Introduction to Topological Superconductivity

Panagiotis Kotetes

NTUA – Lectures on Theoretical Physics 2018

UNIVERSITY OF COPENHAGEN

Niels Bohr Institutet

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Assoc. Prof. @ the Institute of Theoretical Physics
Chinese Academy of Sciences Beijing

Overview

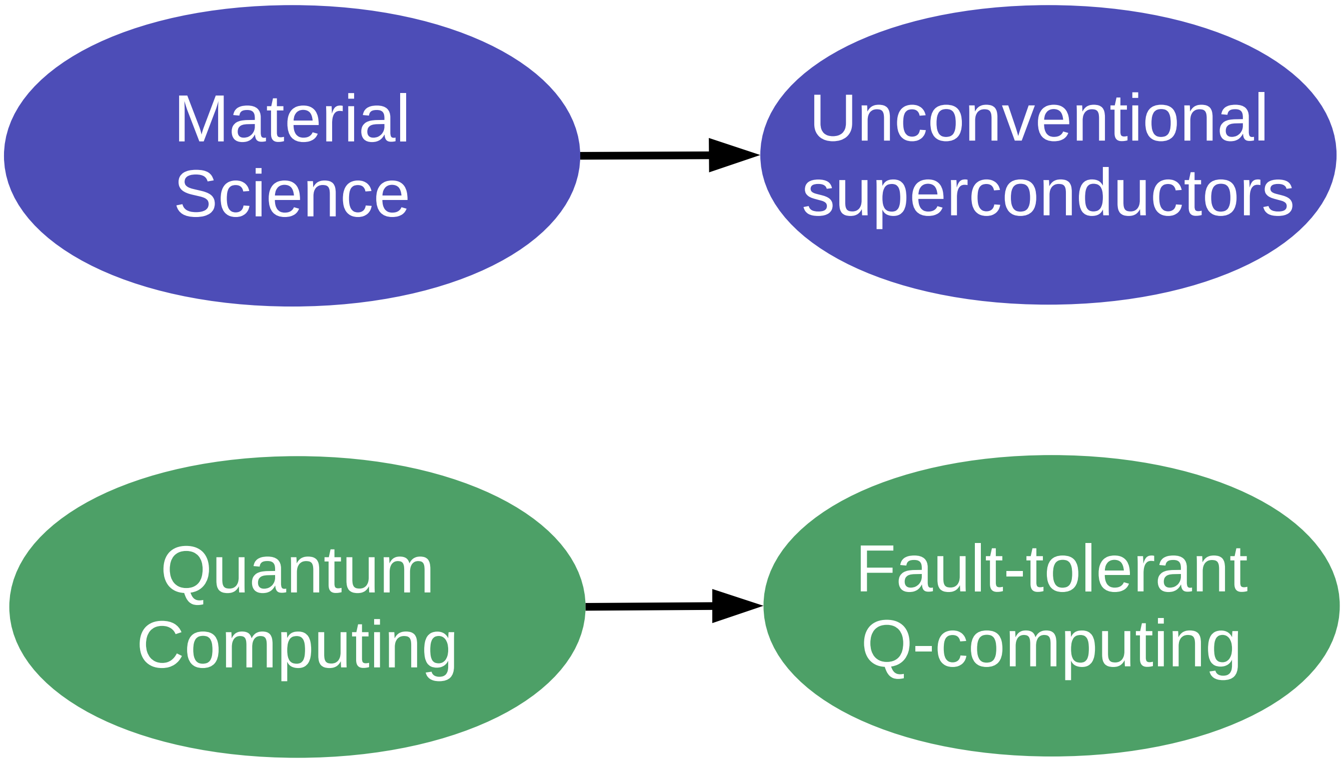
- Motivation – Why look for topological superconductivity?
- Short intro to superconductors (SCs): spin-singlet vs spin-triplet
- Spinful p-wave SCs and the spinless limit: Kitaev's chain
- Majorana fermion quasiparticles and quantum computing
- 4π-periodic Majorana-Josephson effect
- Symmetry and topological classification of phases of matter
- Artificial and effective (intrinsic) p-wave superconductors
- Conclusions and outlook

