Quantum Statistical Physics Out of Equilibrium

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Outline

Problem

- Quantum Quenches
- Thermalisation vs. Equilibration
- Ballistic vs. Diffusive Transport

Motivation

- Theoretical
- Experimental

Models

- Relativistic quantum field theories
- Spin chains / Lattice models
- Quantum liquids

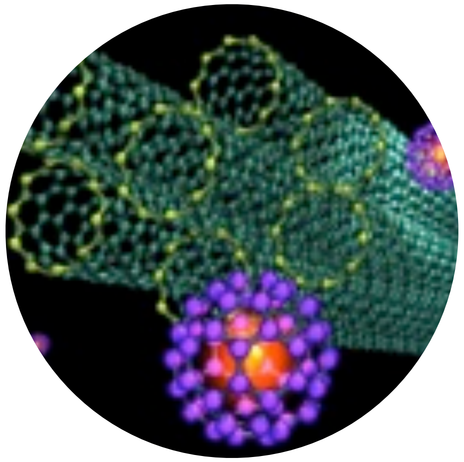
Methods

Dualities

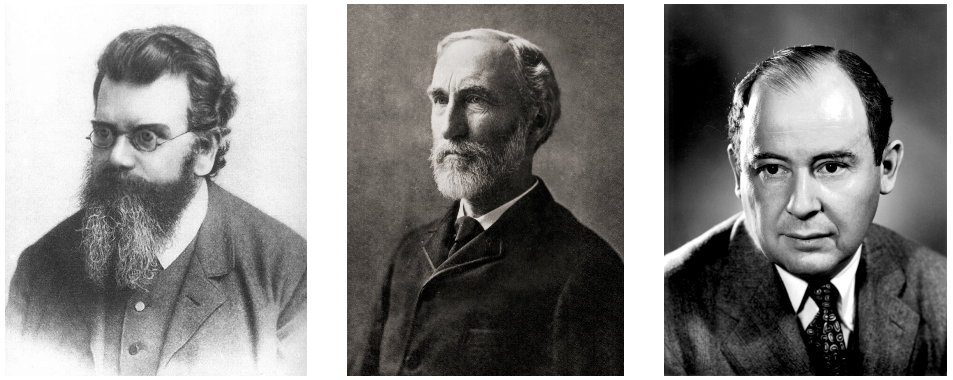
Out-of-equilibrium quantum statistical physics

- Many scientific discoveries of those that have changed human civilisation over the last century are built upon our progress in understanding the statistical physics of matter and quantum systems.
- Yet our understanding has until recently remained limited to systems at thermal equilibrium or close to it, which is rarely the case in physical reality.
- However during the last decade or so the physics community has witnessed an explosion of research activity in out-of-equilibrium quantum physics, opening up countless possibilities for novel applications.

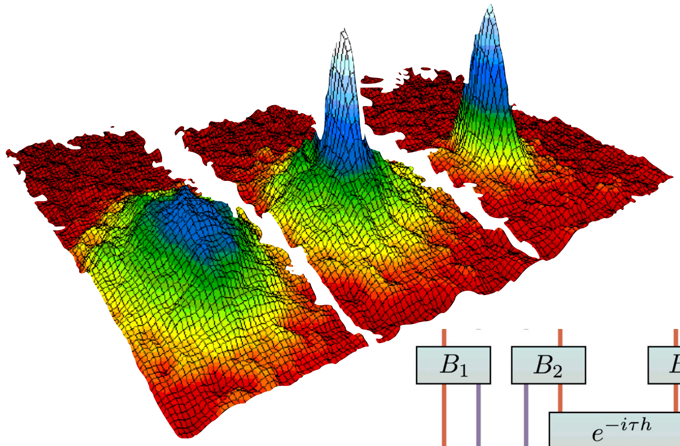


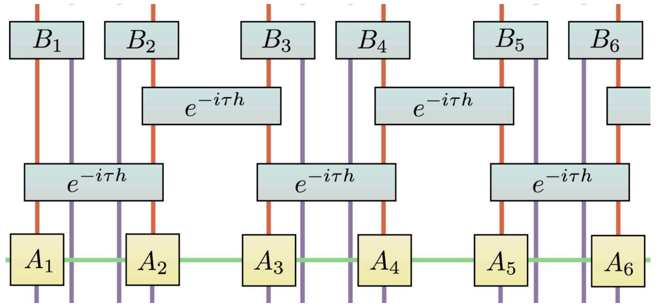



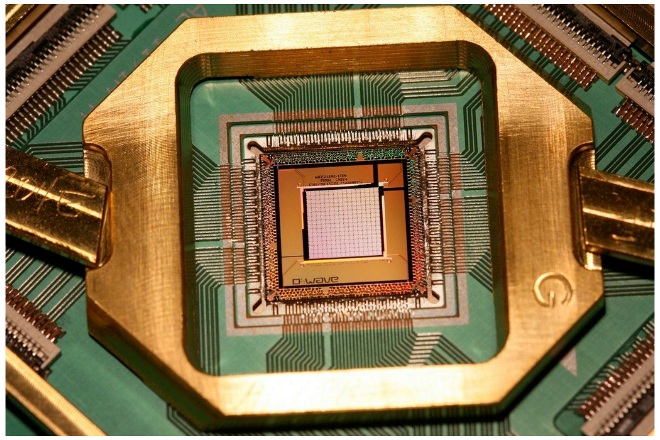
Motivation



- Equilibration in quantum systems is a *fundamental* and *long-standing* question of statistical mechanics
- Reach the *ultimate limits of classical thermodynamics* expectations and unveil *novel quantum effects* at macroscopic level
- Recent progress in experimental (*ultra-cold atoms*) and numerical (*tDMRG, MPS*) techniques for study of quantum many-body dynamics
- Applications to quantum technologies: quantum thermal engines, quantum information processing & computing







Equilibration

Quantum Quenches

- Well-posed theoretical and experimental problem:
  Consider an isolated and thermodynamically large quantum system, prepared in an initial state that is the ground state of some arbitrary Hamiltonian, then let to evolve under a different Hamiltonian

$$H_0|\Psi_0\rangle = 0 \qquad\qquad |\Psi_0\rangle \xrightarrow{e^{-iHt}} ?$$

Calabrese Cardy, PRL (2006)

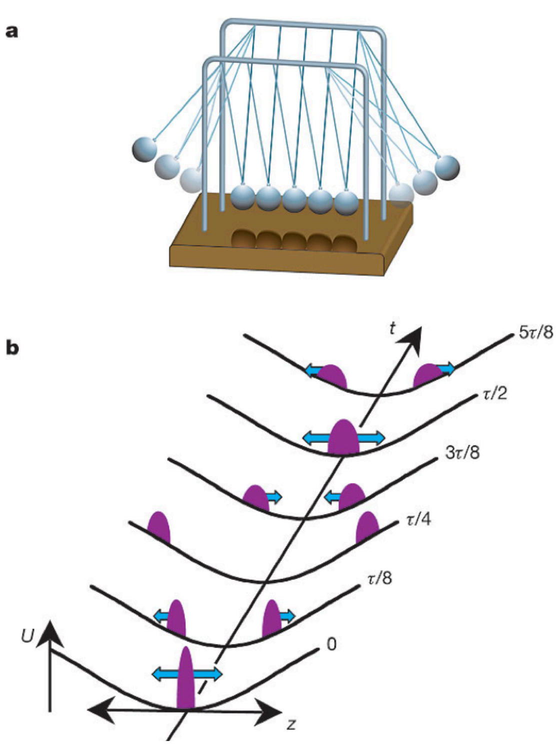
- Questions:
  - Long time behaviour?
  - Does the system tend to equilibrium?
  - If yes, is equilibrium thermal?
  - If not, what type of equilibrium is it?
  - How much and what type of *information* about initial state survives at long times?

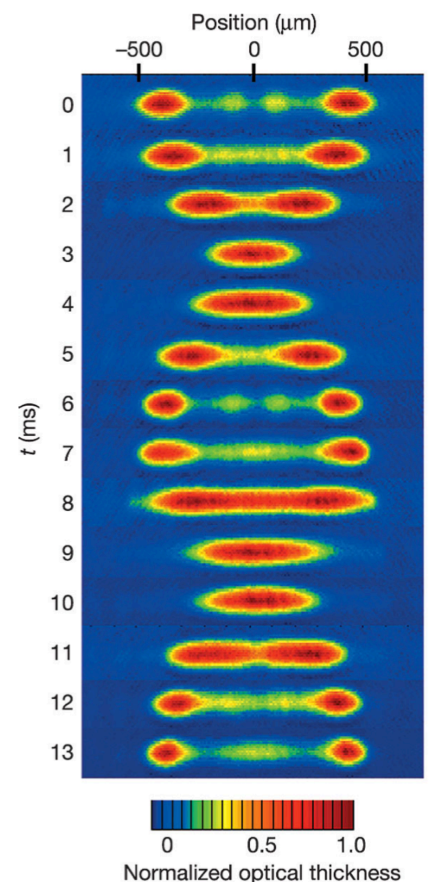
Quantum Newton's Craddle

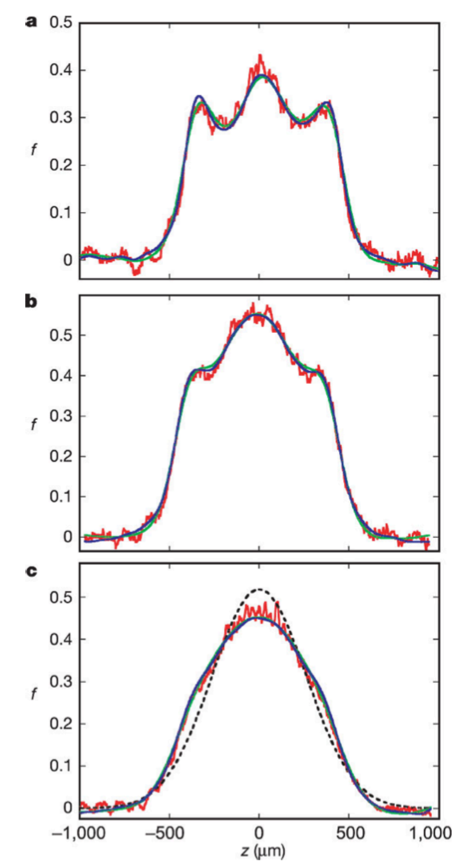
- Experiment:
  A system of 1d non-relativistic bosons with point-like interactions in a harmonic trap prepared in out of equilibrium initial state:
  - does not relax even after many collisions,
  - exhibits non-thermal momentum distribution.

- Lack of thermalisation due to integrability (Lieb-Liniger model)?

Kinoshita et al., Nature (2006)







Quantum Newton’s Craddle

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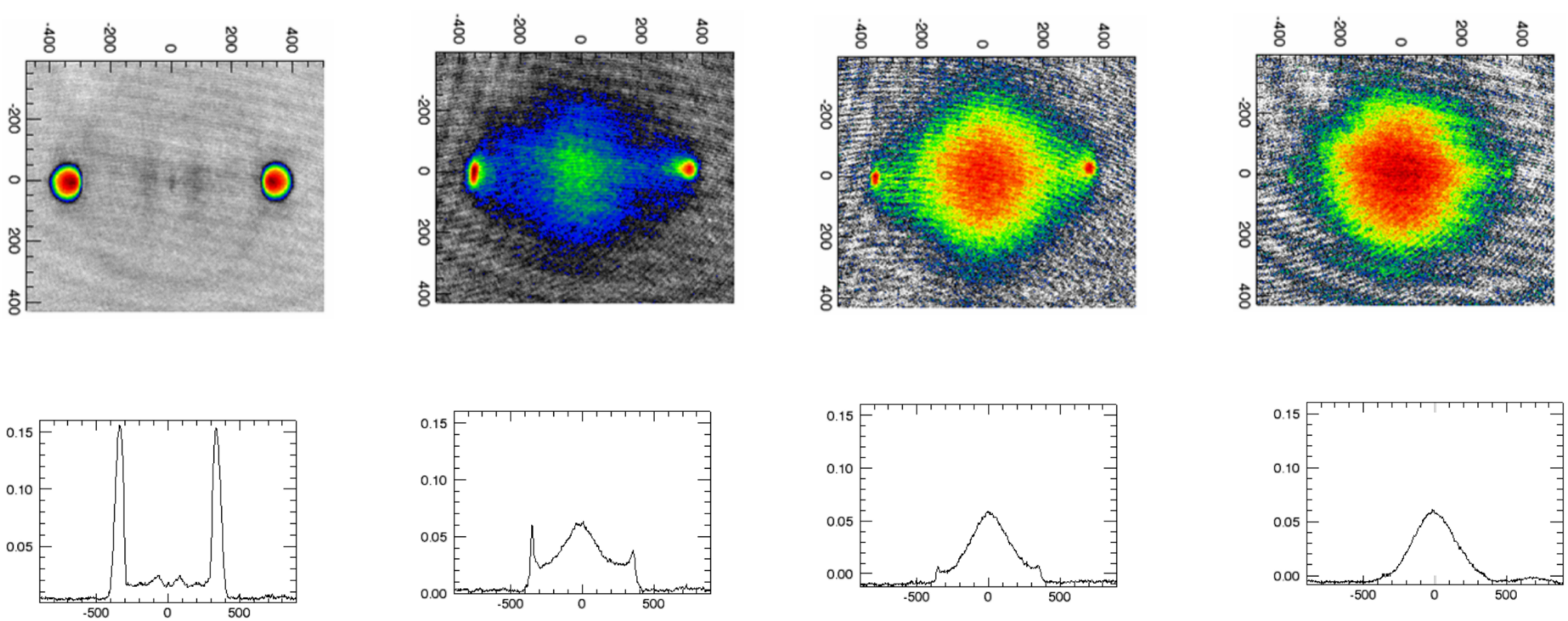
- Lack of thermalisation due to integrability (Lieb-Liniger model)?
  Or dimensionality?

Kinoshita et al., Nature (2006)



GGE

- Integrable models:
  Possess infinite number of local conserved quantities

- Conjecture:

  "Integrable models equilibrate to a Generalised Gibbs Ensemble that is a maximum entropy ensemble determined by all constraints coming from the infinite conserved quantities."

