Quantum Chromodynamics

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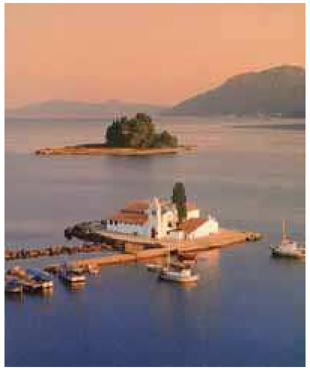

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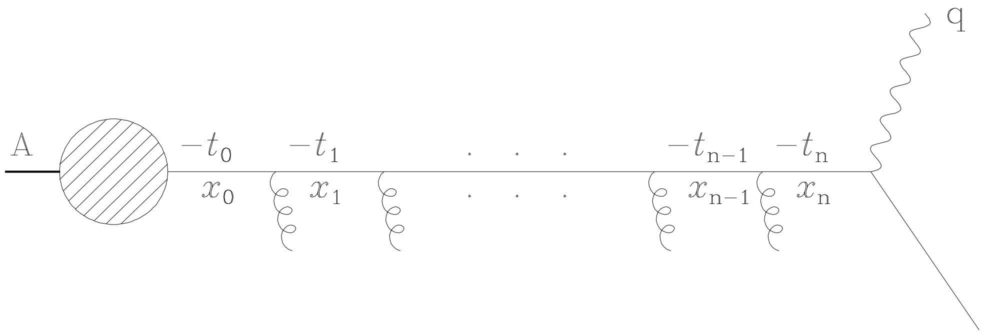
Evolution of quark distribution

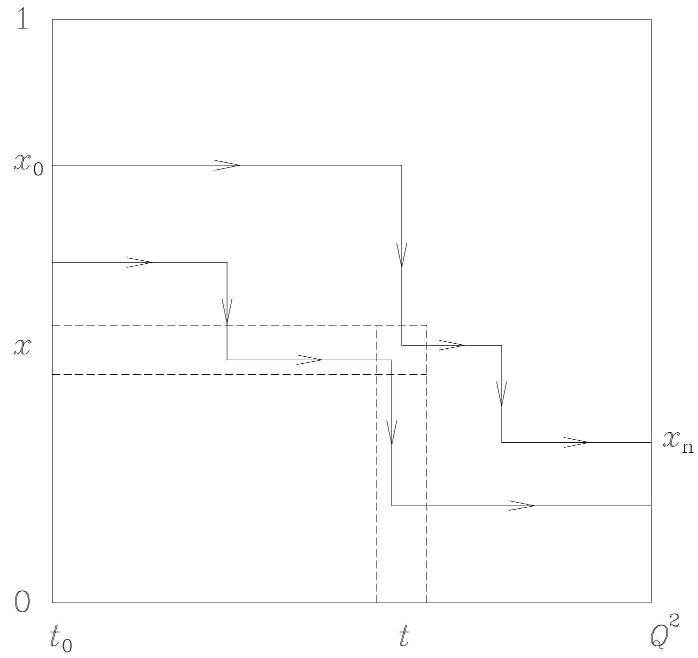
- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.

- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution (also useful for Monte Carlo simulation).



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.
- Consider change in the parton distribution $D(x, t)$ when t is increased to $t + \delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.

- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned} \delta D_{\text{in}}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\,dz\frac{\alpha_s}{2\pi}\hat{P}(z)D(x',t)\,\delta(x-zx') \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}\hat{P}(z)D(x/z,t) \end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned} \delta D_{\text{out}}(x,t) &= \frac{\delta t}{t}D(x,t)\int_0^x dx'\,dz\frac{\alpha_s}{2\pi}\hat{P}(z)\,\delta(x'-zx) \\ &= \frac{\delta t}{t}D(x,t)\int_0^1 dz\frac{\alpha_s}{2\pi}\hat{P}(z) \end{aligned}$$

- Change in population of element is

$$\begin{aligned} \delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t}\int_0^1 dz\,\frac{\alpha_s}{2\pi}\hat{P}(z)\left[\frac{1}{z}D(x/z,t) - D(x,t)\right] . \end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dz\, f(z)\, g(z)_+ \;=\; \int_0^1 dz\, [f(z) - f(1)]\, g(z) \;.$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+ \;,$$

and obtain Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}P(z)D(x/z,t) \;.$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t. In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quark and gluon distributions