Flow towards topological superconductivity

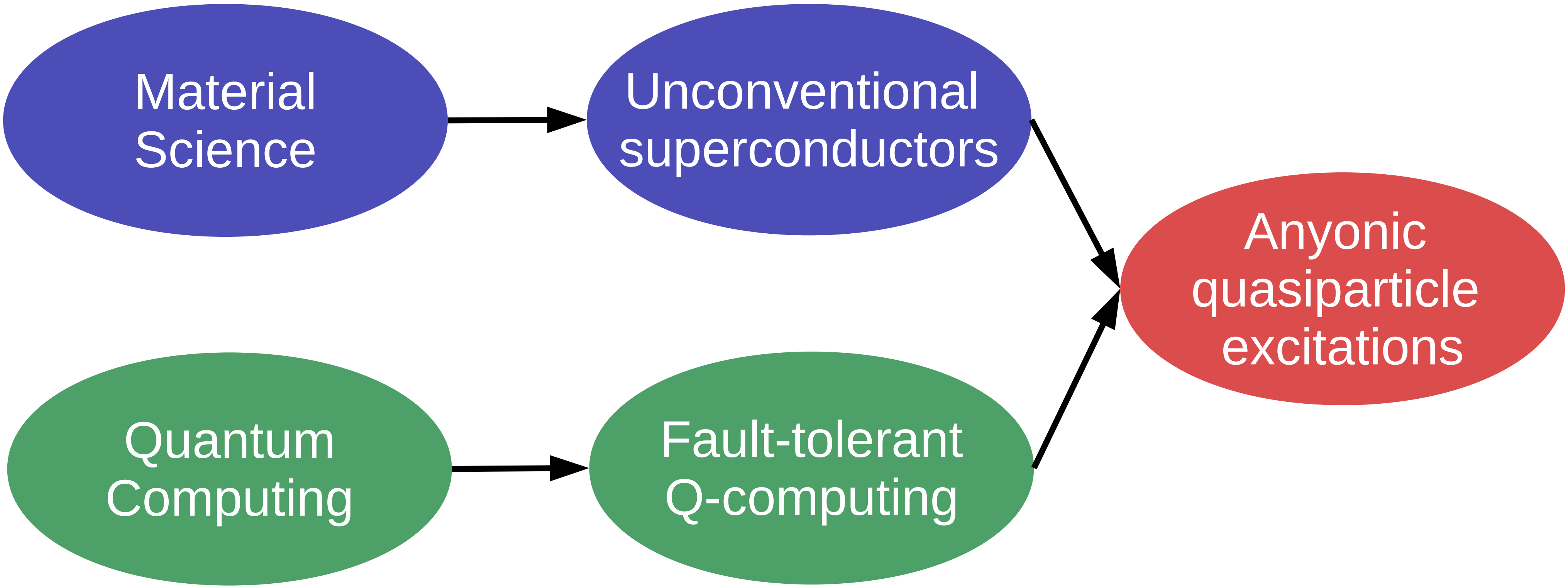
Material
Science

Quantum
Computing

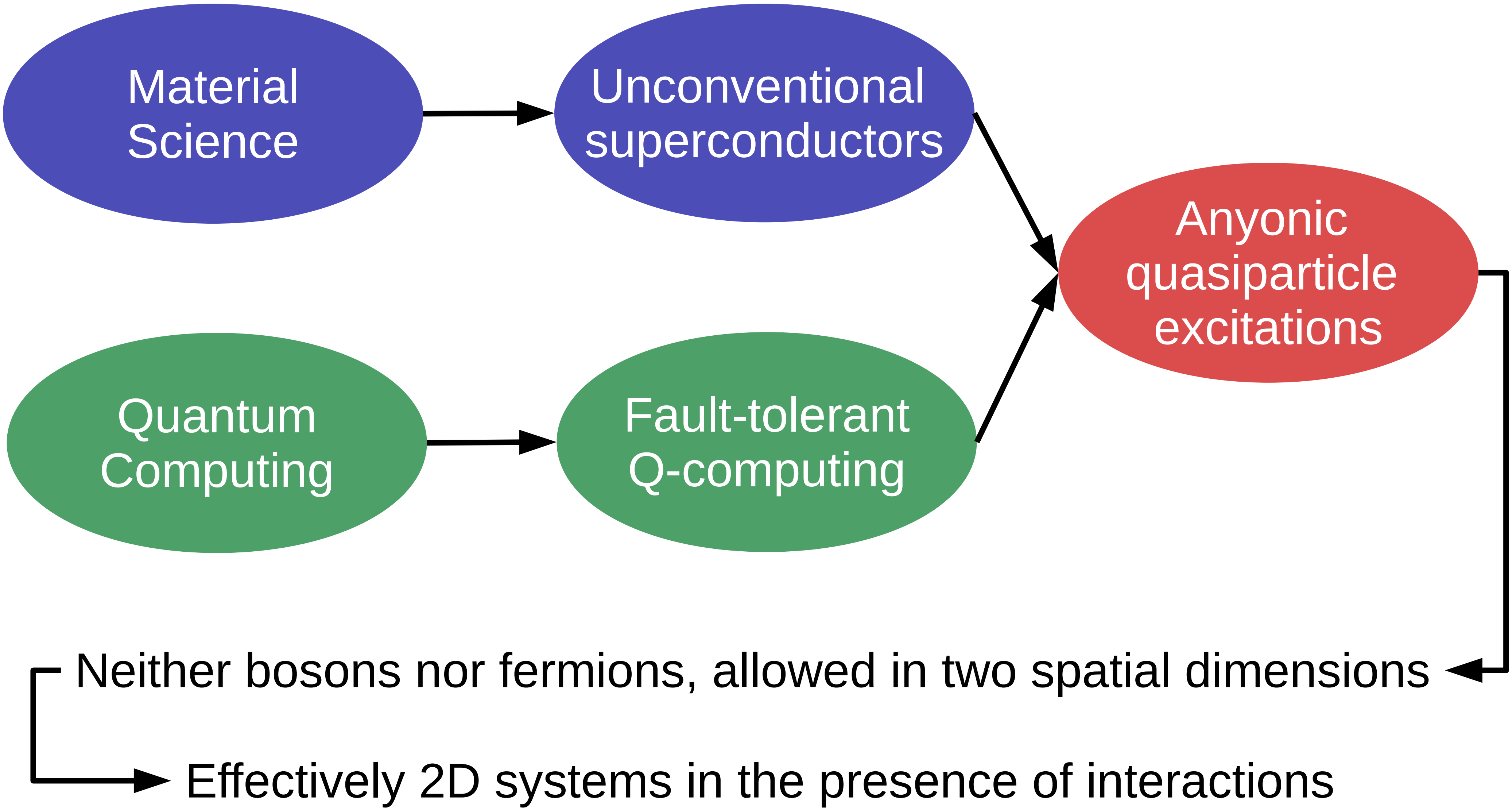
Flow towards topological superconductivity