Rigol, Dunjko, Yurovsky, Olshanii, PRL (2007)

Gibbs ensemble:

$$\rho_{GE} \propto \exp(-\beta H)$$

temperature fixed by the constraint of energy conservation

$$\mathrm{Tr}(\rho_{GE} H) = \langle \Psi_0 | H | \Psi_0 \rangle$$

⇨

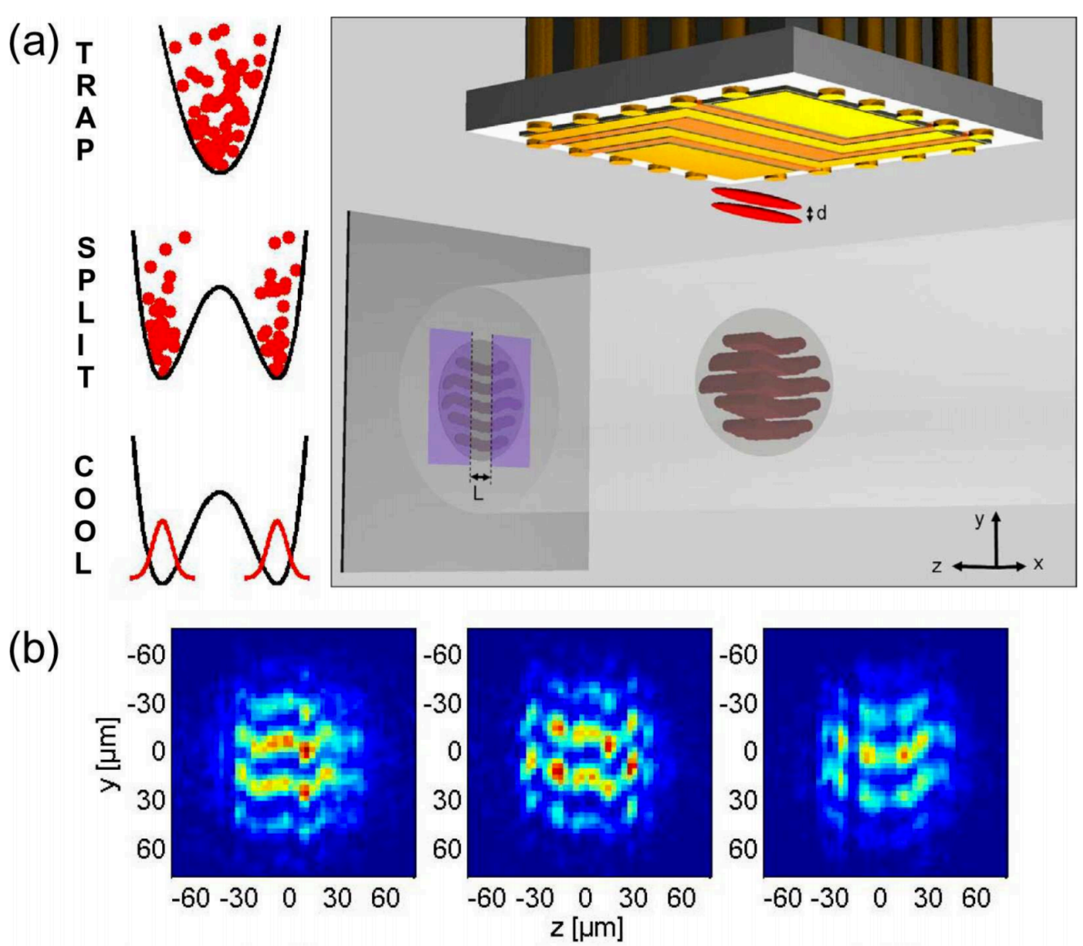
Generalised Gibbs ensemble:

$$\rho_{GGE} \propto \exp\left(-\sum_{n=1}^{\infty} \lambda_n \mathcal{I}_n \right)$$

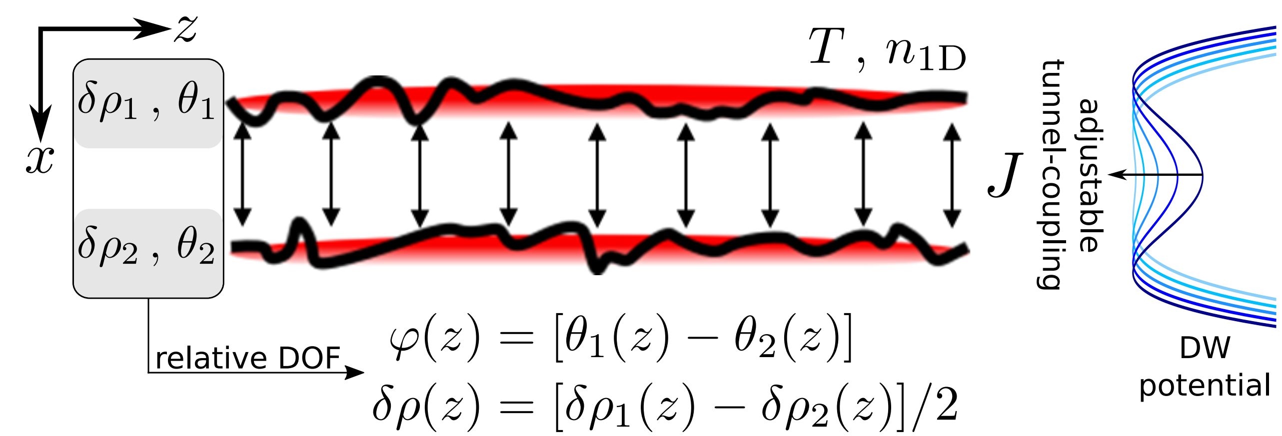
Lagrange multipliers fixed by the constraints of all extra conserved quantities

$$\mathrm{Tr}(\rho_{GGE} \mathcal{I}_n) = \langle \Psi_0 | \mathcal{I}_n | \Psi_0 \rangle$$

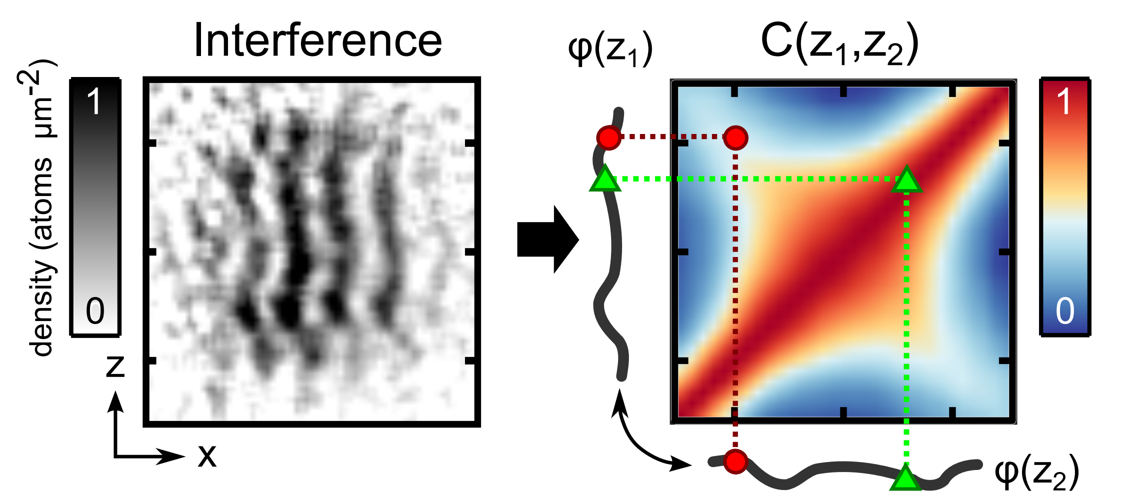
Experimental Observation of GGE



- splitting 1d ultracold atom quasi-condenstate in two coupled subsystems → low-energy physics described by sine-Gordon model



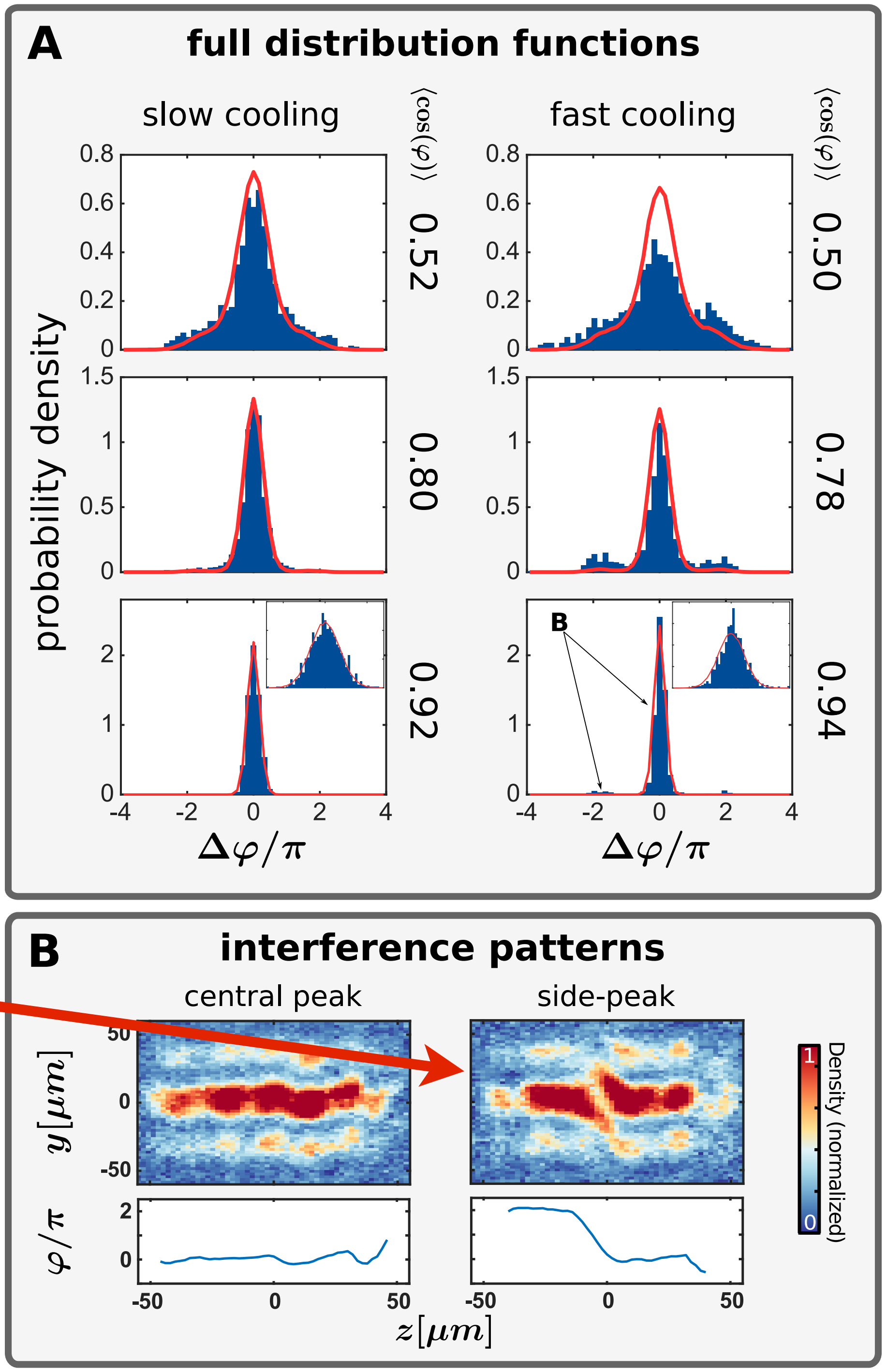
- interference patterns + averaging over many repetitions → direct measurement of multi-point correlation functions of phase field



Schweigler et al., Nature (2017)

Experimental Observation of GGE

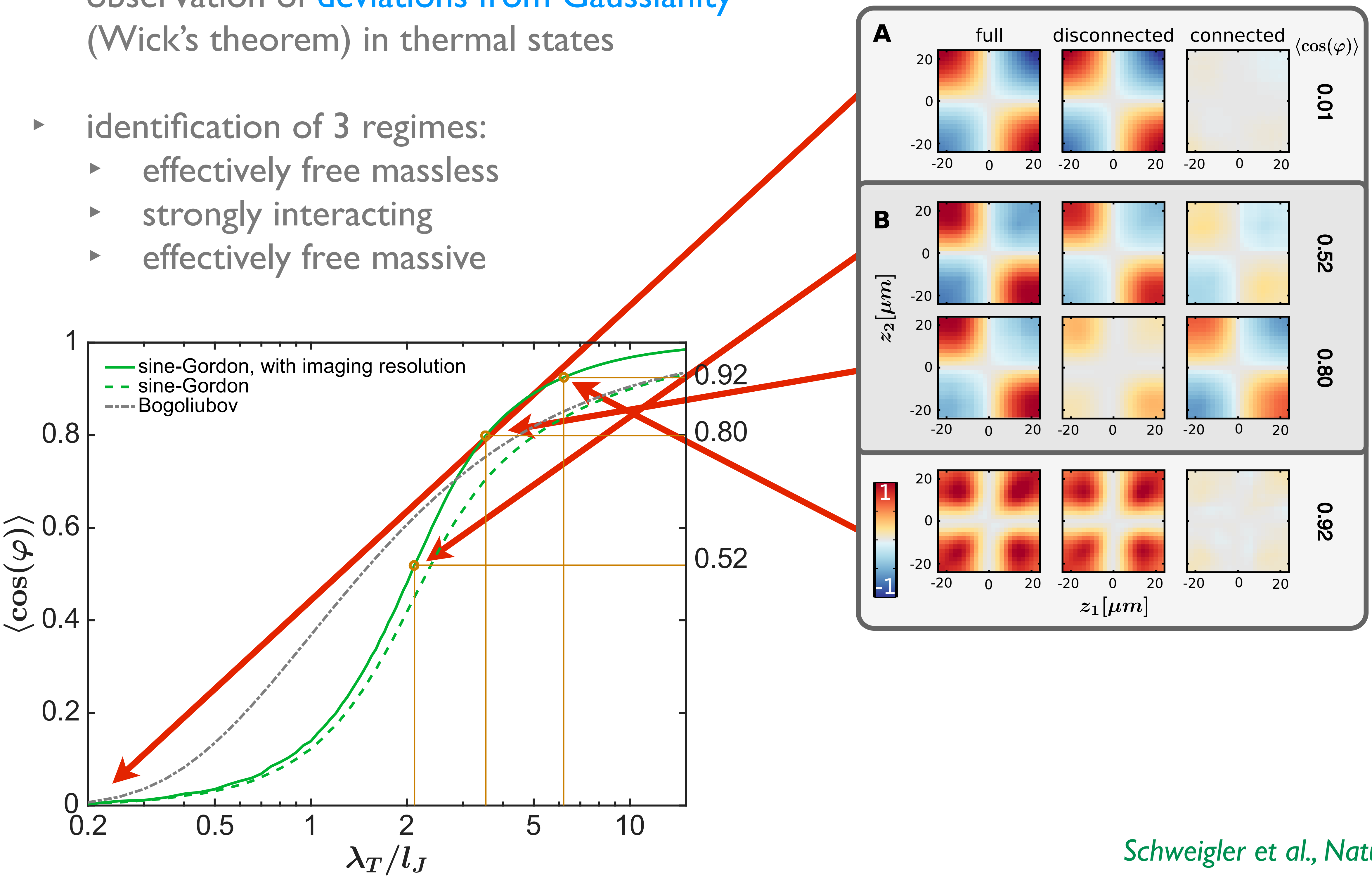
- observation of soliton configurations (2π phase difference between left / right boundaries)



Schweigler et al., Nature (2017)

