A_-^g\right).$$

Therefore we obtain from δA the boundary equation

$$A_+^g = 0 \to A_+ = -(\partial_+ g)g^{-1}.$$

Next the variation of g, using $A_+^g = 0$, gives the boundary term

$$-\frac{k}{2\pi}\int_{\partial M} \mathrm{Tr}\left(F^g g^{-1}\delta g\right) = -\frac{k}{2\pi}\int_{\partial M} \mathrm{Tr}\left(g^{-1} F \delta g\right).$$

Therefore we find from δg that $F_{01} = 0$ on the boundary.

The two boundary equations give

$$A_\mu = -(\partial_\mu g)g^{-1}, \quad \mu = 0, 1.$$

Complete Equations of Motion

The bulk equations of motion result in $F = 0$ as usual. Therefore A is pure gauge. We can now interpret the boundary equation of motion as a boundary condition for the bulk gauge field, and so we have

$$A = -dg\, g^{-1}$$

in M where g is an arbitrary extension of the boundary field g into M.

We see that the on-shell degrees of freedom are given by the boundary field g. We can also see a direct relation to the method of imposing boundary conditions.

Comparing the two approaches

It can be shown that A_+ appears linearly (without derivative of it) in S_T,

$$
\begin{aligned}
S_T = S_{WZW}[g] \quad &+ \quad \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\,\partial_+ g g^{-1} A_- + \frac{k}{4\pi}\int_M \mathrm{Tr}\, A_+ F_{2-} \\
&+ \quad \frac{k}{8\pi}\int_M \mathrm{Tr}\,(A_2 \partial_+ A_- - A_- \partial_+ A_2)
\end{aligned}
$$

so is a (bulk) Lagrange multiplier, imposing the constraint $F_{2-} = 0$ in M. Solving this constraint by writing

$$A_2 = \lambda^{-1}\partial_2\lambda\ ,\ A_- = \lambda^{-1}\partial_-\lambda,$$

$$S_T = S_{WZW}[g] + S_{WZW}[\lambda] + \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(\partial_+ g g^{-1} \lambda^{-1}\partial_-\lambda\right) = S_{WZW}[\lambda g],$$

where we have used the Polyakov-Wiegmann identity in the last step.

Therefore with the identification $U = \lambda g$, we finally arrive at the same action as obtained by the boundary condition approach.

Comments

- Boundary conditions break the gauge symmetry.
- Boundary degrees of freedom g preserve the gauge symmetry.
- After integrating out Lagrange multiplier A_+, action depends only on gauge invariant $U = \lambda g$; $g \to h^{-1}g$ and $\lambda \to \lambda h$.

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For more general actions, will not have a Lagrange multiplier, e.g. matter covariant kinetic terms quadratic in gauge potential.

- Boundary condition approach not convenient.
- Boundary degrees of freedom approach generalises easily.
- Gauge symmetry will remain – no trivial gauge invariant description of degrees of freedom.

ABJM

The bosonic part of the action for the ABJM theory is given by

$$S_{ABJM} = S_{CS} + S_C,$$

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi}\int_M \omega_3(A^{(1)}) - \frac{k}{4\pi}\int_M \omega_3(A^{(2)}) \\ S_C &= -\int_M \mathrm{Tr}\left(D_M C_I^\dagger D^M C^I\right) - \frac{4\pi^2}{3k^2}\int_M \mathrm{Tr}\, V_{Bulk}(C) \\ V_{Bulk}(C) &= C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \\ &\quad + 4 C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6 C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \end{aligned}$$

$A^{(1)}$ and $A^{(2)}$ are gauge potentials for $U(N) \times U(N)$ gauge group.
Matter fields C^I $(I = 1, 2, 3, 4)$ are bifundamental $(N, \bar{N})$.
Covariant derivative acts as $D_M C^I = \partial_M C^I + A_M^{(1)} C^I - C^I A_M^{(2)}$.
In the absence of a boundary, the action is gauge invariant under:

$$A^{(i)} \to A^{(i)h^{(i)}} \ , \ \ C^I \to (h^{(1)})^{-1} C^I h^{(2)}.$$

ABJM with Boundary

As for the Chern-Simons action, a boundary breaks symmetries of the ABJM action.