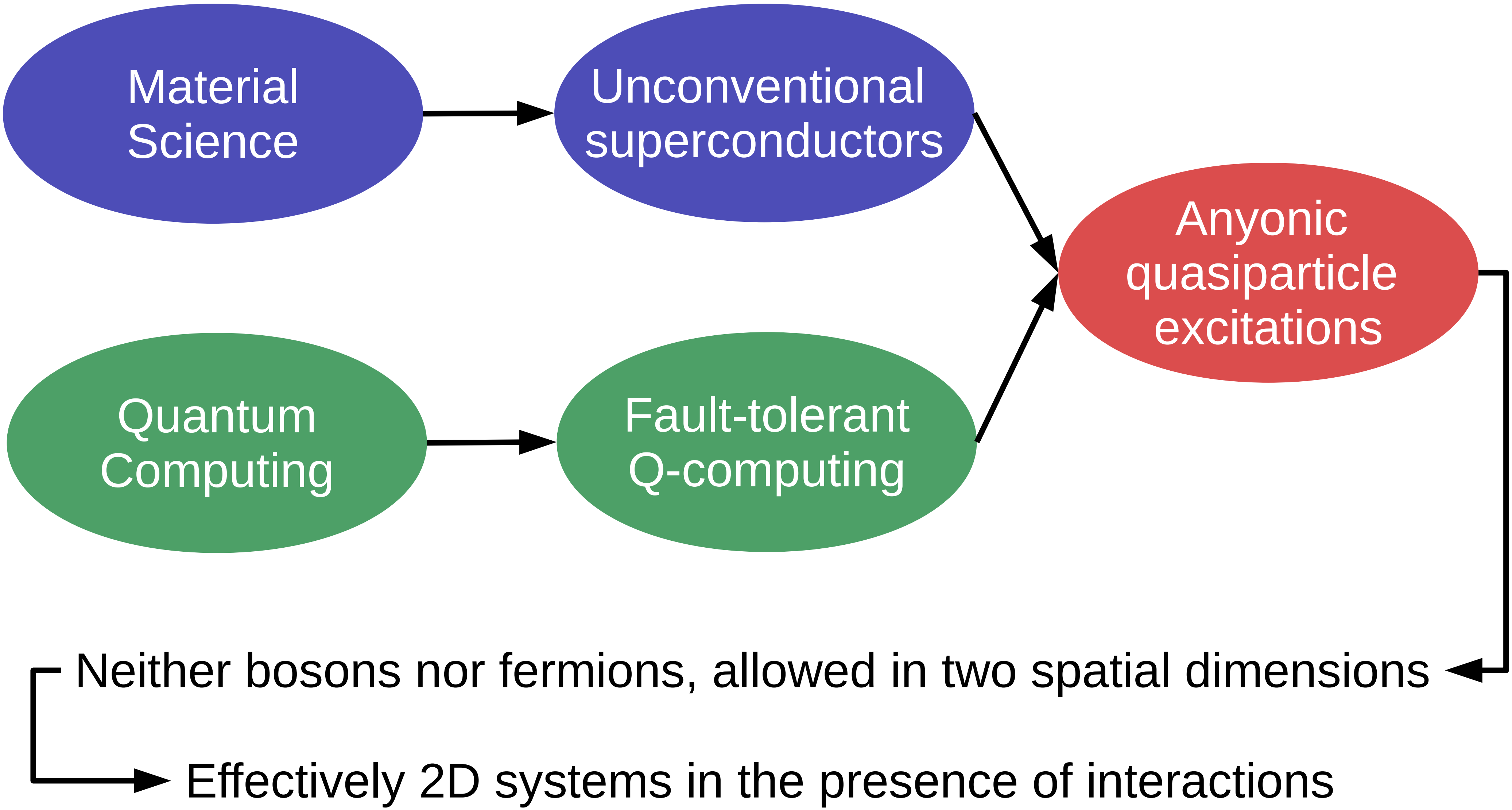
Flow towards topological superconductivity