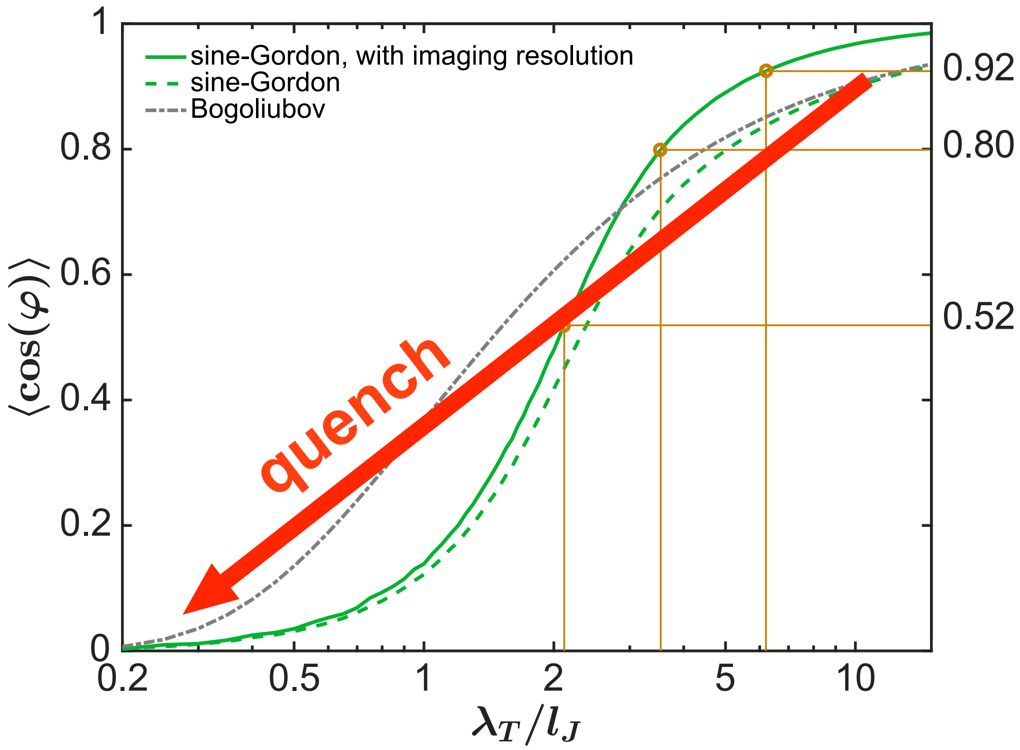
Experimental Observation of GGE

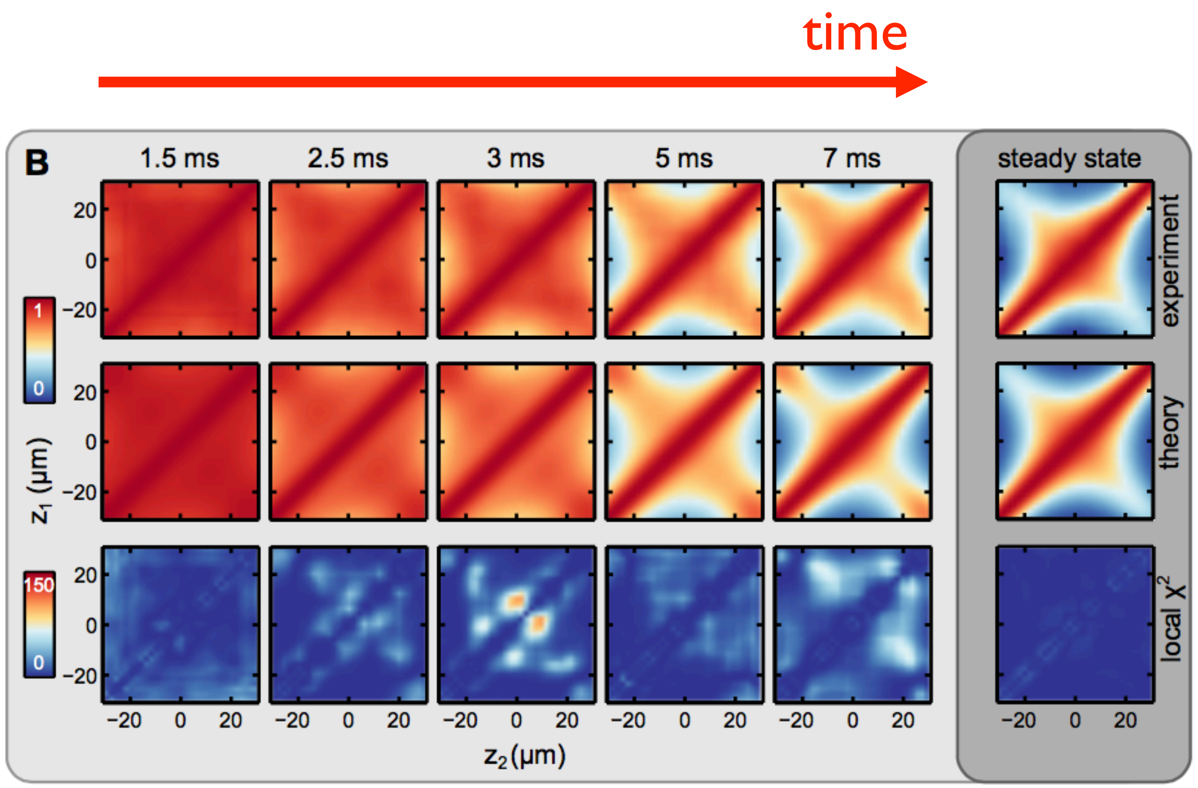
- observation of deviations from Gaussianity (Wick's theorem) in thermal states
- identification of 3 regimes:
  - effectively free massless
  - strongly interacting
  - effectively free massive



Schweigler et al., Nature (2017)

Experimental Observation of GGE





Langen et al., Science (2015)

- Quench from gapped to gapless non-interacting phase
- Observation of dynamics of correlations
- *Non-thermal* steady state: more than one temperature needed to describe steady state
- Agreement between experimental data and theoretical predictions based on a Generalised Gibbs Ensemble

Rigol, Dunjko, Yurovsky, Olshanii, PRL (2007)

Quantum Transport

Inhomogeneous Quenches

- Problem:
  Consider an extended quantum system, prepared in a spatially inhomogeneous (step-like) initial state and let to evolve unitarily under a homogeneous Hamiltonian (e.g. a system initially split in two halves at different temperature, then abruptly joined together).

- Objective:
  Derive asymptotic values of local observables at large times and distances from the origin (e.g. energy/density current).

- Questions:
  - Is transport ballistic? diffusive? other?
  - Relation to conductivity problem

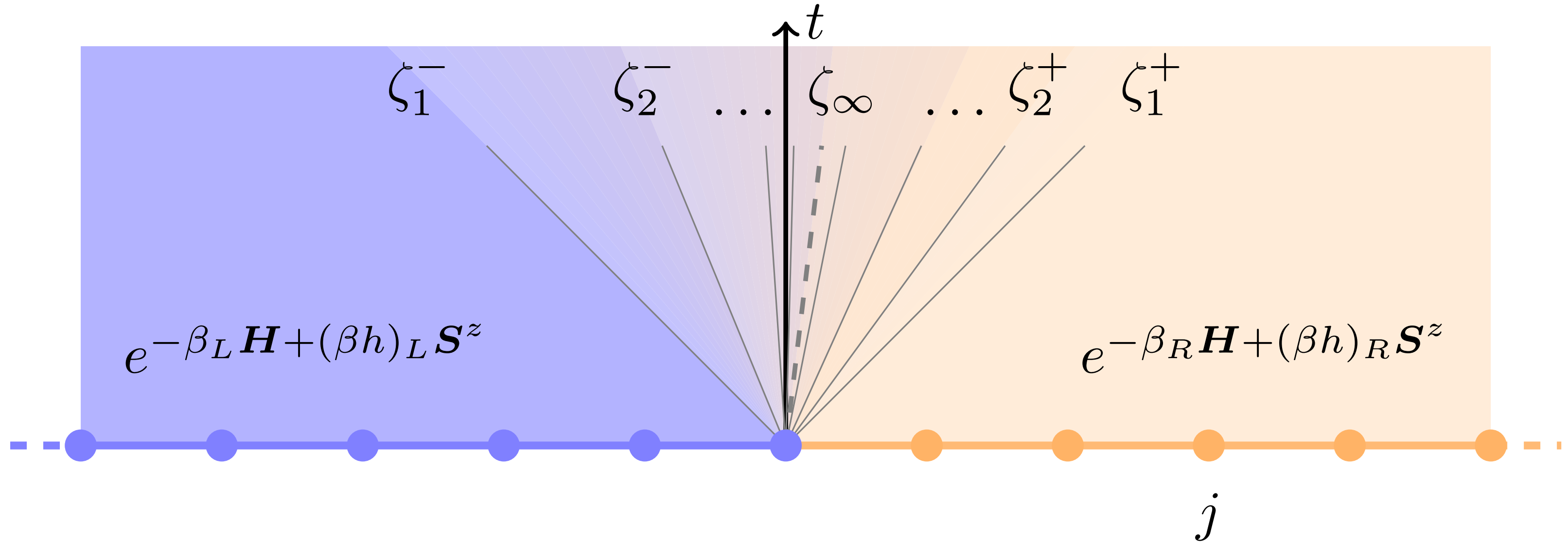
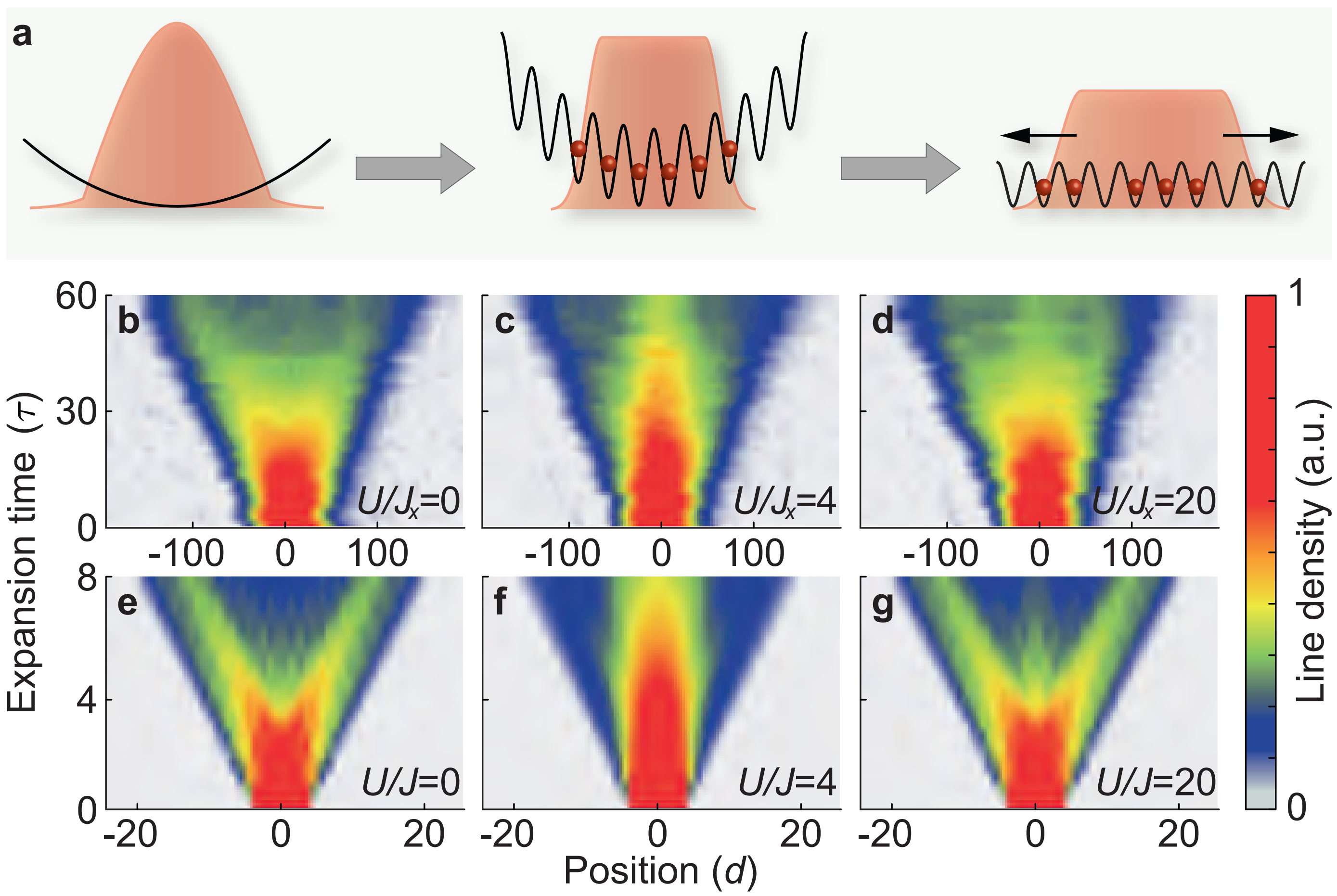


figure from: Piroli De Nardis Collura Bertini Fagotti (2017)

Transport Experiment

- Expansion of initially localised ultracold bosons in homogeneous 1D optical lattices

  (1) Integrable: Ballistic Expansion
  (2) Non-integrable: Diffusive Expansion



Ronzheimer et al. PRL (2017)

Non-Equilibrium Steady State

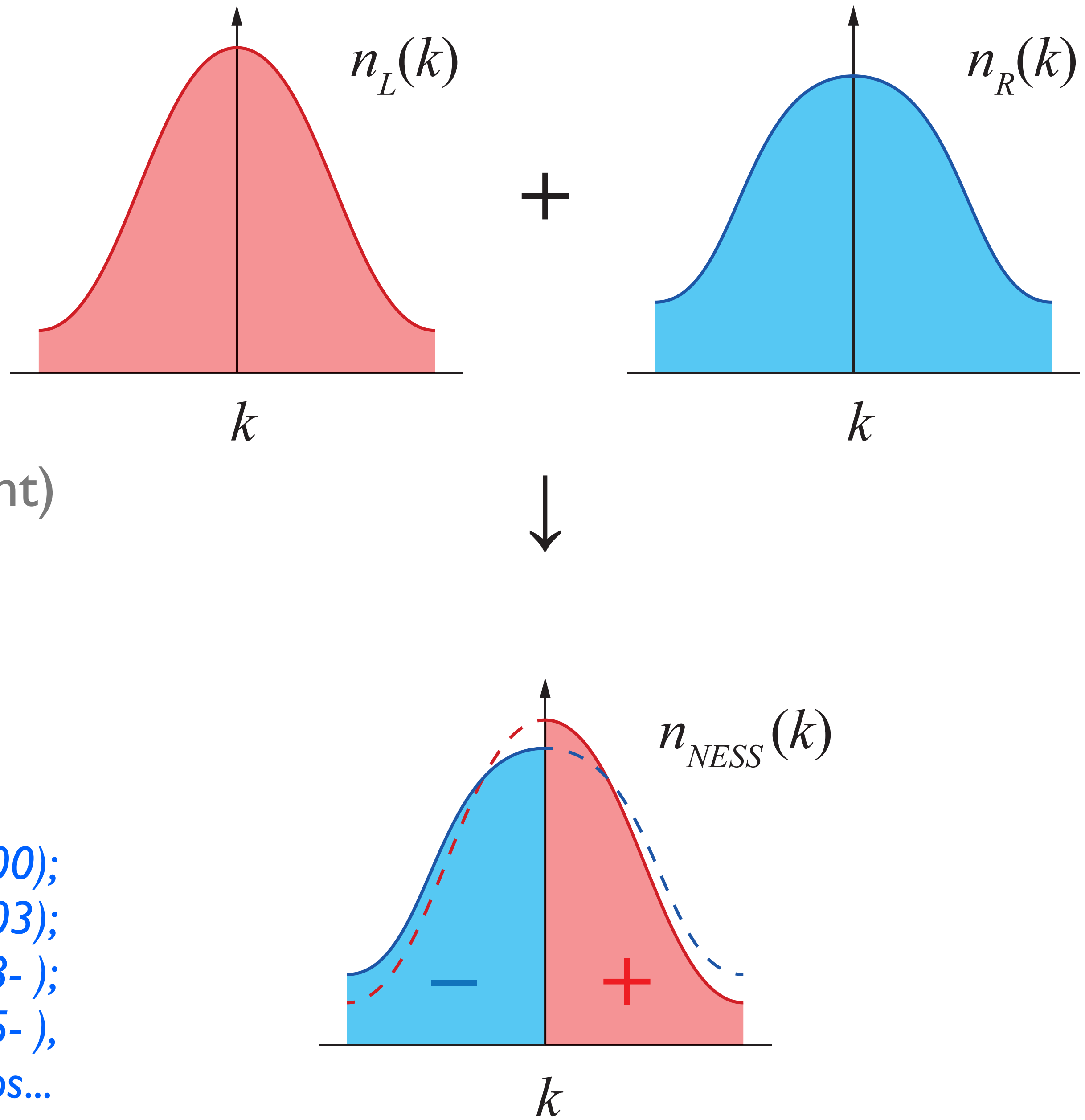
- Long time asymptotics of local observables: Non-Equilibrium Steady State (NESS)

$$\lim_{t\to\infty}\lim_{\mathrm{TDL}}\langle\hat{O}(\{r_i\};t)\rangle = \mathrm{Tr}\left\{\rho_{NESS}\,\hat{O}(\{r_i\})\right\} = \sum_{\Psi}\rho_{NESS}(\Psi)\big\langle\Psi\big|\hat{O}(\{r_i\})\big|\Psi\big\rangle$$

where

$$\rho_0 \propto \mathrm{e}^{-\beta_L H_{0L}} \otimes \mathrm{e}^{-\beta_R H_{0R}}$$

$$\rho_{NESS} \propto \mathrm{e}^{-\beta_L H_{\rightarrow}} \otimes \mathrm{e}^{-\beta_R H_{\leftarrow}}$$



- Initial state:
  product state in coordinate space (left/right)

- Final state:
  product state in momentum space
  (left/right moving modes)