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}P_{ij}(z)D_j(x/z,t) \; .$$

- Quark ($i = q$) can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2 + (1-z)^2\right] \end{aligned}$$

- Gluon can arrive either from $g \to gg$ (2 contributions) or from $q \to qg$ (or $\bar{q} \to \bar{q}g$). Thus number arriving is

$$\begin{aligned}
\delta D_{g,\,\mathsf{in}} &= \frac{\delta t}{t}\int_0^1 dz \frac{\alpha_s}{2\pi}\left\{\hat{P}_{gg}(z)\left[\frac{D_g(x/z,t)}{z} + \frac{D_g(x/(1-z),t)}{1-z}\right]\right. \\
&\quad \left. + \frac{\hat{P}_{qg}(z)}{1-z}\left[D_q\left(\frac{x}{1-z},t\right) + D_{\bar{q}}\left(\frac{x}{1-z},t\right)\right]\right\} \\
&= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}\left\{2\hat{P}_{gg}(z)D_g\left(\frac{x}{z},t\right) + \hat{P}_{qg}(1-z)\left[D_q\left(\frac{x}{z},t\right) + D_{\bar{q}}\left(\frac{x}{z},t\right)\right]\right\},
\end{aligned}$$

- Gluon can leave by splitting into either gg or $q\bar{q}$, so that

$$\delta D_{g,\,\mathsf{out}} = \frac{\delta t}{t} D_g(x,t)\int_0^1 dz \frac{\alpha_s}{2\pi}\left[\hat{P}_{gg}(z) + N_f \hat{P}_{qg}(z)\, dz\right].$$

- After some manipulation we find

$$
\begin{aligned}
P_{gg}(z) &= 2C_A\left[\left(\frac{z}{1-z}+\tfrac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z}+\tfrac{1}{2}z(1-z)\right] - \frac{2}{3}N_f T_R\,\delta(1-z)\,, \\
P_{gq}(z) &= P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z}\,.
\end{aligned}
$$

- Using definition of the plus-prescription, can show that P_{qq} and P_{gg} can be written in more common forms

$$
\begin{aligned}
P_{qq}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right] \\
P_{gg}(z) &= 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+\frac{1}{6}(11C_A-4N_f T_R)\,\delta(1-z)\,.
\end{aligned}
$$

Solution by moments

- Given $D_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.

- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{D}_i(N,t) = \int_0^1 dx\, x^{N-1}\, D_i(x,t)\ .$$

  Inverse Mellin transform is

$$D_i(x,t) = \frac{1}{2\pi i}\int_C dN\, x^{-N}\, \tilde{D}_i(N,t)\ ,$$

  where contour C is parallel to imaginary axis to right of all singularities of integrand.

- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{D}_i(x,t) = \sum_j \gamma_{ij}(N,\alpha_s)\tilde{D}_j(N,t)$$

where anomalous dimensions γ_{ij} are given by moments of splitting functions:

$$\begin{aligned}\gamma_{ij}(N,\alpha_s) &= \sum_{n=0}^{\infty} \gamma_{ij}^{(n)}(N) \left(\frac{\alpha_s}{2\pi}\right)^{n+1} \\ \gamma_{ij}^{(0)}(N) &= \tilde{P}_{ij}(N) = \int_0^1 dz\, z^{N-1}\, P_{ij}(z)\end{aligned}$$

- From above expressions for $P_{ij}(z)$ we find

$$\begin{aligned}\gamma_{qq}^{(0)}(N) &= C_F\left[-\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{k=2}^{N}\frac{1}{k}\right] \\ \gamma_{qg}^{(0)}(N) &= T_R\left[\frac{(2+N+N^2)}{N(N+1)(N+2)}\right] \\ \gamma_{gq}^{(0)}(N) &= C_F\left[\frac{(2+N+N^2)}{N(N^2-1)}\right] \\ \gamma_{gg}^{(0)}(N) &= 2C_A\left[-\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - \sum_{k=2}^{N}\frac{1}{k}\right] - \frac{2}{3}N_f T_R\,.\end{aligned}$$

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_s is

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_s)}$$

- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).