Flow towards topological superconductivity



Flow towards topological superconductivity



Fractional quantum Hall systems and topological superconductors

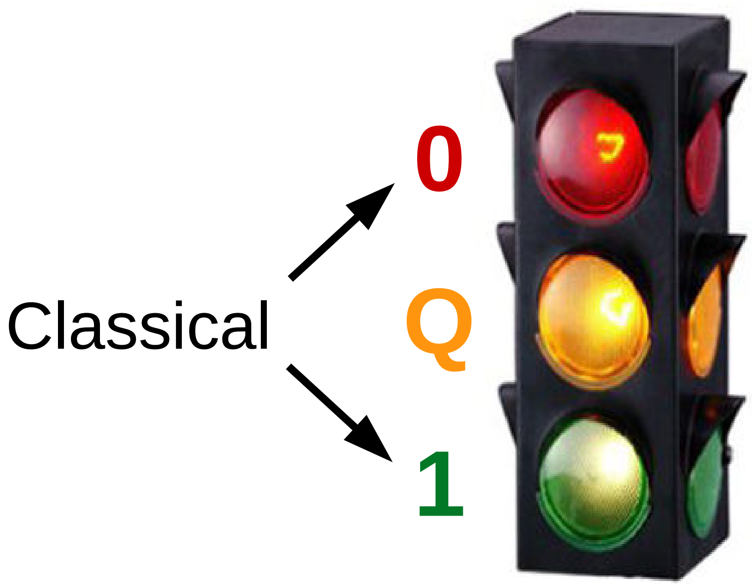
Classical vs Quantum computing

Classical

0

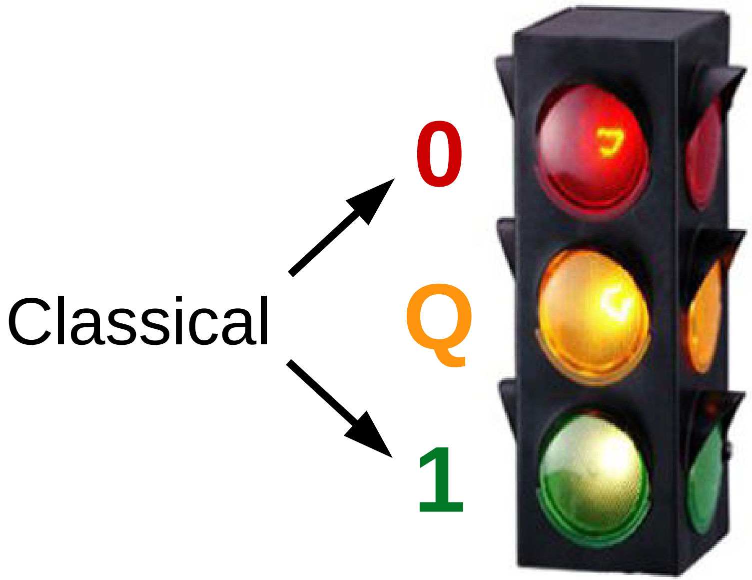
1

Classical vs Quantum computing



Superposition of quantum states

Classical vs Quantum computing



Superposition of quantum states

Candidate qubits:

Electron spin, photon polarization, atomic levels, superconducting devices

Quantum computing:

1. Single-qubit operations
2. Joint-qubit operations

Classical vs Quantum computing

Classical

0

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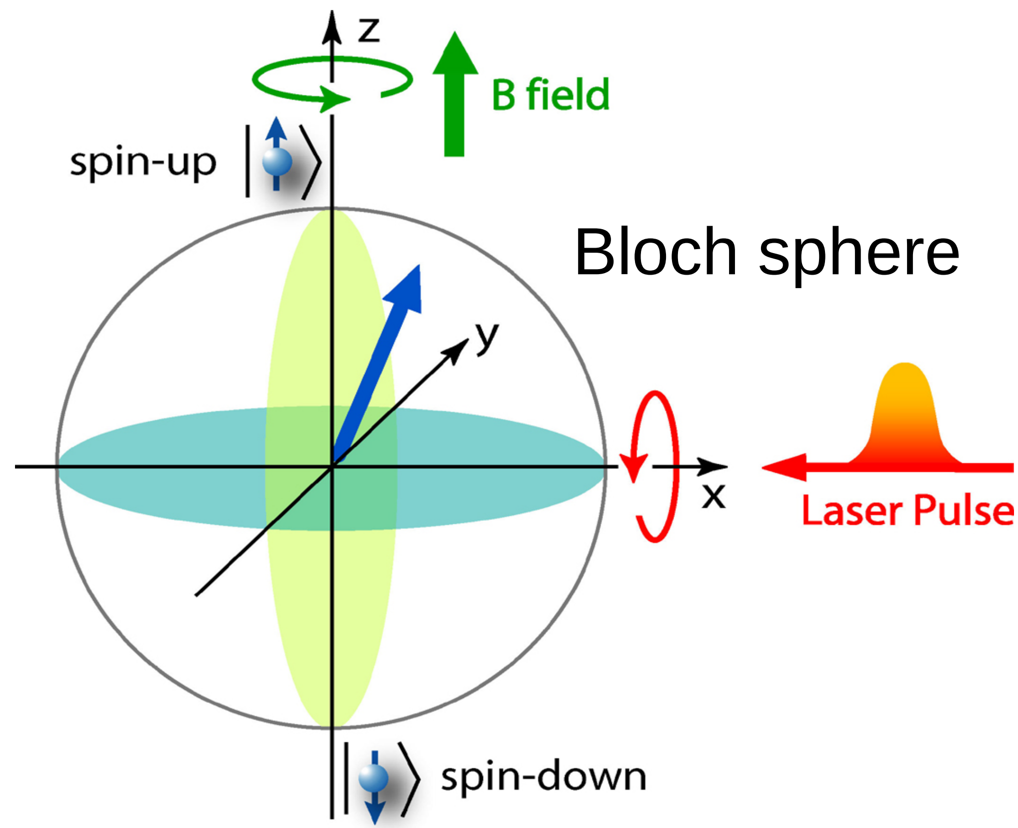
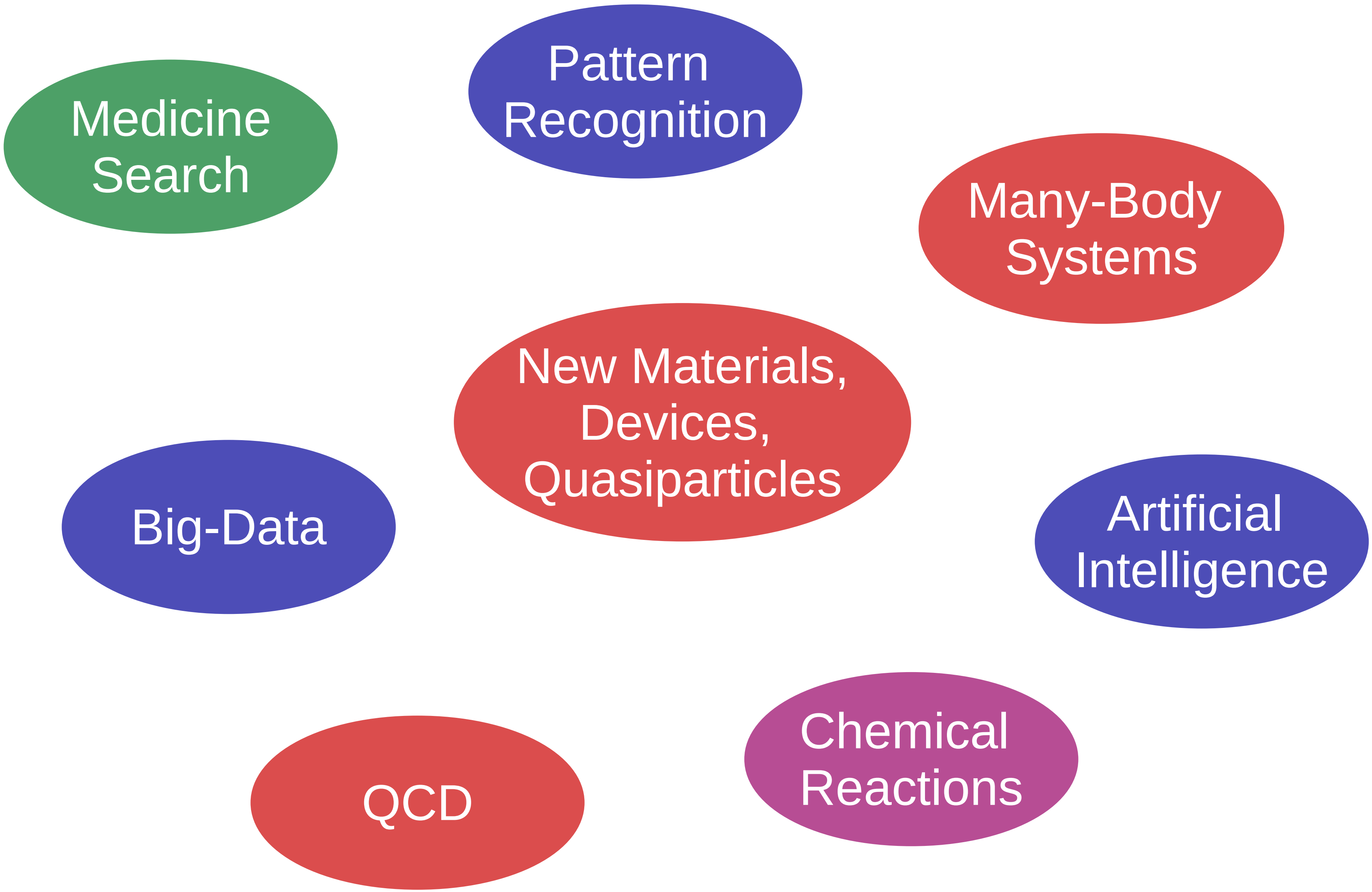