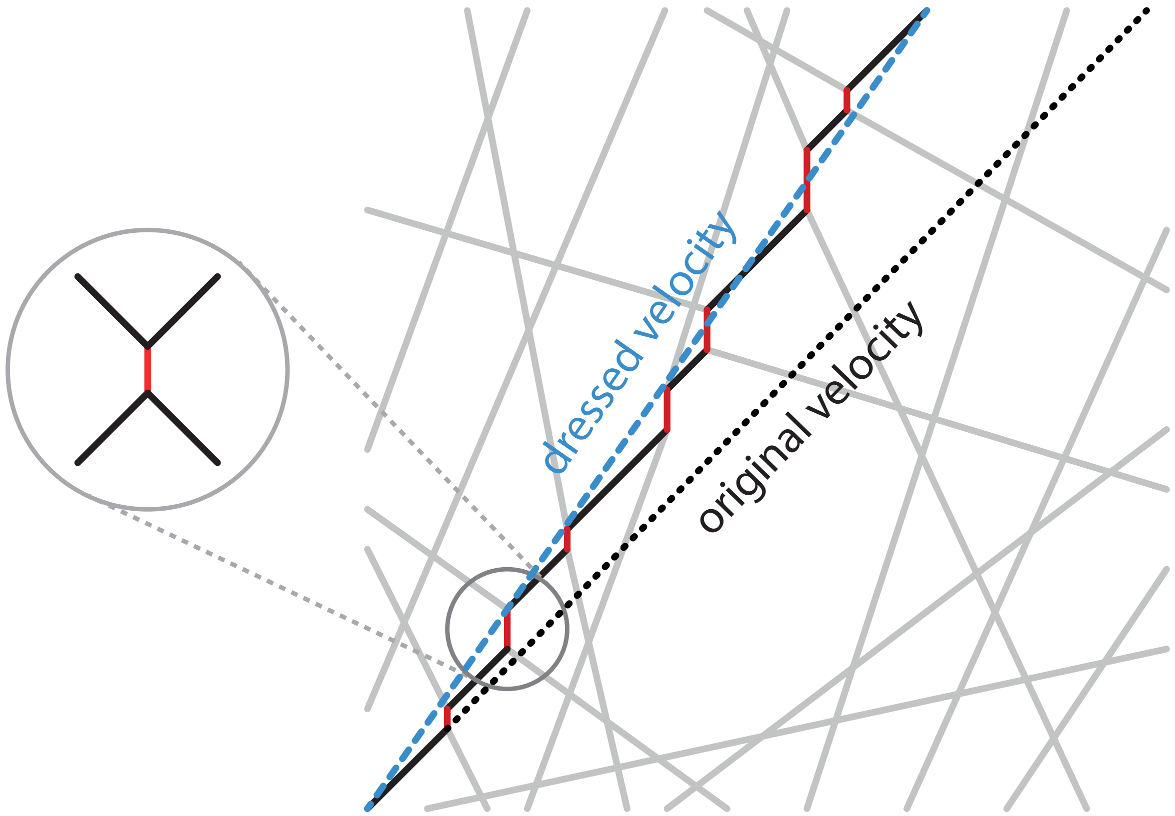
Spohn, Lebowitz (1977); Ruelle (2000); Tasaki (2000);
Araki, Ho (2000); Aschbacher, Pillet (2003);
Bernard, Doyon (2012); + Viti, De Luca (2013-);
+ Dubail, Stephan (2015-),
Sabetta, Misguich, Collura, Karevski, Kormos...

Generalised Hydrodynamics

- Integrable models:
  infinite set of conservation laws expressed as hydrodynamic continuity equations

- Equivalent to classical quasiparticles, moving ballistically and scattering elastically with each other (analogous to Boltzmann or kinetic equation).

- Collisions result in dressing of group velocity:
  cumulative effect of phase shifts due to collisions with other particles



Castro-Alvaredo, Doyon, Yoshimura (2016)
Bertini, Collura, De Nardis, Fagotti (2016)

Models & Methods

Models: QFT

Quantum Field Theory

- Massless free boson $$H_{0FB} = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 \right) \mathrm{d}x$$
- Massive free boson (Klein-Gordon) $$H_{mFB} = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}m^2\phi^2 \right) \mathrm{d}x$$
- ϕ^4 $$H_{\phi^4} = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \right) \mathrm{d}x$$
- sine-Gordon $$H_{sG} = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{m^2}{\beta^2}(1 - \cos\beta\phi) \right) \mathrm{d}x$$

Models: spin chains / lattice models

Spin chains

- Ising

$$H_{\mathrm{TFIsing}} = -J \sum_n \left(S_n^x S_{n+1}^x + h S_n^z \right)$$

$$H_{\mathrm{LFIsing}} = -J \sum_n \left(S_n^x S_{n+1}^x + h_z S_n^z + h_x S_n^x \right)$$

- XX / XY

$$H_{XX} = -J \sum_n \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + h S_n^z \right)$$

$$H_{XY} = -J \sum_n \left[(1+\gamma) S_n^x S_{n+1}^x + (1-\gamma) S_n^y S_{n+1}^y + h S_n^z \right)$$

- XXZ / Heisenberg / XYZ

$$H_{XXZ} = -J \sum_n \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z + h S_n^z \right)$$

$$H_{XXX} = -J \sum_n \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z + h S_n^z \right)$$

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Models: spin chains / lattice models

Lattice models

- Free hopping fermions

$$H_{FF1} = \sum_j \left(c_j^\dagger c_{j+1} + \text{h.c.} + \mu\, n_j \right) \;, \qquad n_j = c_j^\dagger c_j$$

- Free hopping fermions + non-diagonal

$$H_{FF2} = \sum_j \left(c_j^\dagger c_{j+1} + \gamma c_j^\dagger c_{j+1}^\dagger + \text{h.c.} + \mu\, n_j \right)$$

- Interacting fermions

$$H_{int} = \sum_j \left(c_j^\dagger c_{j+1} + \text{h.c.} + n_j n_{j+1} + \mu\, n_j \right)$$

Models: quantum gases / liquids

Quantum liquids

- Free non-relativistic Bose / Fermi gas

$$H_F = \int \left(\partial_x \Psi^\dagger \partial_x \Psi\right) \, \mathrm{d}x \,, \qquad m = \tfrac{1}{2}$$

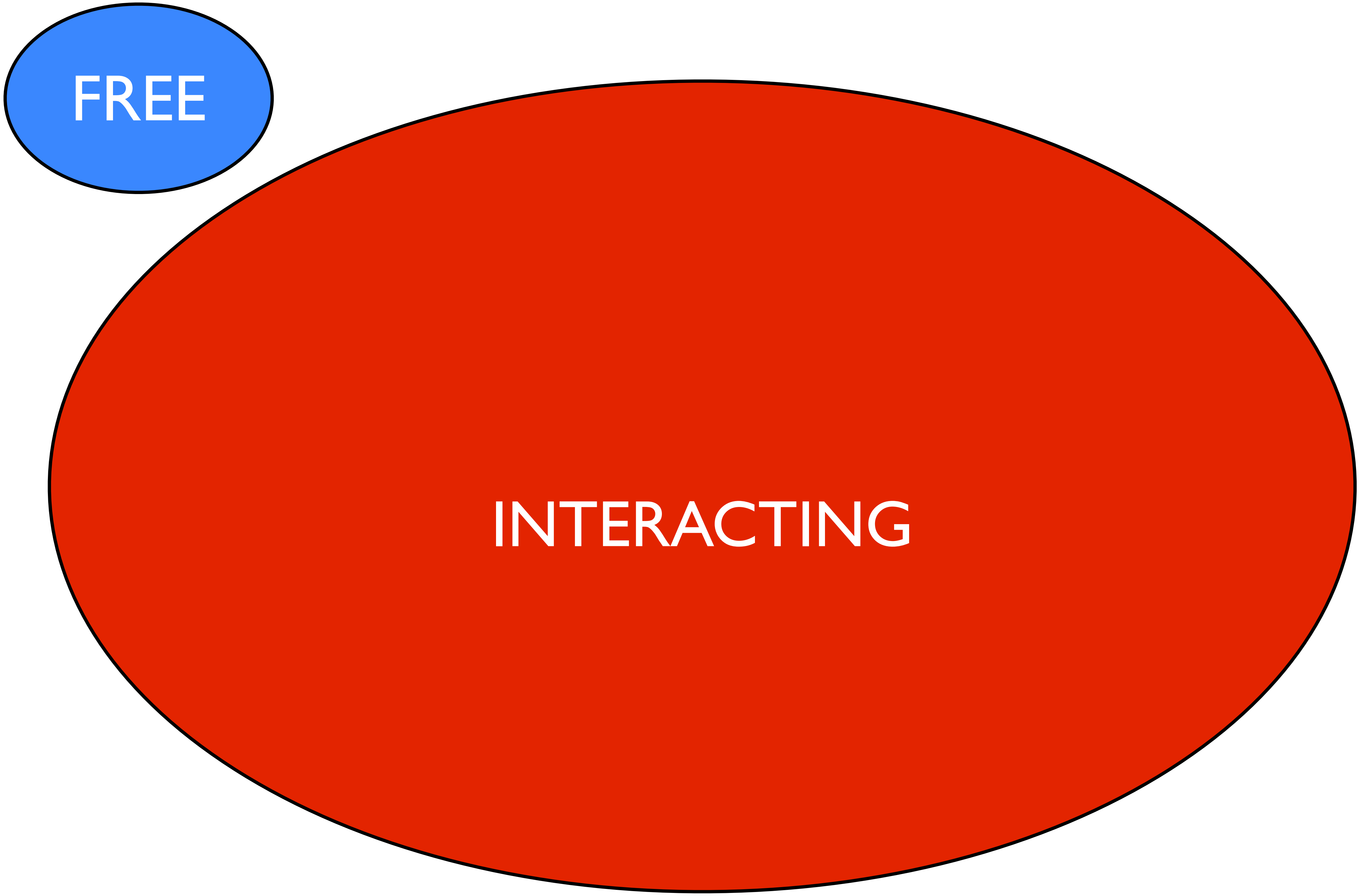
- Interacting Bose gas

$$H_{int} = \int \left(\partial_x \Psi^\dagger \partial_x \Psi\right) \, \mathrm{d}x + \iint V(x - x')\Psi^\dagger(x)\Psi(x)\Psi^\dagger(x')\Psi(x') \, \mathrm{d}x\mathrm{d}x'$$

- Bose gas with point interactions (Lieb-Liniger model)

$$H_{LL} = \int \left(\partial_x \Psi^\dagger \partial_x \Psi + c\, \Psi^\dagger \Psi^\dagger \Psi \Psi\right) \, \mathrm{d}x$$

Model classification



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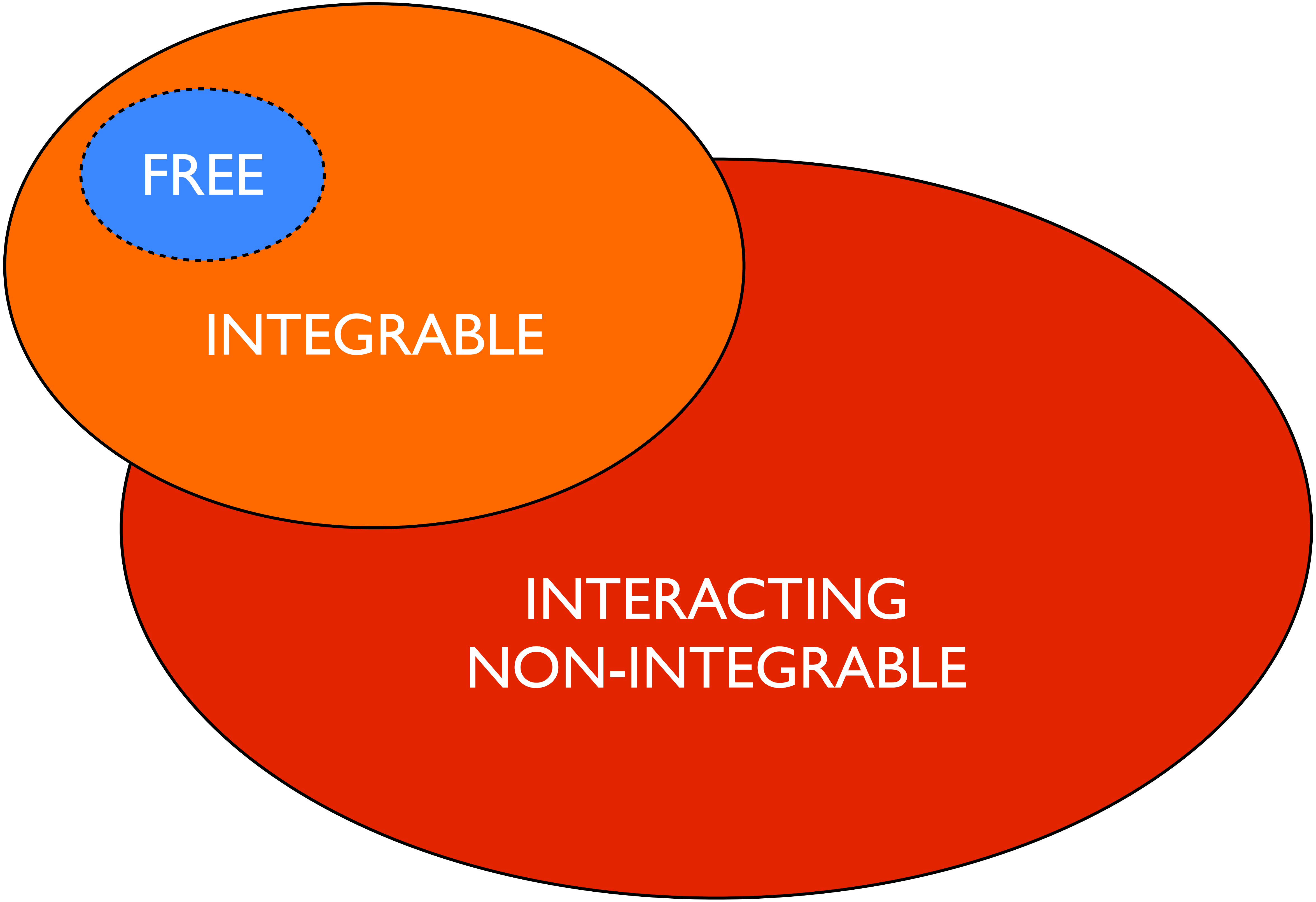
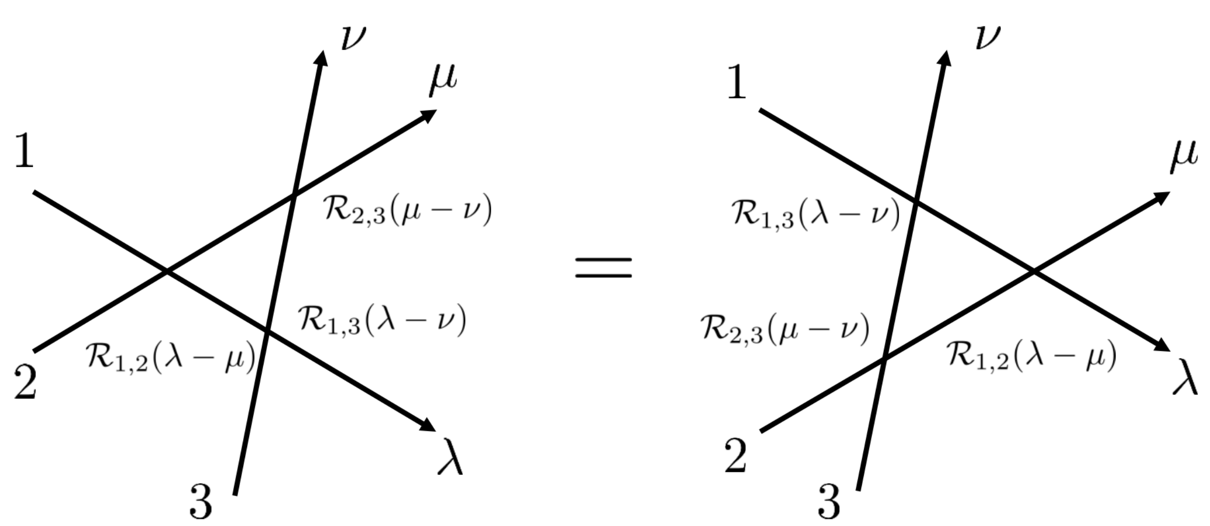