- For running coupling $\alpha_s(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_s(t) = 1/b\ln(t/\Lambda^2)$, we find

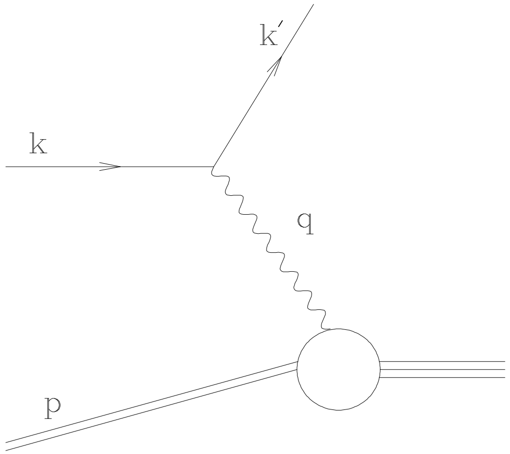
$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_s(t_0)}{\alpha_s(t)}\right)^{d_{qq}(N)}$$

  where $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$.

- Flavour-singlet distribution and quantitative predictions will be discussed later.

Deep inelastic scattering

- Consider lepton-proton scattering via exchange of virtual photon:



- Standard variables are:

$$\begin{aligned} x &= \frac{-q^2}{2p \cdot q} = \frac{Q^2}{2M(E-E')} \\ y &= \frac{q \cdot p}{k \cdot p} = 1 - \frac{E'}{E} \end{aligned}$$

where $Q^2 = -q^2 > 0$, $M^2 = p^2$ and energies refer to target rest frame.

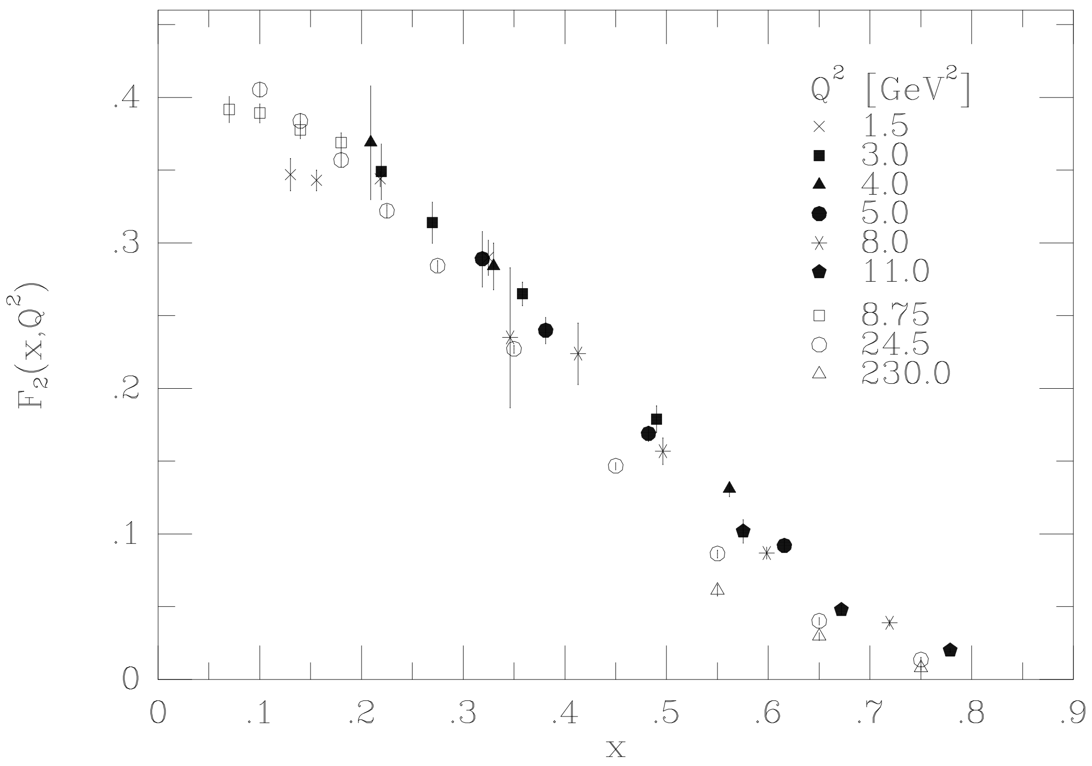
- Elastic scattering has $(p+q)^2 = M^2$, i.e. $x = 1$. Hence deep inelastic scattering (DIS) means $Q^2 \gg M^2$ and $x < 1$.