To maintain gauge and boundary conformal invariance, we can add boundary terms of the form described previously for each of the gauge fields $A^{(1)}$ and $A^{(2)}$. The required boundary actions are

$$S_{Bdry1} = -\frac{k}{8\pi}\int_{\partial M} \mathrm{Tr}\left(g^{-1}D_\mu^{(1)}g\right)^2 + \frac{k}{4\pi}\int_M \left(\omega_3(A^{(1)g}) - \omega_2(A^{(1)})\right)$$

and

$$S_{Bdry2} = -\frac{k}{8\pi}\int_{\partial M} \mathrm{Tr}\left(\hat{g}^{-1}D_\mu^{(2)}\hat{g}\right)^2 - \frac{k}{4\pi}\int_M \left(\omega_3(A^{(2)\hat{g}}) - \omega_2(A^{(2)})\right).$$

Note that although the bulk Chern-Simons terms for $A^{(1)}$ and $A^{(2)}$ differ by a relative sign, as previously mentioned, preserving boundary conformal invariance, we can independently choose the sign of the boundary kinetic terms. Hence we have a well defined quantum field theory, for the boundary fields g and $\hat{g}$, for $k > 0$.

ABJM with Boundary

Note that the presence of a boundary does not spoil the symmetries of the matter action S_C, so we do not need to introduce any further boundary degrees of freedom. Nevertheless we can consider a possible boundary interaction term V_{Bdry} for the matter fields C^I,

$$-\int_{\partial M} V_{Bdry}(C).$$

- The form of V_{Bdry} is constrained by gauge and conformal invariance – must be quartic in C^I.
- Boundary supersymmetry should fix V_{Bdry} but we can find unique form consistent with bulk BPS equations.

Boundary Potential

Reproducing both bulk D-term

$$\frac{k}{2\pi} D_2 Z^I + Z^I (Z^\dagger Z - W W^\dagger) - (Z Z^\dagger - W^\dagger W) Z^I = 0$$

and F-term

$$\frac{k}{4\pi} D_2 Z^I - \epsilon^{IJ} \epsilon_{KL} W^{\dagger K} Z^\dagger_J W^{\dagger L} = 0, \quad C^I \sim \{Z^I, W^{I\dagger}\}$$

BPS equations as boundary equations is uniquely achieved by the quartic boundary potential

$$V_{Bdry} := V_D + V_F$$

where

$$V_D = \frac{\pi}{k} \mathrm{Tr}[(Z Z^\dagger - W^\dagger W)^2 - (Z^\dagger Z - W W^\dagger)^2],$$

$$V_F = -\frac{2\pi}{k} \mathrm{Tr}[\epsilon_{IJ} \epsilon^{KL} Z^I W_K Z^J W_L] + h.c.$$

Boundary ABJM Equations of Motion

As for the Chern-Simons theory, we find on the boundary

$$A_\mu^{(1)} = -(\partial_\mu g)g^{-1} \quad \text{and} \quad A_\mu^{(2)} = -(\partial_\mu \hat{g})\hat{g}^{-1} \quad , \quad \mu = 0, 1.$$

together with

$$D_2 C^I = -\delta V_{Bdry}/\delta C_I^\dagger$$

However, the bulk equations of motion are not $F = 0$ but

$$\begin{aligned} \frac{k}{2\pi} F^{(1)} &= *C^I D C_I^\dagger - h.c. \\ \frac{k}{2\pi} F^{(2)} &= *D C_I^\dagger C^I - h.c. \end{aligned}$$

so we cannot simply solve for the gauge potentials.

Self-Dual String Action

Can now derive self-dual string action from cylindrical membranes between two M5-branes with separation L. In terms of the M2 tension T_3 and string tension $T_2 = T_3 L$, with b.c. $W = 0$, and after some rescaling we find

$$
\begin{aligned}
S_{SDS} &= -T_2 \int d^2x \left[(D\mathcal{Z})^2 + T_3^2 V_M(\mathcal{Z}) + \frac{T_3^2}{T_2} V_{Bdry}(\mathcal{Z}) \right] \\
&\quad + 2S^{(-)}_{WZW}[g] + \frac{k}{2\pi}\int d^2x\, \partial_+ g g^{-1} A^{(1)}_- - \frac{k}{4\pi}\int d^2x\, A^{(1)\,2}_\mu \\
&\quad + 2S^{(+)}_{WZW}[\hat{g}] - \frac{k}{2\pi}\int d^2x\, \partial_- \hat{g}\hat{g}^{-1} A^{(2)}_+ - \frac{k}{4\pi}\int d^2x\, A^{(2)\,2}_\mu .
\end{aligned}
$$

Conclusions & Outlook

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- CS + Boundary $\rightarrow$ WZW: Boundary conditions or new boundary fields.
- New boundary fields is a general method which can be used for ABJM.
- Derive full SUSY boundary theory.
- Couple to background fields, including M5-brane 2-form potential.
- Application to self-dual string solitons in M5.
- Relevance to fundamental M5-brane theory degrees of freedom?