Integrable models

- Characterised by presence of infinite set of local conserved quantities

$$Q_n = \int q_n(x)\,\mathrm{d}x$$

- Multi-particle collisions can be decomposed into sequence of two-particle collisions and the order is irrelevant (Yang-Baxter equation)



- Collisions are elastic: no production or destruction of particles

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- Interacting fermions

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Methods: analytical

- Quantum Field Theory
  - Conformal Field Theory
  - Integrable Field Theory
  - Renormalisation Group Theory
- Bosonisation - Luttinger liquid theory (gapless phase of all 1-dim models)
- Bethe Ansatz (integrable models)
- Random Matrix Theory (non-integrable models)
- Semiclassical / Kinetic / Hydrodynamic approaches (all classes)

Methods: numerical

- time-dependent Density Matrix Renormalisation Group (tDMRG)
  Tensor Network / Matrix Product State methods
  (1-dim spin chains / lattice models)
- Truncated Conformal Space Approach (QFT / continuous models)
- ABACUS (integrable models)
- Quantum Boltzmann Equation (all classes)

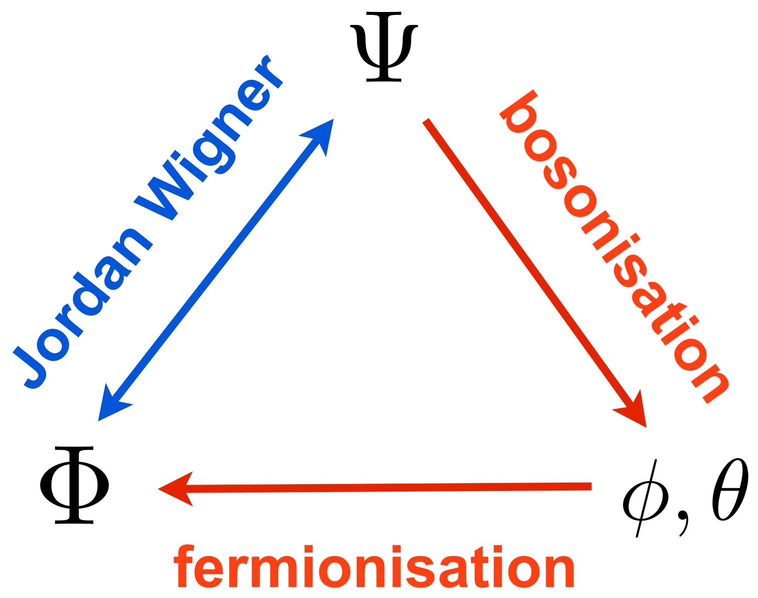
Dualities

- **Boson-Fermion correspondence** (fermions ↔ bosons)

$$\hat{\Phi}^{\dagger}_{\sigma}(x) = \hat{F}_{\sigma}\frac{1}{\sqrt{2\pi a}}\mathrm{e}^{2\pi\mathrm{i}\sigma\rho_0 x}\mathrm{e}^{\mathrm{i}(\sigma\hat{\vartheta}(x)-\hat{\varphi}(x))}$$

- **Jordan - Wigner transformation** (bosons → hard-core bosons / spins → fermions)

$$\hat{\Phi}^{\dagger}(x) = \hat{\Psi}^{\dagger}(x)\exp\left\{\mathrm{i}\pi\int_{-\infty}^{x}\mathrm{d}x'\hat{\rho}(x')\right\}$$



Dualities

Quantum Field Theory

- Massless free boson
- Massive free boson (Klein-Gordon)
- ϕ^4
- sine-Gordon
- ...in gapless phase

Spin chains

- Ising
- XX / XY
- XXZ Heisenberg XYZ

Lattice models

- Free hopping fermions
- Free hopping fermions + non-diagonal
- Interacting fermions

Quantum liquids

- Free non-relativistic Bose / Fermi gas
- Interacting Bose gas
- Bose gas with point interactions (Lieb-Liniger model)
- ...at large interaction (Tonks-Girardeau)

End of Introduction

Quantum Statistical Physics Out of Equilibrium

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University of Ljubljana

Lectures on Theoretical Physics 2018
NTUA
Athens
18-19 December 2018

Outline

Summary of 1st lecture

Gaussification

proof of relaxation to GGE in the special case of Gaussian dynamics

Intro to integrability:

the Lieb-Liniger model

Intro to Bosonisation:

Luttinger liquid approximation of the Lieb-Liniger model

Summary

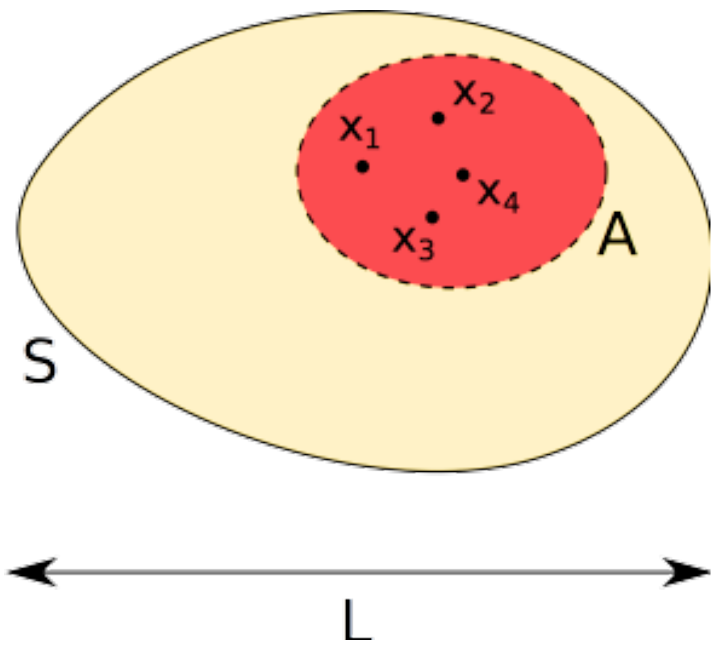
Quantum Quenches

- Well-posed theoretical and experimental problem:
  Consider an isolated and thermodynamically large quantum system, prepared in an initial state that is the ground state of some arbitrary Hamiltonian, then let to evolve under a different Hamiltonian

$$H_0|\Psi_0\rangle = 0 \qquad\qquad |\Psi_0\rangle \xrightarrow{e^{-iHt}} ?$$

Calabrese Cardy, PRL (2006)

- Questions:
  - Long time behaviour?
  - Does the system tend to equilibrium?
  - If yes, is equilibrium thermal?
  - If not, what type of equilibrium is it?
  - How much and what type of *information* about initial state survives at long times?

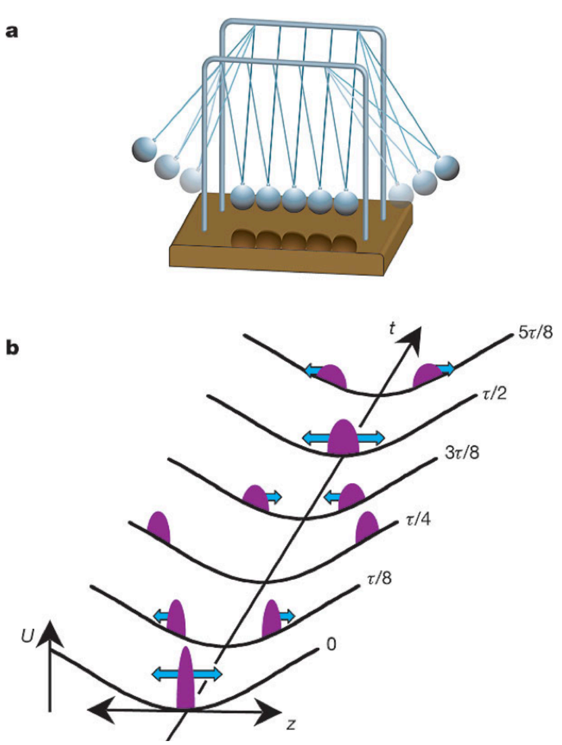


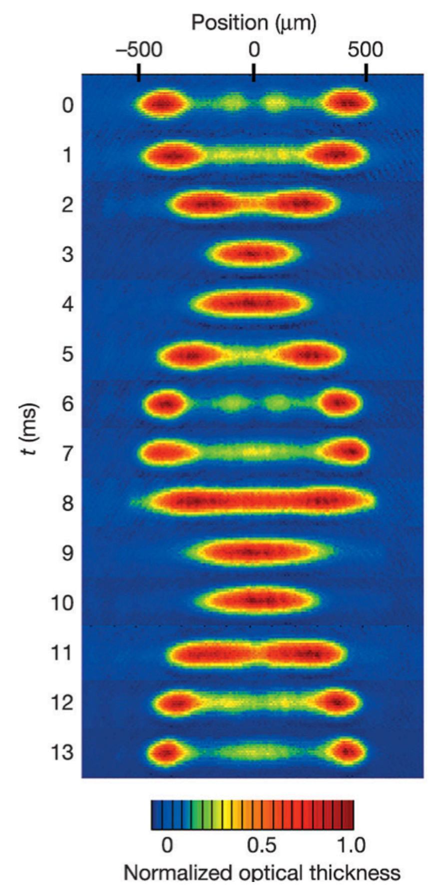
Quantum Newton's Craddle

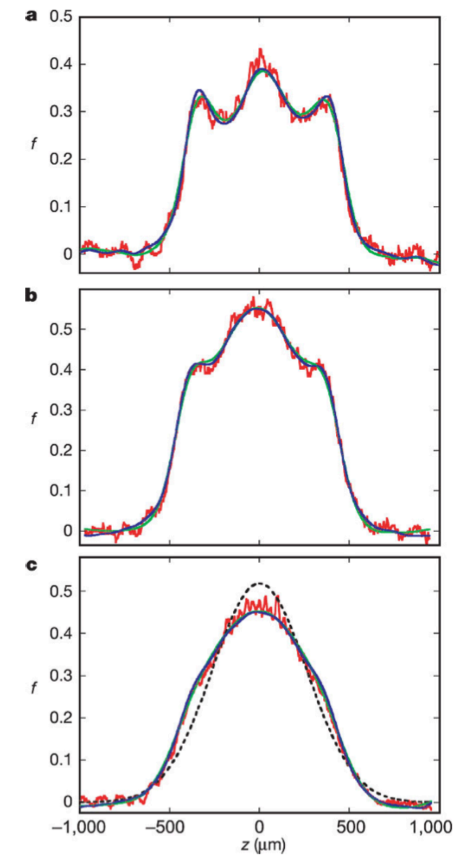
- Experiment:
  A system of 1d non-relativistic bosons with point-like interactions in a harmonic trap prepared in out of equilibrium initial state:
  - does not relax even after many collisions,
  - exhibits non-thermal momentum distribution.

- Lack of thermalisation due to integrability (Lieb-Liniger model)?

Kinoshita et al., Nature (2006)





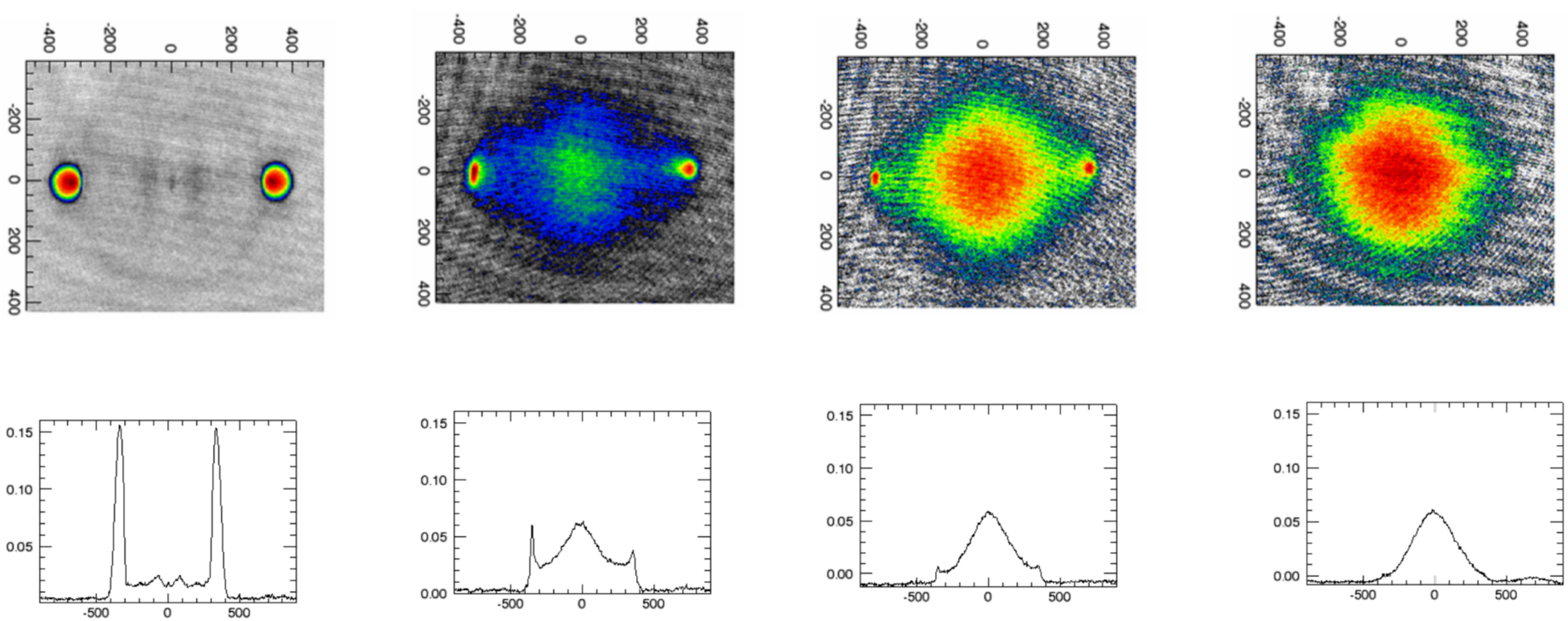


Quantum Newton's Craddle

- Experiment:
  A system of 1d non-relativistic bosons with point-like interactions in a harmonic trap prepared in out of equilibrium initial state:
  - does not relax even after many collisions,
  - exhibits non-thermal momentum distribution.

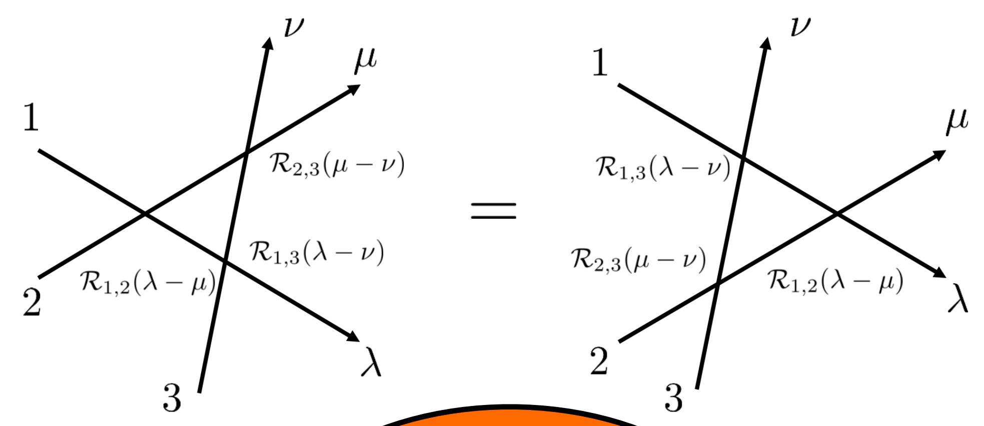
- Lack of thermalisation due to integrability (Lieb-Liniger model)?
  Or dimensionality?