- Structure functions $F_i(x, Q^2)$ parametrise target structure as 'seen' by virtual photon. Defined in terms of cross section

$$
\begin{aligned}
\frac{d^2\sigma}{dxdy} \quad = \quad & \frac{8\pi\alpha^2 ME}{Q^4}\left[\left(\frac{1+(1-y)^2}{2}\right)2xF_1\right. \\
& \left. +(1-y)(F_2 - 2xF_1) - (M/2E)xyF_2\right] .
\end{aligned}
$$

- Bjorken limit is Q^2, $p \cdot q \to \infty$ with x fixed. In this limit structure functions obey approximate Bjorken scaling law, i.e. depend only on dimensionless variable x:

$$
F_i(x, Q^2) \longrightarrow F_i(x).
$$



- Figure shows F_2 structure function for proton target. Although Q^2 varies by two orders of magnitude, in first approximation data lie on universal curve.

- Bjorken scaling implies that virtual photon is scattered by *pointlike constituents* (partons) — otherwise structure functions would depend on ratio Q/Q_0, with $1/Q_0$ a length scale characterizing size of constituents.

- Parton model of DIS is formulated in a frame where target proton is moving very fast — *infinite momentum frame*.

  - ⋆ Suppose that, in this frame, photon scatters from pointlike quark with fraction ξ of proton's momentum. Since $(\xi p+q)^2 = m_q^2 \ll Q^2$, we must have $\xi = Q^2/2p{\cdot}q = x$.
  - ⋆ In terms of Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$, spin-averaged matrix element squared for massless $eq \to eq$ scattering (related by crossing to $e^+e^- \to q\bar{q}$) is

$$\overline{\sum}|\mathcal{M}|^2 = 2e_q^2 e^4 \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

    where $\overline{\sum}$ denotes average (sum) over initial (final) colours and spins.
  - ⋆ In terms of DIS variables, $\hat{t} = -Q^2$, $\hat{u} = \hat{s}(y-1)$ and $\hat{s} = Q^2/xy$. Differential cross section is then

$$\frac{d^2\hat{\sigma}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi).$$

⋆ From structure function definition (neglecting M)

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left\{[1+(1-y)^2]F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right\}.$$

⋆ Hence structure functions for scattering from parton with momentum fraction ξ is

$$\hat{F}_2 = xe_q^2\delta(x-\xi) = 2x\hat{F}_1\ .$$

⋆ Suppose probability that quark q carries momentum fraction between ξ and $\xi + d\xi$ is $q(\xi)\,d\xi$. Then

$$\begin{aligned} F_2(x) &= \sum_q \int_0^1 d\xi\ q(\xi)\ xe_q^2\delta(x-\xi) \\ &= \sum_q e_q^2 xq(x) = 2xF_1(x)\ . \end{aligned}$$

⋆ Relationship $F_2 = 2xF_1$ (Callan-Gross relation) follows from spin-$\frac{1}{2}$ property of quarks ($F_1 = 0$ for spin-0).

- Proton consists of three valence quarks (uud), which carry its electric charge and baryon number, and infinite sea of light $q\bar{q}$ pairs.

- Probed at scale Q, sea contains all quark flavours with $m_q \ll Q$. Thus at $Q \sim 1$ GeV expect

$$F_2^{em}(x) \simeq \frac{4}{9}x[u(x) + \bar{u}(x)] + \frac{1}{9}x[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)]$$

where

$$\begin{aligned} u(x) &= u_V(x) + \bar{u}(x) \\ d(x) &= d_V(x) + \bar{d}(x) \\ s(x) &= \bar{s}(x) \end{aligned}$$