Non-commutative D3-brane theory

Consider a D3-brane (0123) with a constant NSNS B-field $B_{12} = B$. Quantising open strings leads to non-commutative geometry on D3 WV. Specifically we have an open string metric

$$G = \mathrm{diag}(-1, 1 + B^2, 1 + B^2, 1)$$

and non-commutativity parameter $\theta^{12} = B/(1 + B^2)$

$$[X^1, X^2] = \theta^{12}.$$

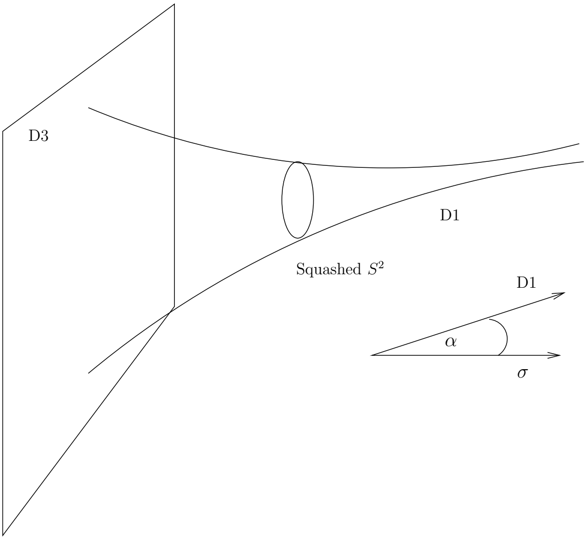
The multiple D1 action (non-Abelian DBI) has a modified Nahm equation as BPS equation:

$$\begin{aligned} \partial_\sigma X^i &= i\epsilon^{ij}[X^j, X^3], \quad i, j = 1, 2, \\ \partial_\sigma X^3 &= i(1 + B^2)\Big([X^1, X^2] + i\theta\Big) \end{aligned}$$

where θ is exactly the expected non-commutativity parameter θ^{12}.

Modified Nahm Equation

Modified Nahm equation correctly describes geometry of D1/D3 squashed fuzzy S^2 funnel at angle α for $B = \tan\alpha$.



Nahm Equation as D1 Boundary Condition

We can also derive this modified Nahm equation as a boundary condition for the matrix D-string action. Without a B-field we have the boundary condition

$$\partial_\sigma X^i = \frac{1}{2}\epsilon^{ijk}F_{jk} = \frac{i}{2}\epsilon^{ijk}[X^j, X^k]$$

where F is a matrix sampling the U(1) field strength on the D3-brane at the string endpoints, and the second equality is a matching condition since the smooth configuration can be described by either the D3 or multiple D1 actions.

Introducing a B-field modifies the BPS equation and gives the expected modified Nahm equation.

Nahm Equation as F1 Boundary Condition

Without a B-field a matrix string action gives the Nahm equation as a boundary condition, in the same way as the S-dual D1 case. Introducing a B-field modifies the string action in a different way. If we allow an open string metric $(G_{\mu\nu}) = \mathrm{diag}(-g_0, g_1, g_2, g_3)$ with $g_0 = g_3$ and $g_1 = g_2$. Then

$$\partial_\sigma X^i + B^i{}_j \partial_\tau X^j = \frac{i}{2} \frac{g_j g_k}{g_0 g_1} \epsilon_{ijk} \left[X^j, X^k \right]$$

Relation to NC Geometry

We can decompose this boundary condition, and solve the τ dependence by writing

$$X^i(\tau,\sigma) = X_0^i(\tau,\sigma)\mathbf{1} + Y^i(\sigma)$$

so

$$\partial_\sigma X_0^i + B^i{}_j \partial_\tau X_0^j = 0$$

which is exactly the mixed boundary condition encountered when quantising the open string in a B-field background. The result of the quantisation is that at the boundary

$$\left[X_0^i(\tau,\sigma_0), X_0^j(\tau,\sigma_0)\right] = i\theta^{ij}$$

where here

$$\theta^{12} = \theta = \frac{B}{1+B^2}$$

Assuming Y^i and X_0^j commute we now have

$$\begin{aligned}\partial_\sigma Y^i &= i\epsilon^{ij}[Y^j, Y^3], \quad i,j = 1,2, \\ \partial_\sigma Y^3 &= i\frac{g_1}{g_0}\left([Y^1, Y^2] + i\theta\right)\end{aligned}$$

which gives precisely the modified Nahm equation if $g_1/g_0 = 1 + B^2$. In fact this is consistent with the expected open string metric where

$$g_0 = 1 \text{ and } g_1 = 1 + B^2$$

Constant 3-form C-field potential

For M2 on M5 with constant C-field so that on M5

$$H_{012} = \frac{1}{4}\sin\alpha \ , \quad H_{345} = \frac{1}{4}\tan\alpha$$

the M2 makes angle α with normal to M5. Geometric configuration of such fuzzy S^3 funnel can be described by modified Basu-Harvey equation