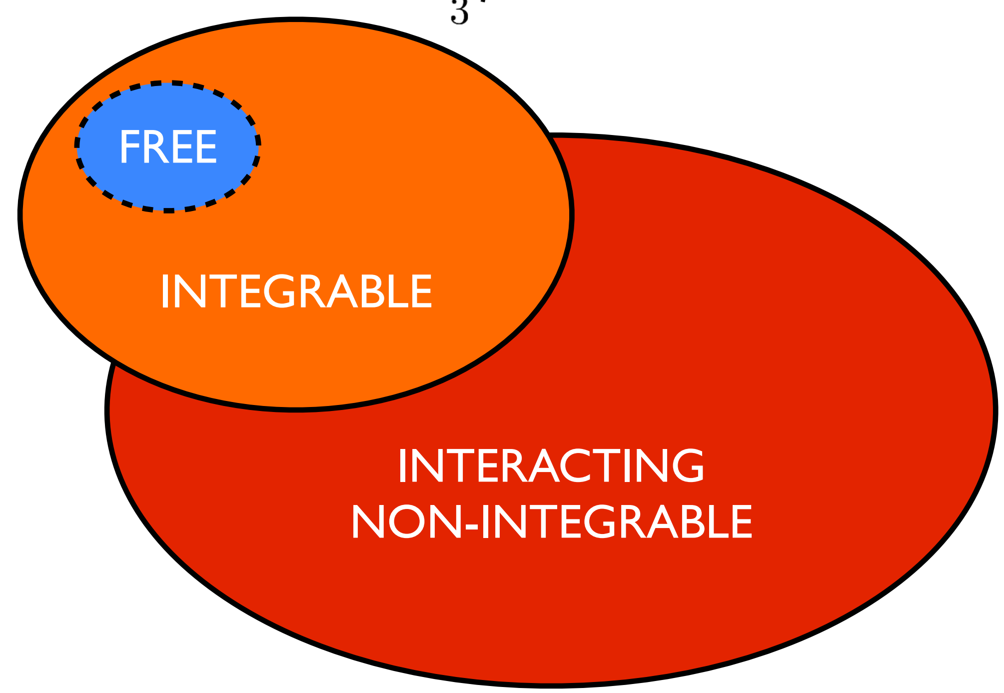
Kinoshita et al., Nature (2006)



Integrability & Equilibration

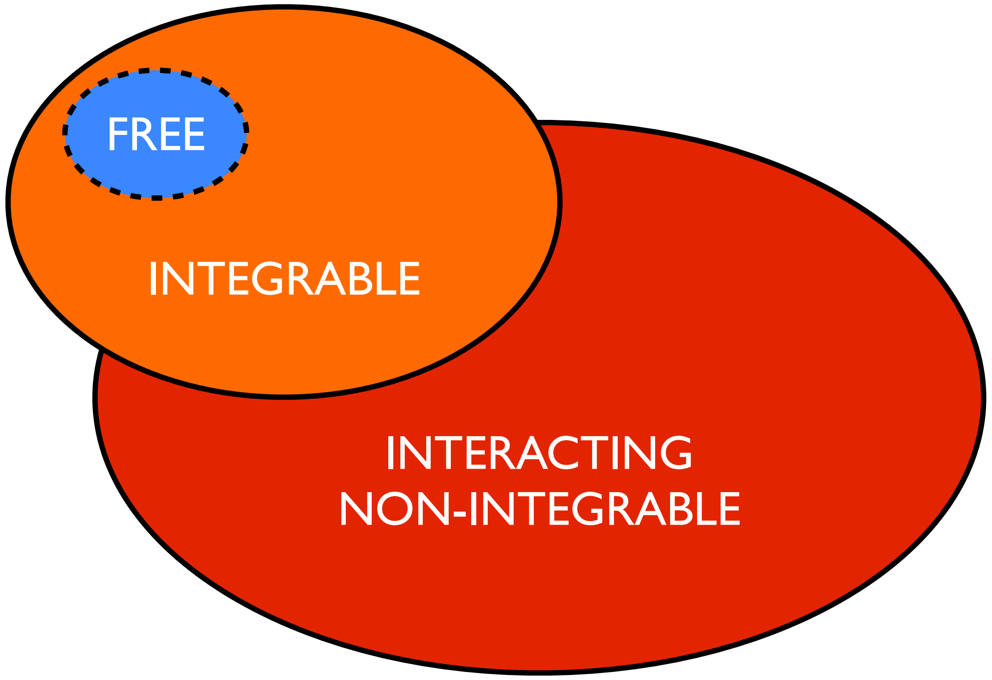
- Integrable models:
  - characterised by presence of *infinite* set of *local conserved quantities* (beyond total momentum and energy) $$Q_n = \int q_n(x)\, \mathrm{d}x$$ ...which means that they do not thermalize when brought out of equilibrium, but rather expected to relax to a generalised statistical ensemble (GGE)
  - elastic particle scattering
  - exactly solvable by Bethe-Ansatz
  - one-dimensional
  - may possess non-trivial quasi-particle excitations: solitons & breathers
  - serve as non-trivial models of many-body dynamics: less trivial than free models, yet possible to analyse exactly





Integrability & Equilibration

- Examples:
  - All non-interacting models
  - Models that can be *mapped* into non-interacting ones (Ising spin chain in transverse field, XY model, hard-core boson gas)
  - Heisenberg model, more generally XYZ spin chain
  - sine/sinh-Gordon model, Thirring model
  - 1d Bose gas with point-like interactions (Lieb-Liniger model)



Generalised Gibbs Ensemble

- Conjecture:
  "In integrable models local observables equilibrate to a Generalised Gibbs Ensemble that is a maximum entropy ensemble determined by all constraints coming from the infinite number of conserved quantities."

  Rigol, Dunjko, Yurovsky, Olshanii, PRL (2007)

Gibbs ensemble:

$$\rho_{GE} \propto \exp(-\beta H)$$

temperature fixed by constraint of energy conservation

$$\mathrm{Tr}(\rho_{GE} H) = \langle \Psi_0 | H | \Psi_0 \rangle$$

⇒

Generalised Gibbs ensemble:

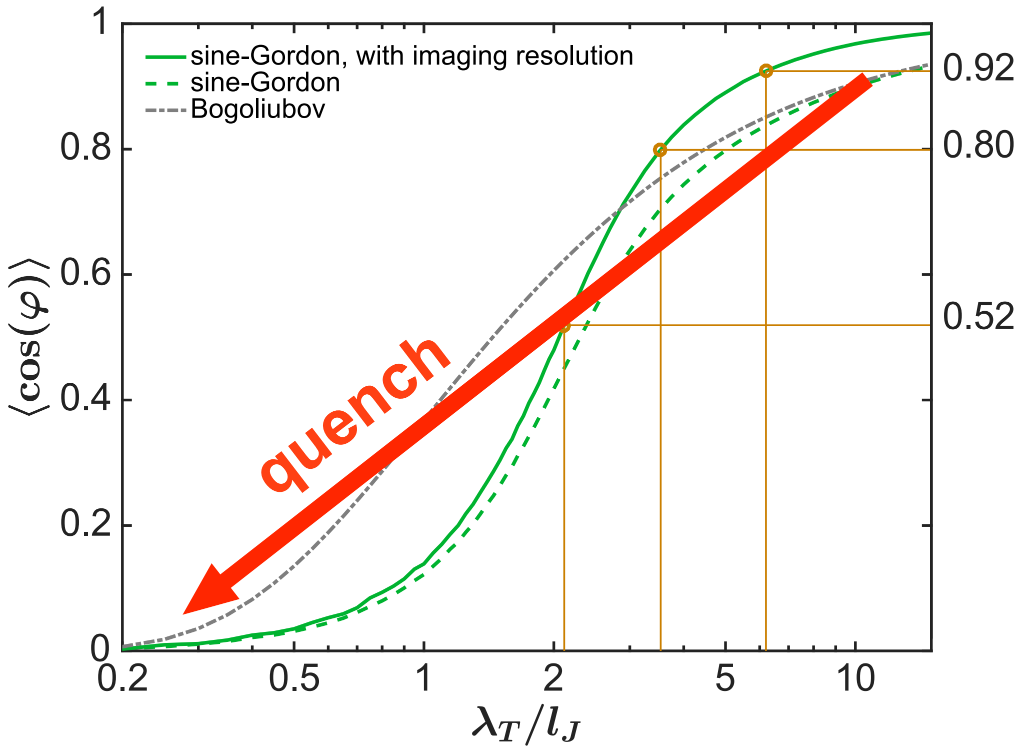
$$\rho_{GGE} \propto \exp\left(- \sum_{n=1}^{\infty} \lambda_n \mathcal{I}_n \right)$$

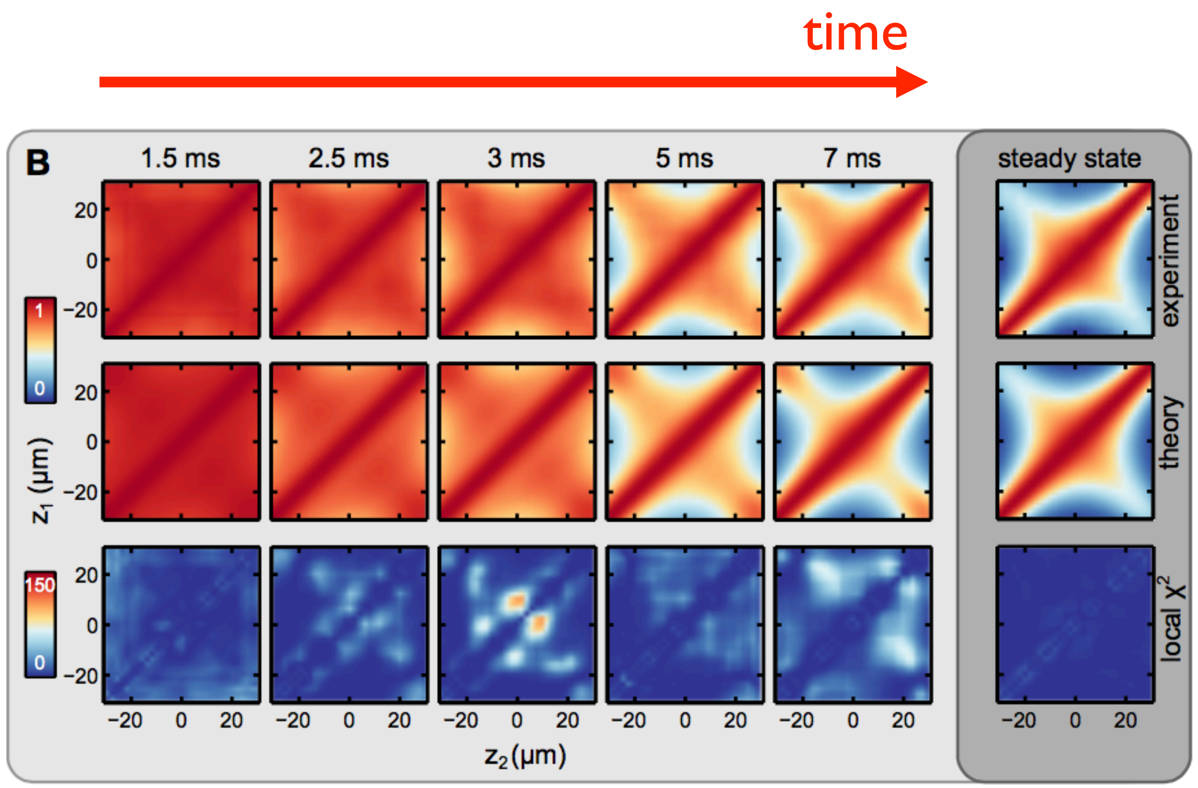
Lagrange multipliers fixed by constraints of all conserved quantities

$$\mathrm{Tr}(\rho_{GGE} \mathcal{I}_n) = \langle \Psi_0 | \mathcal{I}_n | \Psi_0 \rangle$$

- Very economic: number of local conserved quantities increases only polynomially with system size (compare with exponential number of initial state's independent parameters)
- Successfully verified analytically or numerically in large number of special cases
- But: complete set of relevant charges (local & quasi-local) not known for most models

Experimental Observation of GGE





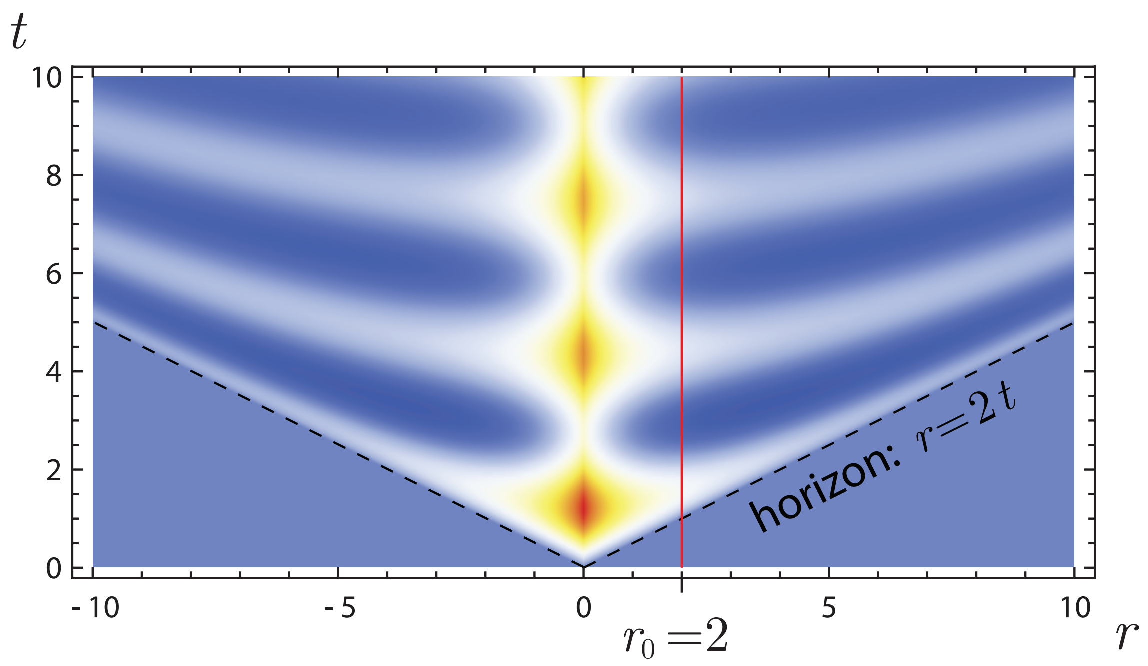
Langen et al., Science (2015)

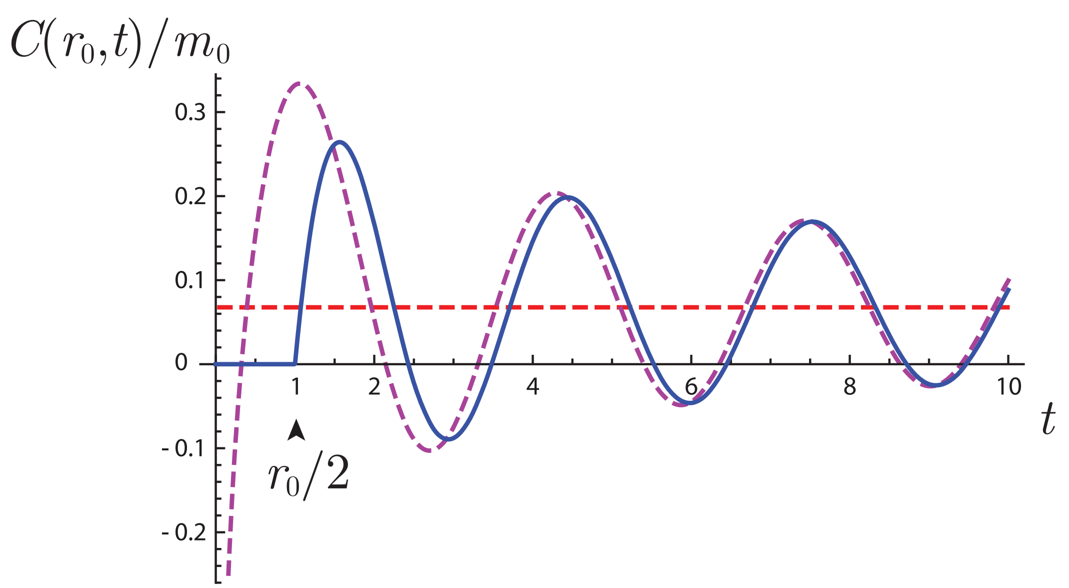
- Quench from gapped to gapless non-interacting phase
- Observation of dynamics of correlations
- *Non-thermal* steady state: more than one temperature needed to describe steady state
- Agreement between experimental data and theoretical predictions based on a Generalised Gibbs Ensemble