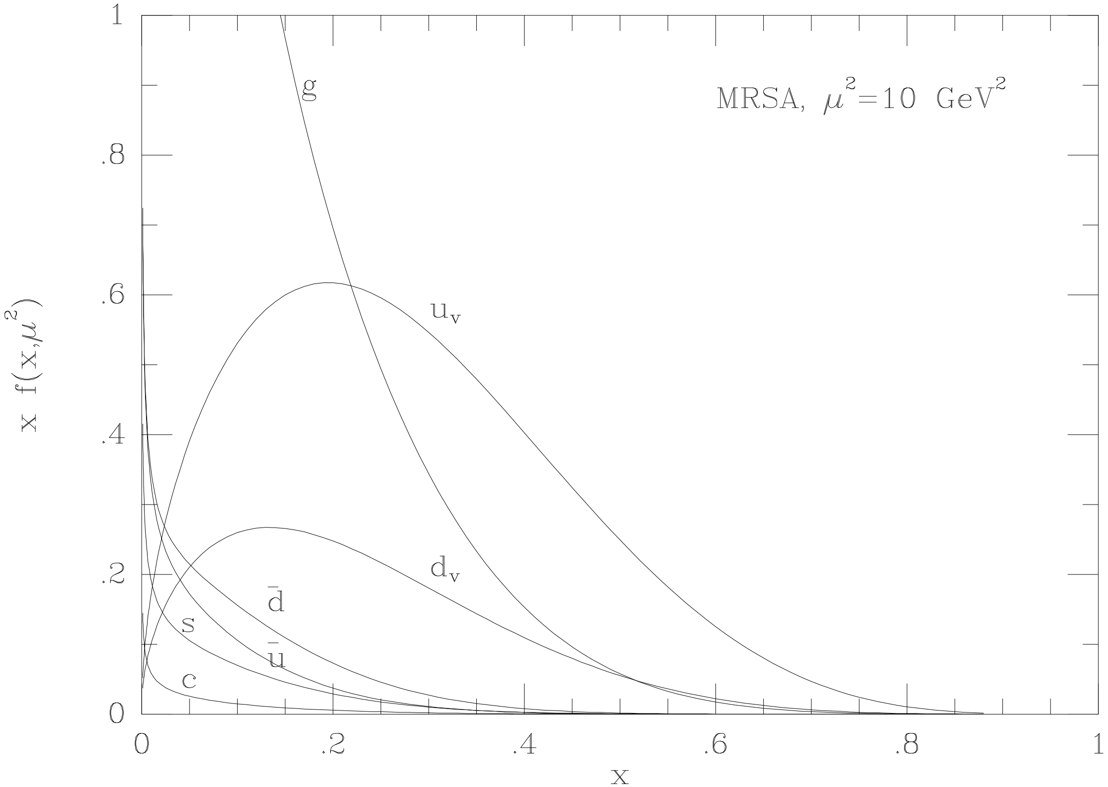
with sum rules

$$\int_0^1 dx\; u_V(x) = 2\;, \quad \int_0^1 dx\; d_V(x) = 1\;.$$

- Experimentally one finds

$$\sum_q \int_0^1 dx\, x[q(x) + \bar{q}(x)] \simeq 0.5 \; .$$

Thus quarks only carry about 50% of proton's momentum. Rest is carried by *gluons*. Although not directly measured in DIS, gluons participate in other hard scattering processes such as large-p_T jet and prompt photon production.



- Figure shows typical set of parton distributions extracted from fits to DIS data, at $Q^2 = 10$ GeV^2.

Scaling violation

- Bjorken scaling is not exact. Structure functions decrease at large x and grow at small x with increasing Q^2. This is due to Q^2 dependence of parton distributions, considered earlier. In present notation, they satisfy DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}q(x,t) = \frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{dz}{z}P(z)q\left(\frac{x}{z},t\right) \equiv \frac{\alpha_s(t)}{2\pi}P \otimes q$$

  where P is $q \to qg$ splitting function.

- Taking into account other types of parton branching that can occur in addition to $q \to qg$, we obtain coupled evolution equations

$$\begin{aligned}
t\frac{\partial q_i}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\left[P_{qq} \otimes q_i + P_{qg} \otimes g\right] \\
t\frac{\partial \bar{q}_i}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\left[P_{qq} \otimes \bar{q}_i + P_{qg} \otimes g\right] \\
t\frac{\partial g}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\left[P_{gq} \otimes \sum (q_i + \bar{q}_i) + P_{gg} \otimes g\right] .
\end{aligned}$$

- Lowest-order splitting functions derived earlier. More generally they are power series in α_s, same for deep inelastic scattering (spacelike branching) and jet fragmentation (timelike branching) in leading order, but differing in higher orders. Consequently, behaviour of structure functions at small x is different from that of jet fragmentation functions.

- For the present, concentrate on larger x values ($x \gtrsim 0.01$), where PT expansion converges better.