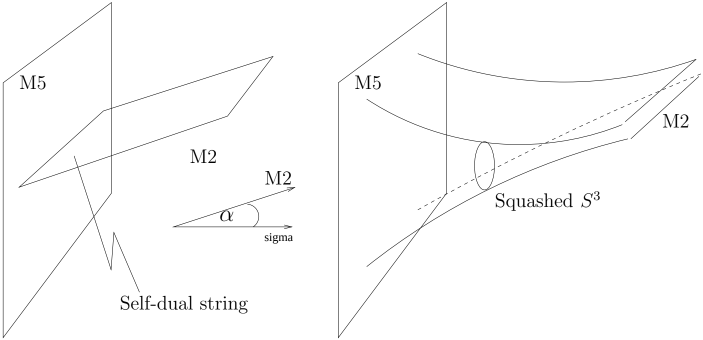
$$\partial_2 \phi^i = \frac{i\cos\alpha}{3!}\epsilon_{ijkl}[\phi^j, \phi^k, \phi^l] - \delta_2^i \tan\alpha$$

where we have rescaled the X^i as

$$\phi^i = \begin{cases} (1+\tan^2\alpha)^{1/2} X^i, & \text{for } i = 3,4,5, \\ X^i, & \text{for } i = 2. \end{cases}$$

Modified Basu-Harvey Equation

Modified Basu-Harvey equation correctly describes geometry of M2/M5 squashed fuzzy S^3 funnel at angle α for $C_{345} \sim \tan\alpha$.



Modified Basu-Harvey Boundary Condition

After modifying the BLG theory to include coupling to C-field and open membrane metric, and boundary coupling to 2-form potential on M5, the boundary conditions lead to

$$\partial_2 X^i + C^i{}_{jk}\partial_0 X^j * \partial_1 X^k = \frac{1}{3!\sqrt{-G}}\epsilon_{ijkl}G_{jj}G_{kk}G_{ll}\tilde{F}^{jkl}$$

We also allow the generalised relation

$$\tilde{F}^{ijk} = if[X^i, X^j, X^k] \ , \quad f = 1 \ \text{ for } \ C = 0$$

which leads to

$$\partial_2 X^i + C^i{}_{jk}\partial_0 X^j * \partial_1 X^k = \frac{if}{3!\sqrt{-G}}\epsilon_{ijkl}G_{jj}G_{kk}G_{ll}[X^i, X^j, X^k]$$

We now substitute

$$X^i(\tau,\sigma_1,\sigma_2) = X_0^i(\tau,\sigma_1,\sigma_2)\mathbf{1} + Y^i(\sigma_2)$$

Modified Basu-Harvey Boundary Condition

$$\partial_2 X_0^i + C_{ijk}\partial_0 X_0^j \partial_1 X_0^k = 0$$

$$\begin{aligned} \partial_2 Y^2 &= if(\frac{g_1}{g_0})^{3/2}\left([X_0^3, X_0^4, X_0^5] + [Y^3, Y^4, Y^5]\right) \\ \partial_2 Y^i &= if(\frac{g_1}{g_0})^{1/2}\epsilon^{i2jk}\left([X_0^2, X_0^j, X_0^k] + [Y^2, Y^j, Y^k]\right), \quad i, j, k \neq 2 \end{aligned}$$

This produces the modified Basu-Harvey equation if we identify $f = \cos\alpha$, $g_1/g_0 = 1 + \tan^2\alpha$ and

$$[X_0^2, X_0^j, X_0^k] = 0 \ , \ [X_0^j, X_0^k, X_0^l] = i\Theta^{jkl}$$

$$\Theta^{jkl} = \epsilon^{jkl}\frac{C}{(1+C^2)^2}$$

Outlook

It would be useful to get more evidence for our predicted quantum geometry. Through compactification and dualities we can relate this system to D-branes. We would expect:

- Relation to NC geometry since $C_{(3)} \longrightarrow B_{(2)}$
- Possible new D-brane geometries in RR backgrounds.

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There are many open questions:

- Realisation and consequences of this new quantum geometry?
- 3-bracket quantum geometry by novel open membrane quantisation?
- 3-bracket construction of single/multiple M5-brane action?