Mass quench in Klein-Gordon

$$H_{KG} = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}m^2\phi^2 \right) \mathrm{d}x$$

- in Fourier space: infinite set of independent harmonic oscillators
- solve in Schroedinger (using Bogoliubov transformation and squeezed states) or Heisenberg picture (EoM: linear harmonic oscillator)





- 2pt correlation function:

$$C_{qq}(r,t) = \int \frac{dk}{2\pi} e^{ikr} \left(\frac{E_k^2 + E_{0k}^2}{4E_k^2 E_{0k}} + \frac{E_k^2 - E_{0k}^2}{4E_k^2 E_{0k}} \cos 2E_k t \right)$$

- Horizon effect
- Equilibration to a non-thermal state

$$C_{t\to\infty}(r) = \int \frac{dk}{2\pi} e^{ikr} \, \frac{E_k^2 + E_{0k}^2}{4E_k^2 E_{0k}}$$

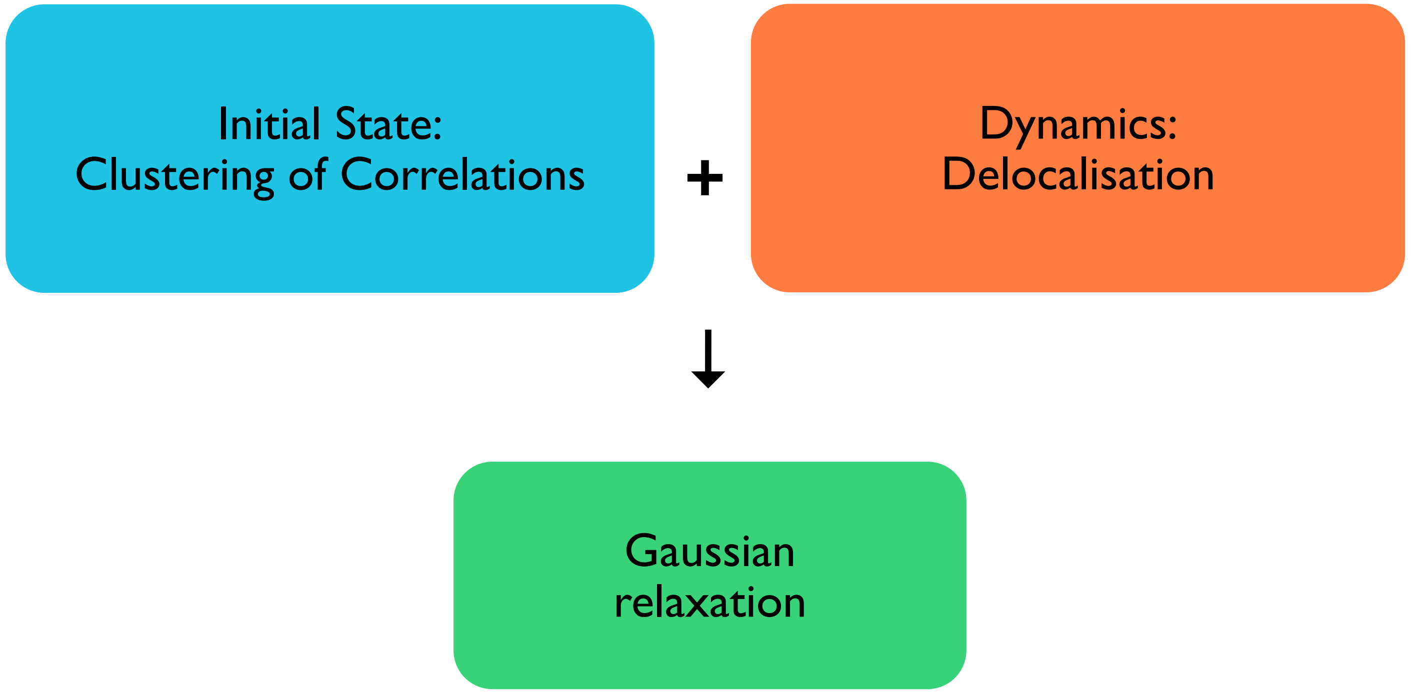
Gaussification
in interacting-to-free quantum quenches

Gaussification in interacting-to-free quantum quenches

"A quantum quench from a general interacting Hamiltonian to a non-interacting one, results in relaxation to a Gaussian GGE, under the conditions of clustering of initial correlations and delocalising dynamics."

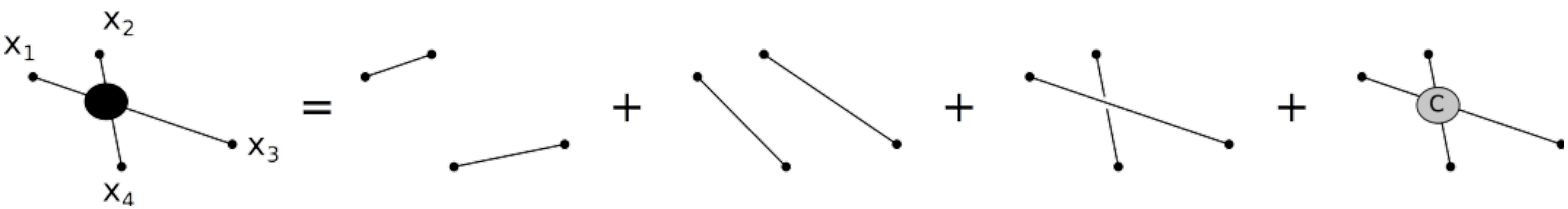
Cramer Eisert (2010), Gluza Krumnow Friesdorf Gogolin Eisert (2016), Sotiriadis Calabrese (2014), Sotiriadis (2016-17), Doyon (2017)

- All memory of initial non-Gaussian correlations (connected correlation functions of order > 2) erased by Gaussian dynamics!
- Later generalised to dynamics under genuinely interacting integrable spin chains



Connected Correlation Function

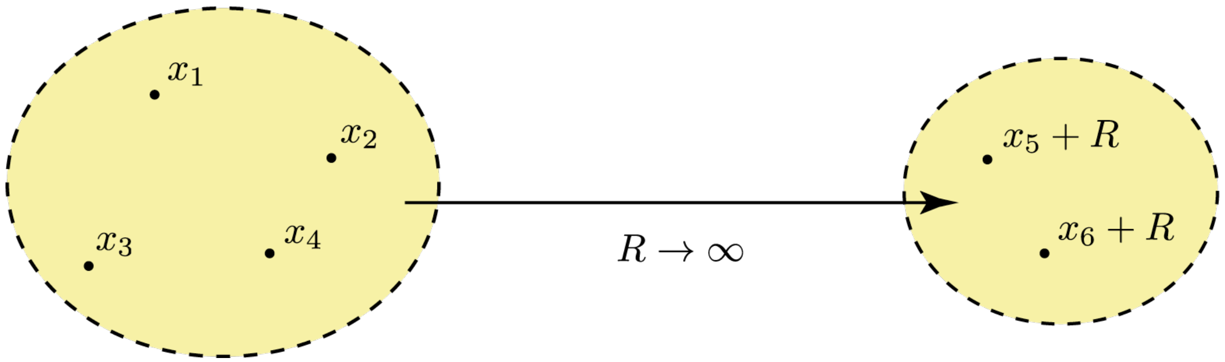
- Connected correlation functions (aka cumulants):



- Gaussian states:
  All connected correlation functions of order higher than 2 vanish (i.e. all higher order correlation functions can be decomposed into combinations of 2pt functions: Wick's theorem)

Proof of Gaussification

- 1st condition
  Clustering of initial correlations:
  Initial correlations between two groups of points *far* from each other must factorise
  - generally valid - expresses locality of interactions in pre-quench Hamiltonian



$$\lim_{R\to\infty} \left\langle \prod_i \phi(x_i) \prod_j \phi(x_j + R) \right\rangle = \left\langle \prod_i \phi(x_i) \right\rangle \left\langle \prod_j \phi(x_j) \right\rangle$$

- 2nd condition
  Delocalising dynamics:
  initially local fields spread with time under the action of post-quench Hamiltonian
  - typically valid for non-interacting dynamics due to non-linear dispersion
  - non-trivial - not necessarily true for all integrable systems!

- Physical mechanism:
  Information determining large time values of local observables originates from spatially distant points, thus independent $\rightarrow$
  Gaussification: reminiscent of classical central limit theorem

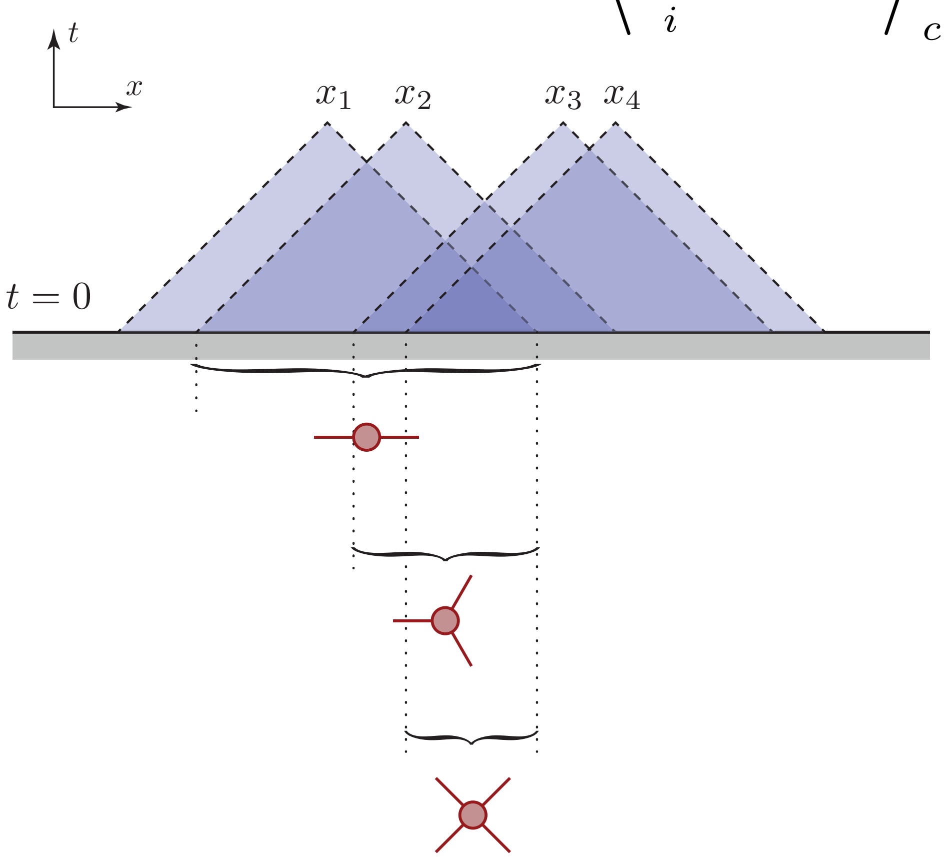
Method

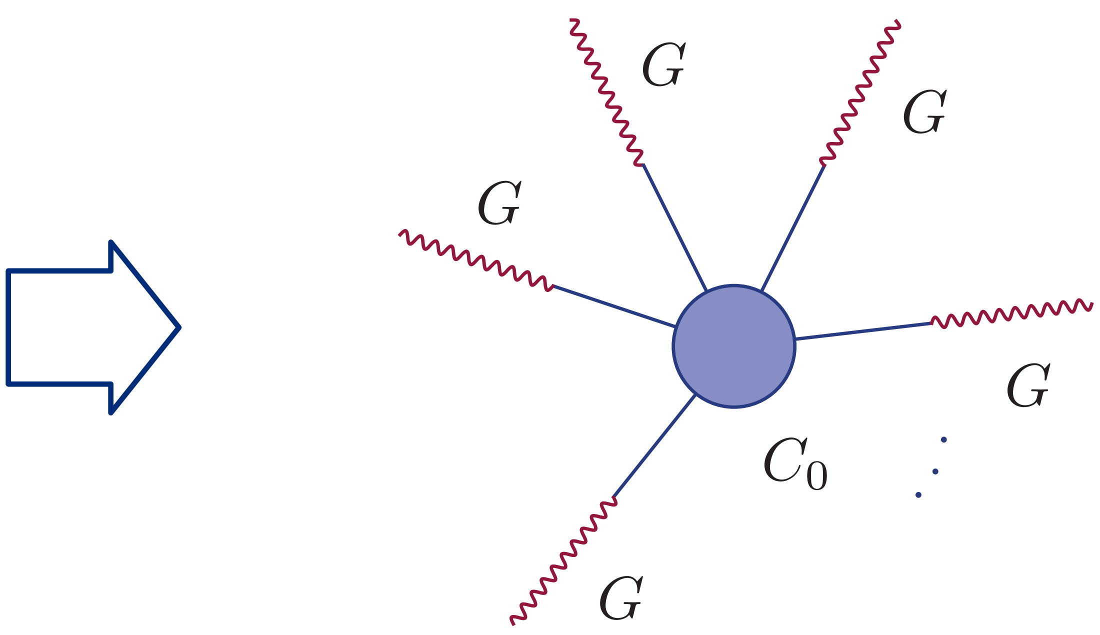
- Diagrammatic method:
  - Express time-evolved field in terms of initial fields by exact solution of Heisenberg equations of motion (always possible for free dynamics)

$$\hat{\phi}(x,t) = \sum_{\alpha} \int \mathrm{d}x \, G_{\alpha,\beta}(x-x',t)\hat{\Phi}_{\beta}(x')$$

  - Use cumulant expansion of initial state: extract large time decay of connected correlations from large distance decay of initial correlations (clustering) + large time decay of field propagators (delocalisation)

$$C(\{x_i\},t) \equiv \left\langle \prod_i \hat{\phi}(x_i,t) \right\rangle_c = \sum_{\alpha_i} \prod_i \int \mathrm{d}x_i \, G_{\alpha_i,\beta_i}(x_i - x_i', t) \left\langle \prod_i \hat{\Phi}_{\beta_i}(x_i') \right\rangle_c$$





Interacting dynamics:
the Lieb-Liniger case

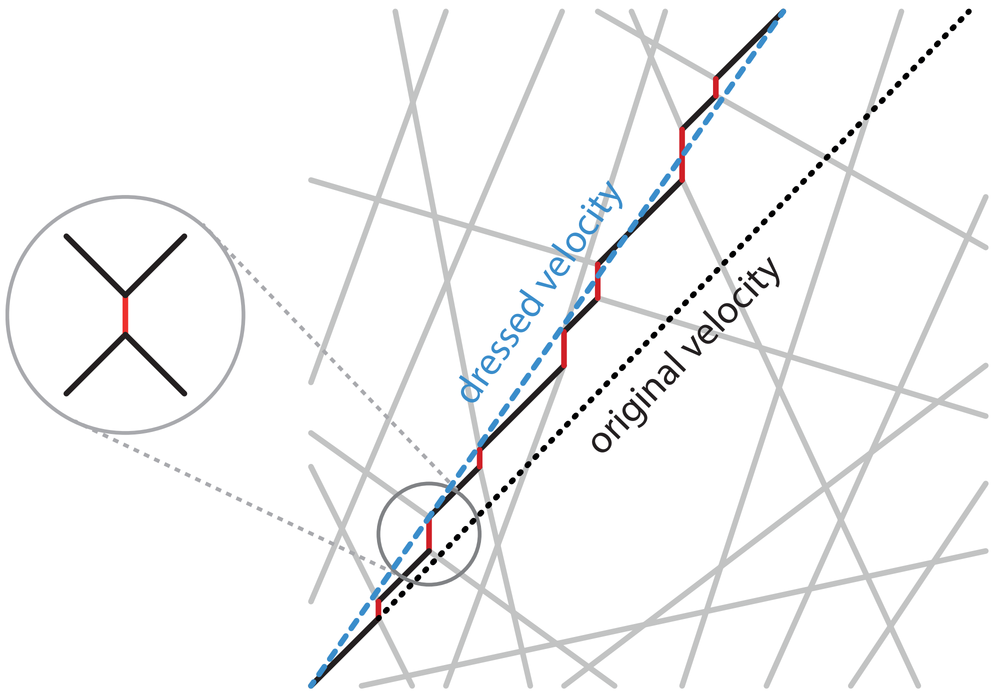
Lieb-Liniger Dynamics

- Lieb-Liniger model:
  one-dimensional system of non-relativistic bosons with point-like interactions