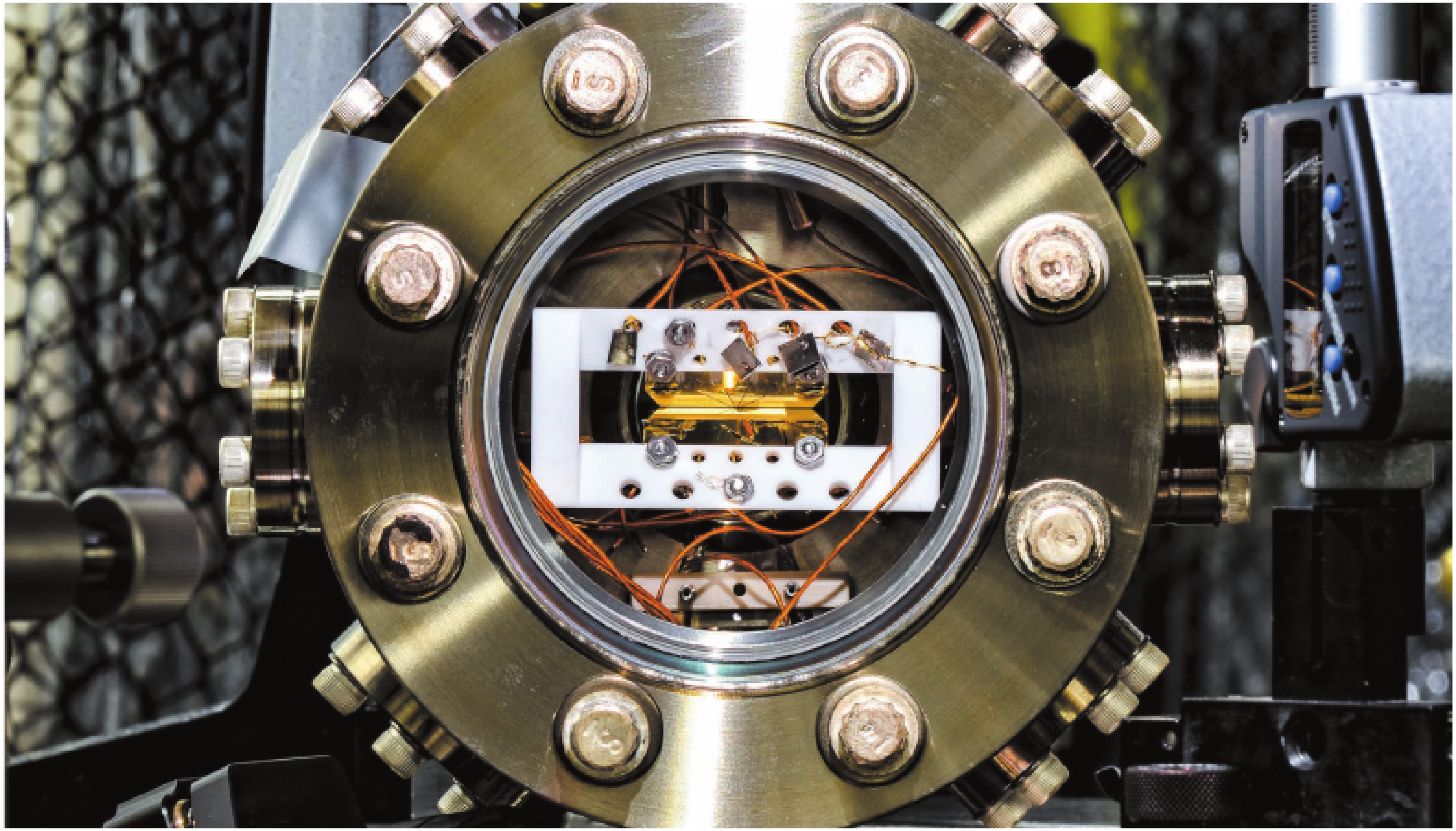
Quantum computing:

1. Single-qubit operations
2. Joint-qubit operations

The Quantum-Computer Promise



The Quantum-Computer Status

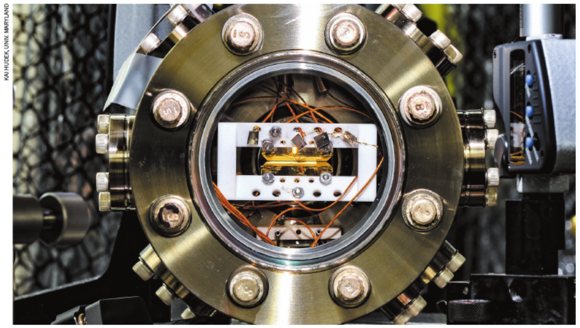


KAI HUDEK, UNIV. MARYLAND

Davide Castelvecchi Nature news 2017

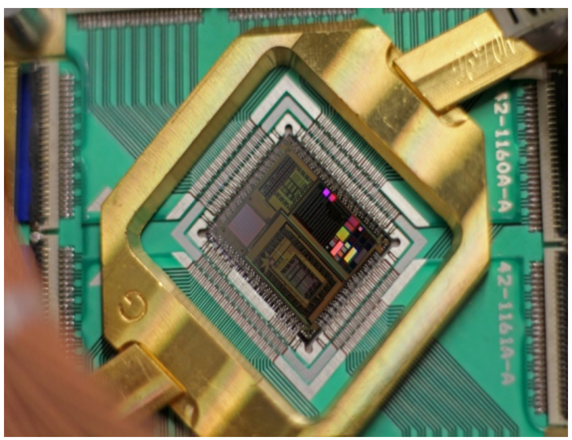
Quantum computers ready to leap out of the lab

The Quantum-Computer Status



Trapped ions

Monroe Maryland @ IonQ

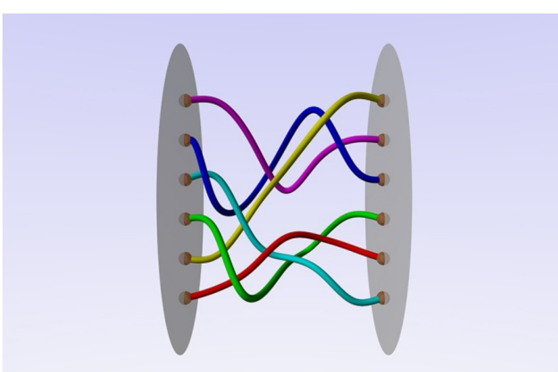


Superconducting qubits

Martinis UCSB @ Google

Schoelkopf Yale @ Quantum Circuits

Rigetti and IBM



Majorana qubits

Marcus and Kouwenhoven @ Microsoft

Topological vs conventional qubits

- Conventional qubits are vulnerable to noise and decoherence
  1. Superconducting qubits suffer from noise e.g. 1/f
  2. Spin qubits in quantum dots couple to nuclear spins

- Topological qubits are *in principle* immune to decoherence & noise

Topological vs conventional qubits

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  1. Superconducting qubits suffer from noise e.g. 1/f
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- Topological qubits are *in principle* immune to decoherence & noise

- Topological qubits rely on many-body ground-state degeneracies

Kitaev, Annals of Phys. (2003) and Nayak *et al.*, Rev. Mod. Phys. (2008)

$$\left.\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}\right|_{t>0} = \widehat{U}(t) \left.\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}\right|_{t=0}$$

Single qubit

Evolution operator

TQC and Anyon Braiding

$$\left(\begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right)\Bigg|_{t>0} = \widehat{U}(t) \left(\begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right)\Bigg|_{t=0}$$

Ground state degeneracy due to zero-energy quasiparticles

TQC and Anyon Braiding

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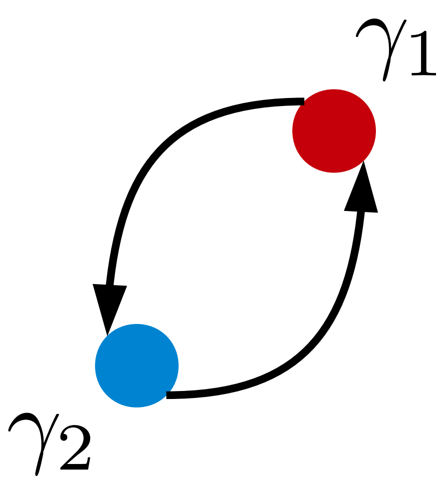