- Recall solution of evolution equations for flavour non-singlet combinations V, e.g. $q_i - \bar{q}_i$ or $q_i - q_j$. Mixing with gluons drops out and

$$t \frac{\partial}{\partial t} V(x,t) = \frac{\alpha_s(t)}{2\pi} P_{qq} \otimes V \ .$$

Taking moments (Mellin transform)

$$\tilde{V}(N,t) = \int_0^1 dx\, x^{N-1}\, V(x,t)$$

we find

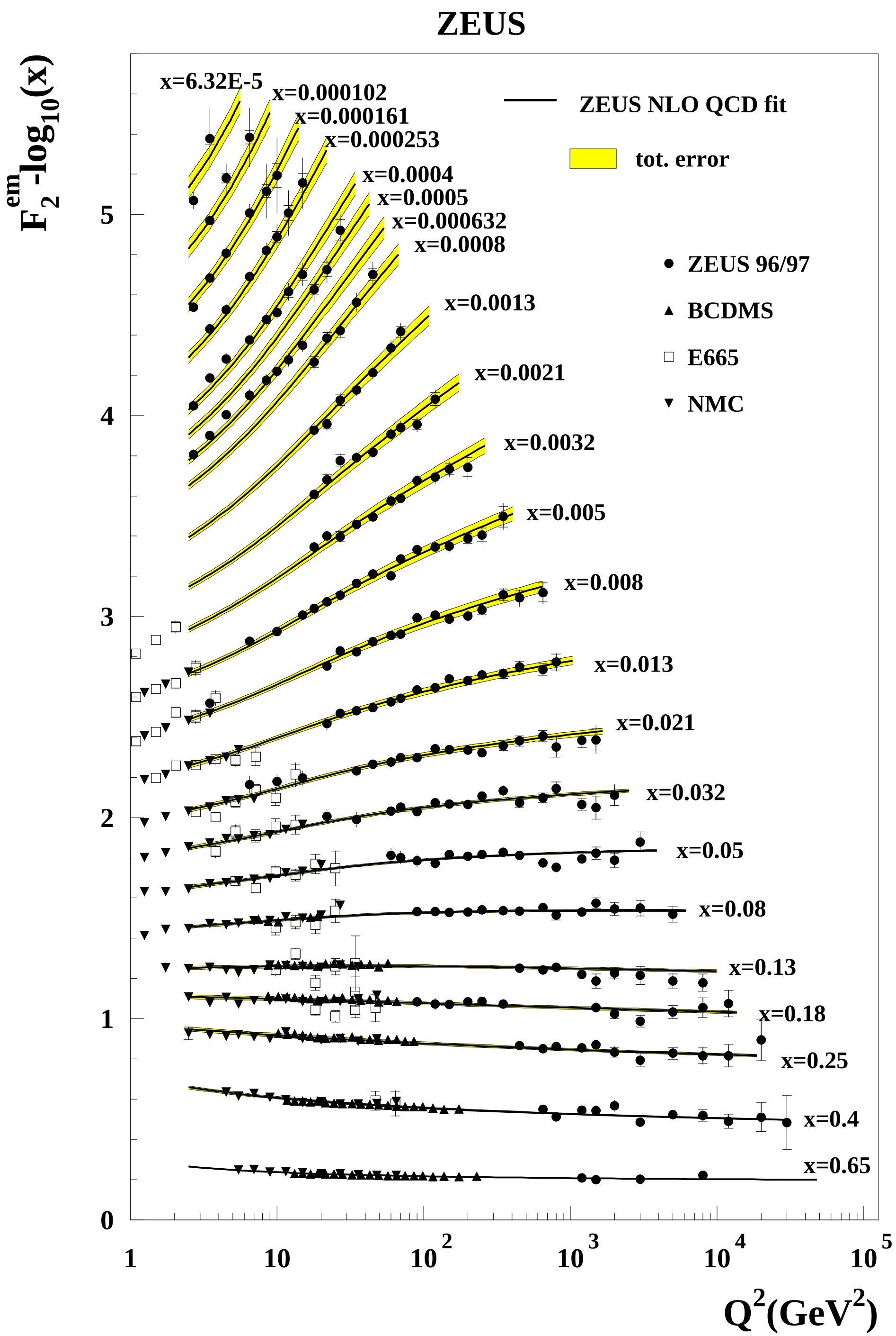
$$t\frac{\partial}{\partial t}\tilde{V}(N,t) = \frac{\alpha_s(t)}{2\pi}\gamma_{qq}^{(0)}(N)\,\tilde{V}(N,t)$$

where $\gamma_{qq}^{(0)}(N)$ is Mellin transform of $P_{qq}^{(0)}$. Solution is

$$\tilde{V}(N,t) = \tilde{V}(N,0)\left(\frac{\alpha_s(0)}{\alpha_s(t)}\right)^{d_{qq}(N)}$$

where $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$.

- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases V *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data.