- Hamiltonian
$$H_{LL} = \int \left(\partial_x \Psi^\dagger \partial_x \Psi + c\Psi^\dagger \Psi^\dagger \Psi \Psi\right) \, \mathrm{d}x$$

- Despite integrability, exact derivation of equilibrium state possible only for special cases of initial states

- Dynamics can be understood semiclassically through kinetic / Boltzmann-type equation



Castro-Alvaredo, Doyon, Yoshimura (2016)
Bertini, Collura, De Nardis, Fagotti (2016)

Quench Action Method

- Exact eigenstates and energy eigenvalues known by Bethe Ansatz

$$\langle \boldsymbol{x}|\psi(\boldsymbol{\lambda})\rangle \propto \sum_{P \text{ perm.s}} (-1)^{[P]} \exp\left(\mathrm{i}\sum_i \lambda_{P_i} x_i\right) \prod_{j>i} \left[\lambda_{P_j} - \lambda_{P_i} - \mathrm{i}c\,\mathrm{sign}(x_j - x_i)\right]$$

where “rapidities” λ given by Behe Ansatz equations

$$\exp\left(i\lambda_i L\right) \prod_{j=1}^{N} \left(\frac{\lambda_i - \lambda_j - ic}{\lambda_i - \lambda_j + ic}\right) = -1$$

- Time evolution after a quench

$$\langle\Omega|e^{+iHt}\mathcal{O}e^{-iHt}|\Omega\rangle = \sum_{E,E'} \langle\Omega|E'\rangle\langle E'|\mathcal{O}|E\rangle\langle E|\Omega\rangle e^{-i(E-E')t}$$

- Problem 1: overlaps of initial state in post-quench eigenstates not known
no general solution

Quench Action Method

- Problem 2: summation over exponentially many energy eigenstates

$$\langle\Omega|e^{+iHt}\mathcal{O}e^{-iHt}|\Omega\rangle = \sum_{E,E'} \langle\Omega|E'\rangle\langle E'|\mathcal{O}|E\rangle\langle E|\Omega\rangle e^{-i(E-E')t}$$

- Quench Action method:
  - in thermodynamic limit, write sums as functional integrals over macrostates characterised by rapidity densities $\varrho(\lambda)$

$$e^{-iHt}|\Omega\rangle = \sum_{E} |E\rangle\langle E|\Omega\rangle e^{-iEt}$$

$$= \int \mathcal{D}\rho(\lambda)\ |\rho(\lambda)\rangle\ e^{-iE[\rho]t} e^{\log\langle\rho|\Omega\rangle + S[\rho]}$$

  - find macrostate that maximises the action: best representation of initial state
  - steady state is given by this saddle-point macrostate

$$\lim_{t\to\infty} \langle\Omega|e^{+iHt}\mathcal{O}e^{-iHt}|\Omega\rangle = \langle\Phi_s|\mathcal{O}|\Phi_s\rangle$$

Caux Essler (2013)

Bosonisation:
mapping interacting models to free

Bosonization in Lieb-Liniger model

- One-dimensional interacting Bose gas

$$\hat{H} = \int dx\, \partial_x \hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x) + \int dx dx'\, V(x-x') \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \hat{\Psi}^\dagger(x') \hat{\Psi}(x')$$

- Introduce density/phase fields $\hat{\Psi}^\dagger(x) = \sqrt{\hat{\rho}(x)} \mathrm{e}^{\mathrm{i}\hat{\phi}(x)}$ and $\hat{\rho}(x) = \rho_0 + \frac{1}{\pi} \partial_x \hat{\theta}(x)$
  with commutation relations $[\partial_x \hat{\theta}(x), \hat{\phi}(x')] = -[\hat{\theta}(x), \partial_{x'} \hat{\phi}(x')] = \mathrm{i}\pi\delta(x-x')$

- Keeping only quadratic terms in the gradients

$$\hat{H}_{Lm} = \frac{v}{2\pi} \int dx \left[K \left(\partial_x \hat{\phi} \right)^2 + \frac{1}{K} \left(\partial_x \hat{\theta} \right)^2 \right]$$

  standard Luttinger model = massless free boson CFT [Haldane (1981)]

- Local fields correspond to *derivatives* of bosonisation fields (and vertex operators): density/current

$$\hat{\rho}(x) = \hat{\Psi}^\dagger(x) \hat{\Psi}(x) = \rho_0 + \frac{1}{\pi} \partial_x \hat{\theta}(x)$$

$$\hat{j}(x) = -\mathrm{i} \left(\hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x) - \partial_x \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \right) = -2\sqrt{\hat{\rho}(x)} \left(\partial_x \hat{\phi}(x) \right) \sqrt{\hat{\rho}(x)} \sim -2\rho_0 \partial_x \hat{\theta}(x)$$

Bosonization glossary

$$\hat{H}_{Lm} = \frac{v}{2\pi}\int dx \left[K\left(\partial_x\hat{\phi}\right)^2 + \frac{1}{K}\left(\partial_x\hat{\theta}\right)^2\right] = \frac{v}{2\pi}\sum_{\sigma=\pm}\int dx\ \left(\partial_x\hat{\varphi}_\sigma\right)^2$$

<table>
<tr><th>original bosons</th><th colspan="2">bosonisation
density/phase fields</th><th colspan="2">fermionic
quasiparticle field</th></tr>
<tr><td>contact interaction</td><td rowspan="2">free</td><td>linear dispersion</td><td rowspan="3">free</td><td>linear dispersion</td></tr>
<tr><td>long-range interaction</td><td>non-linear dispersion</td><td rowspan="2">non-linear dispersion</td></tr>
<tr><td>kinetic term
(next to leading order)</td><td rowspan="2">non-free</td><td>chiral interaction</td></tr>
<tr><td>nonlinear dispersion</td><td>non-chiral interaction</td><td>non-free</td><td>perturbative interaction</td></tr>
</table>

Bosonization Dynamics

$$\hat{H}_{Lm} = \frac{v}{2\pi}\int dx \left[K\left(\partial_x\hat{\phi}\right)^2 + \frac{1}{K}\left(\partial_x\hat{\theta}\right)^2\right] = \frac{v}{2\pi}\sum_{\sigma=\pm}\int dx\,\left(\partial_x\hat{\varphi}_\sigma\right)^2$$

- Equations of motion: wave equation

$$\partial_t^2\hat{\phi} = v^2\partial_x^2\hat{\phi}$$

- Solution: d'Alembert formula

$$\hat{\phi}(x,t) = \frac{1}{2}\Big(\hat{\phi}(x-vt,0) + \hat{\phi}(x+vt,0)\Big) + \frac{1}{2v}\int_{x-vt}^{x+vt} dx'\,\partial_t\hat{\phi}(x',0)$$

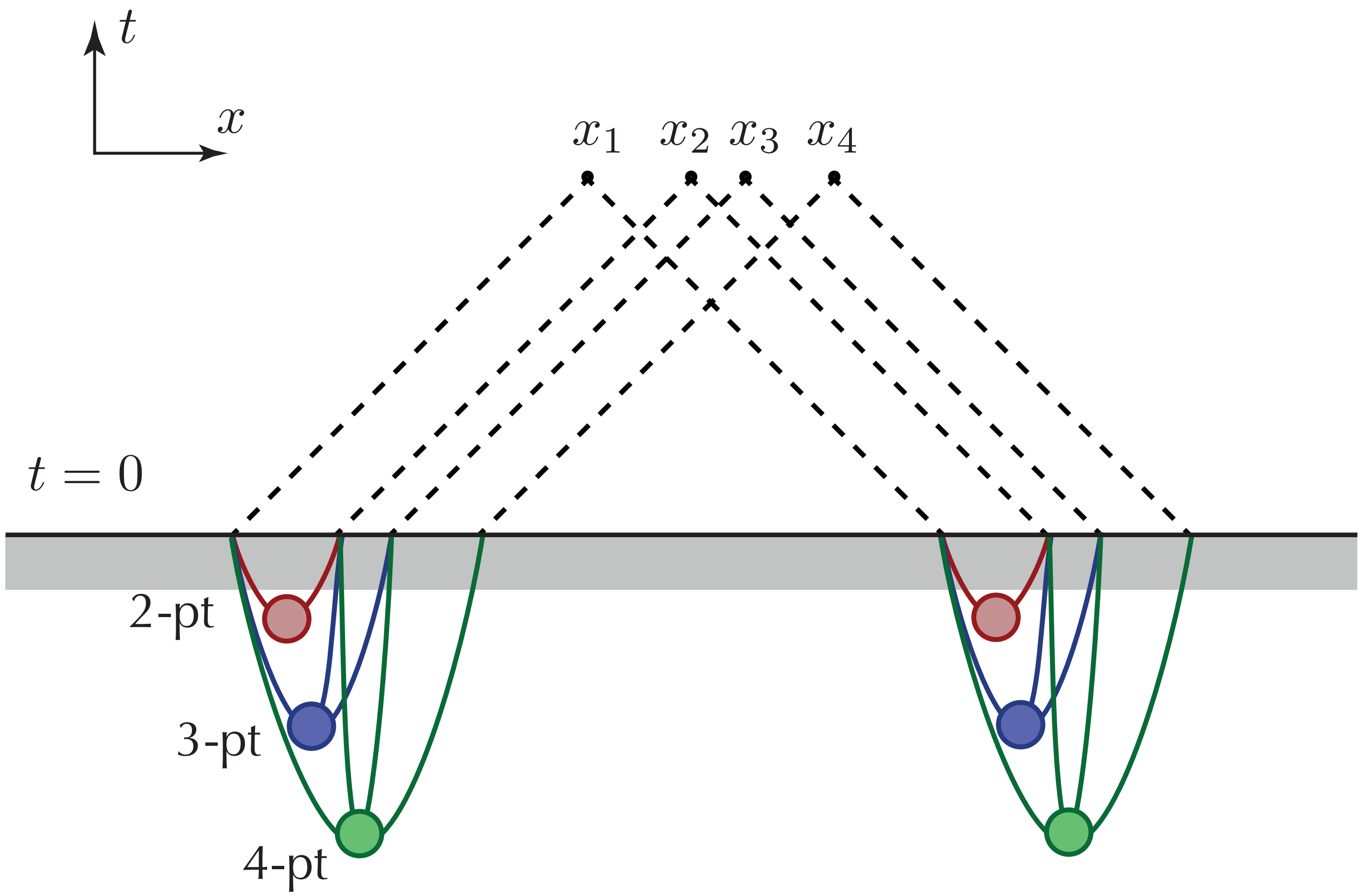
- Large time asymptotics of correlations of *local* observables (field derivatives)

$$\partial_x\hat{\varphi}(x,t) = \frac{1}{2}\sum_{\sigma=\pm}\partial_x\hat{\varphi}_\sigma(x+\sigma vt,0)$$

- Large time connected correlations decompose into two contributions from left and right asymptotics of initial correlations, but don't vanish generally

$$\lim_{t\to\infty}\left\langle\prod_{i=1}^{n}\partial_x\hat{\varphi}(x_i,t)\right\rangle_c = \lim_{R\to\infty}\left[\left\langle\prod_{i=1}^{n}\partial_x\hat{\varphi}_-(x_i-R)\right\rangle_c + \left\langle\prod_{i=1}^{n}\partial_x\hat{\varphi}_+(x_i+R)\right\rangle_c\right]$$

Bosonization Dynamics



Memory of all initial correlations preserved up to infinite times: no Gaussification

Nonlinear Dispersion effects

$$\hat{H}_{\text{disp}} = \frac{\beta}{2\pi} \int dx \left[K \left(\partial_x^2 \hat{\phi} \right)^2 + \frac{1}{K} \left(\partial_x^2 \hat{\theta} \right)^2 \right] = \frac{\beta}{2\pi} \sum_{\sigma=\pm} \int dx \, \left(\partial_x^2 \hat{\varphi}_\sigma \right)^2$$

- Equations of motion

$$\partial_t \hat{\varphi}_\pm(x,t) = \pm \left(v \partial_x \hat{\varphi}_\pm(x,t) + \beta \partial_x^3 \hat{\varphi}_\pm(x,t) \right)$$

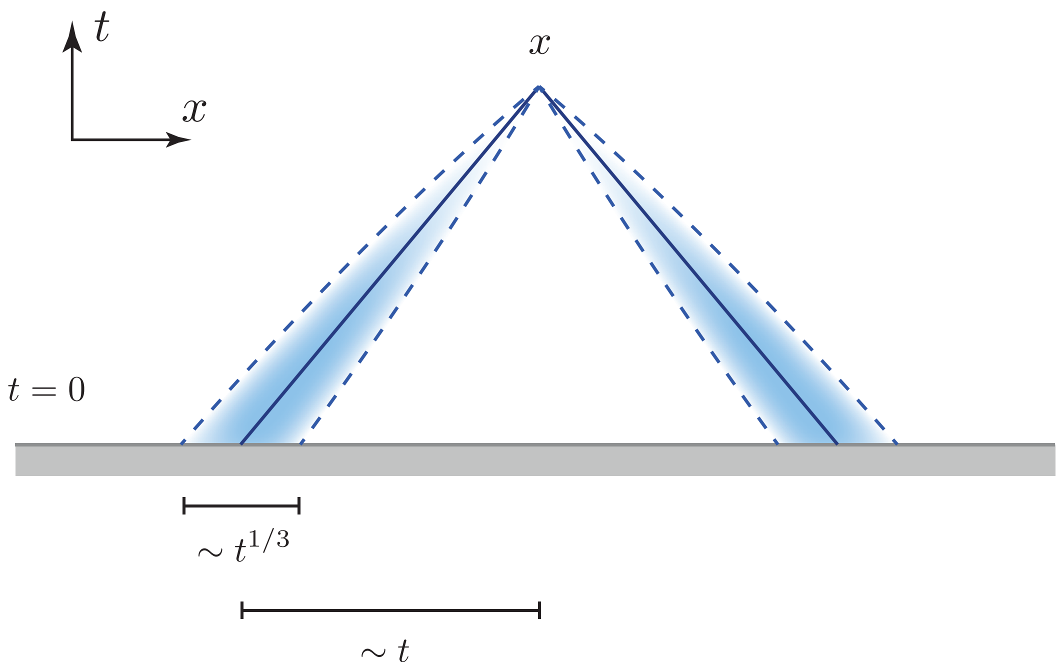
- Solution

$$\partial_x \hat{\phi}(x,t) = \int dx' \left(\partial_t G(x-x',t) \partial_{x'} \hat{\phi}(x',0) + \partial_x G(x-x',t) \partial_t \hat{\phi}(x',0) \right)$$

$$G(x,t) = \int \frac{dk}{2\pi} e^{ikx} \frac{\sin \omega(k) t}{\omega(k)}$$

$$\omega(k) = |k| \left(v + \beta k^2 \right) = v|k| f(k)$$

- Propagator still exhibits light-cone form but also dispersive spreading → decays with time uniformly in space



Initial clustering + Uniform decay of propagator with time → Gaussification

Thank you for your attention