ZEUS
ZEUS NLO QCD fit
tot. error
ZEUS 96/97
BCDMS
E665
NMC
x=6.32E-5
x=0.000102
x=0.000161
x=0.000253
x=0.0004
x=0.0005
x=0.000632
x=0.0008
x=0.0013
x=0.0021
x=0.0032
x=0.005
x=0.008
x=0.013
x=0.021
x=0.032
x=0.05
x=0.08
x=0.13
x=0.18
x=0.25
x=0.4
x=0.65
F_2^{em}-log$_{10}$(x)
Q^2(GeV^2)

- For flavour-singlet combination, define

$$\Sigma = \sum_i (q_i + \bar{q}_i) \ .$$

Then we obtain

$$\begin{aligned} t\frac{\partial\Sigma}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\left[P_{qq}\otimes\Sigma + 2N_f P_{qg}\otimes g\right] \\ t\frac{\partial g}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\left[P_{gq}\otimes\Sigma + P_{gg}\otimes g\right] \ . \end{aligned}$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution g: evolution equation for moments has matrix form

$$t\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & 2N_f\gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{g} \end{pmatrix}$$

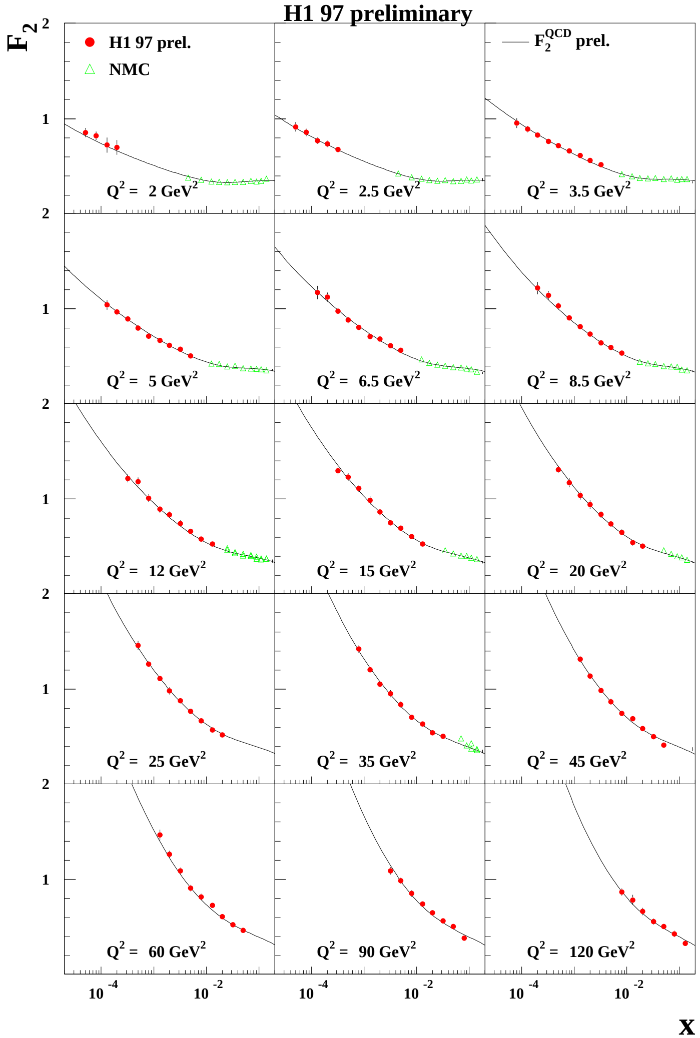
- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$\gamma_\pm = \frac{1}{2}[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg} - \gamma_{qq})^2 + 8N_f\gamma_{gq}\gamma_{qg}}] \; .$$

- Expressing $\tilde{\Sigma}$ and $\tilde{g}$ as linear combinations of eigenvectors $\tilde{\Sigma}_+$ and $\tilde{\Sigma}_-$, we find they evolve as superpositions of terms of above form with $\gamma_\pm$ in place of γ_{qq}.

Small x

- At small x, corresponding to $N \to 1$, $\gamma_+ \to \gamma_{gg} \to \infty$, $\gamma_- \to \gamma_{qq} \to 0$. Therefore structure functions grow rapidly at small x.



- Higher-order corrections also become large at small x:

$$
\begin{aligned}
\gamma_{qq}^{(1)}(N) &\to \frac{40C_F N_f T_R}{9(N-1)} \\
\gamma_{qg}^{(1)}(N) &\to \frac{40C_A T_R}{9(N-1)} \\
\gamma_{gq}^{(1)}(N) &\to \frac{9C_F C_A - 40C_F N_f T_R}{9(N-1)} \\
\gamma_{gg}^{(1)}(N) &\to \frac{(12C_F - 46C_A)N_f T_R}{9(N-1)} \,.
\end{aligned}
$$

- Thus we find

$$
\begin{aligned}
\gamma_+ &\to \frac{2C_A}{N-1}\frac{\alpha_s}{2\pi}\left[1 + \frac{(26C_F - 23C_A)N_f}{18C_A}\frac{\alpha_s}{2\pi} + \dots\right] \\
&= \frac{2C_A}{N-1}\frac{\alpha_s}{2\pi}\left[1 - 0.64N_f\frac{\alpha_s}{2\pi} + \dots\right]
\end{aligned}
$$

where neglected terms are either non-singular at $N=1$ or higher-order in α_s. Thus NLO correction is relatively small.

- In general one finds (Balitsky, Fadin, Kuraev, Lipatov, BFKL) that for small x ($N \to 1$)

$$\gamma_+ \to \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{\gamma^{(n,m)}}{(N-1)^m} \left(\frac{\alpha_s}{2\pi}\right)^n$$

- In x space LO BFKL equation (or BFKL Pomeron) resums terms of the form

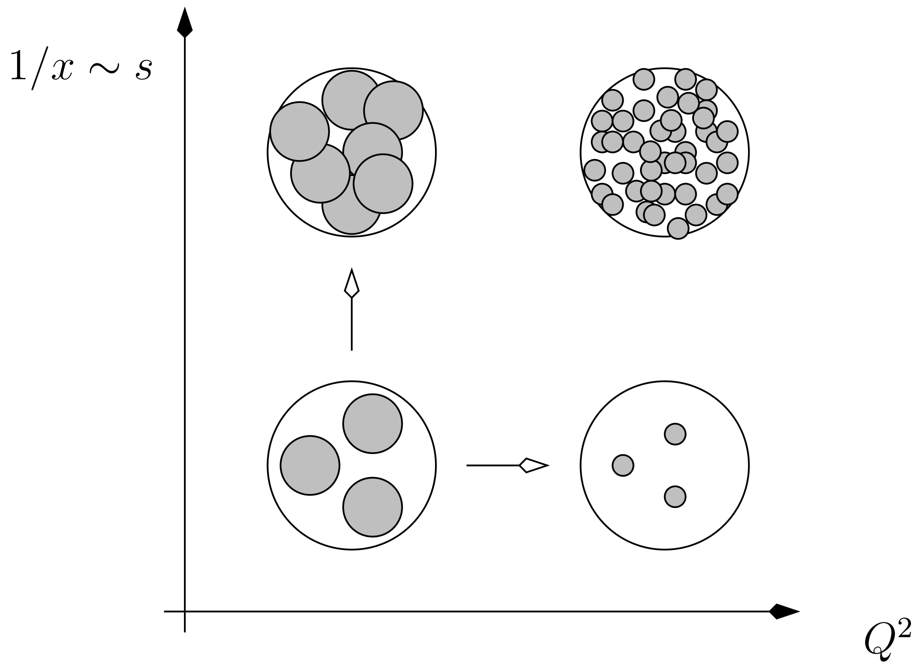
$$\left(\alpha_s \log \frac{1}{x}\right)^n$$

- It happens that $\gamma^{(2,2)}$ (and $\gamma^{(3,3)}$) are zero.
  - ⋆ This is probably why significant deviations from NLO QCD have not yet been seen in DIS at small x, whereas they are obvious in jet fragmentation.
  - ⋆ Anomalous dimension at small x is much less singular than the timelike (jet fragmentation) case, where $m \le 2n-1$ and $\gamma^{(2,3)}$ and $\gamma^{(3,5)}$ are not zero.
    Crucial difference is coherence (angular ordering), which suppresses soft gluon emission in low-x fragmentation, but does not suppress low-x spacelike branching in DIS.

High Energy Scattering

- Interesting regime of QCD is high energy limit, $s \to \infty$(or $x \to 0$), with t fixed.

- At high energies there is large phase space for emission of soft gluons. Therefore colliding hadrons evolve into a dense system of partons, now often called the color glass condensate.



- When partons start to overlap one expects recombination effects to become important. This corresponds to nonlinearities in evolution equations.

- As a consequence the growth of structure functions should be tamed at small x. Can we see the corresponding saturation effects already at present collider energies, for example at HERA?!