Spatial exchange

Topological quantum computing only if quasiparticles are anyons

Non-Abelian exchange statistics

$$\gamma_1 \mapsto +\gamma_2$$
$$\gamma_2 \mapsto -\gamma_1$$

Ising anyons

Majorana fermion quasiparticles

TQC and Anyon Braiding

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}\Bigg|_{t>0} = \widehat{U}(t) \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}\Bigg|_{t=0}$$

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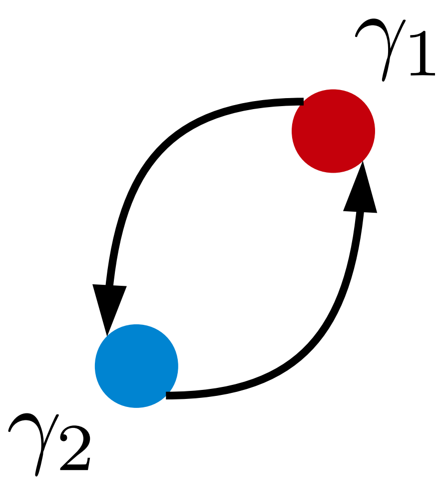
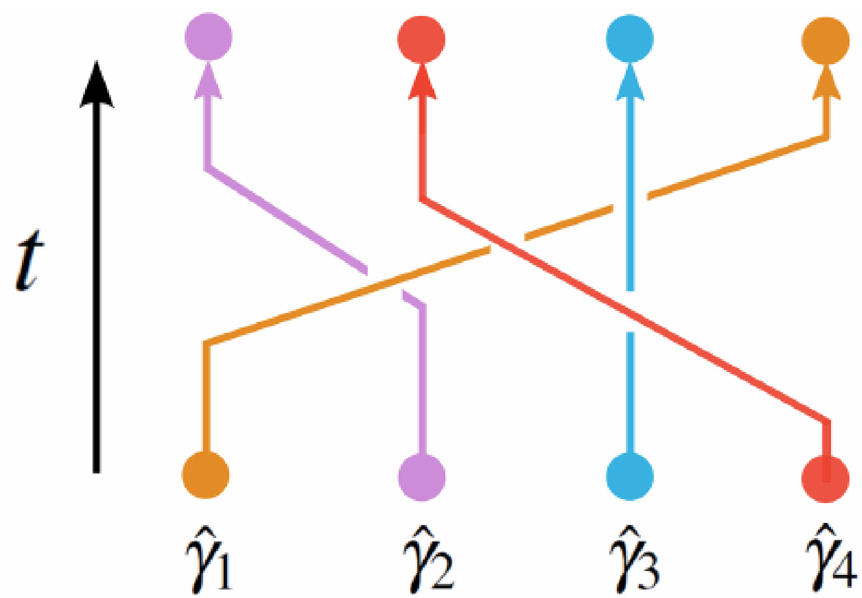


Spatial exchange

Logical qubit – Braiding



Majorana fermions

Ettore Majorana 1937

Self-conjugate solutions of the Dirac equation

$$\gamma_E = \gamma^\dagger_{-E}$$

Majorana fermions

Neutrino is the major
Majorana-particle candidate

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Ettore Majorana 1937

Hypothesis' test experiment

Ordinary vs Neutrinoless
double beta decay

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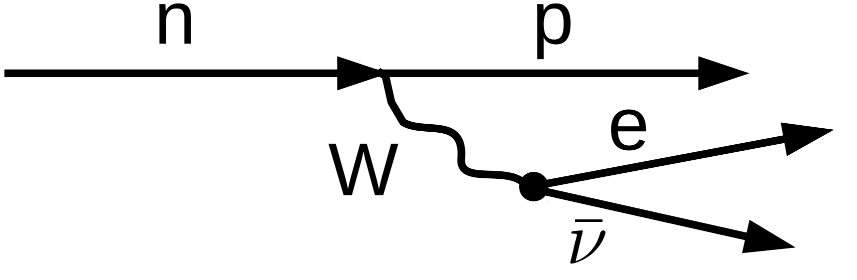

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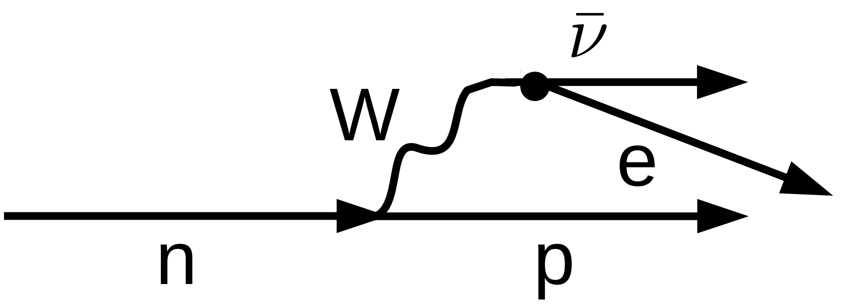

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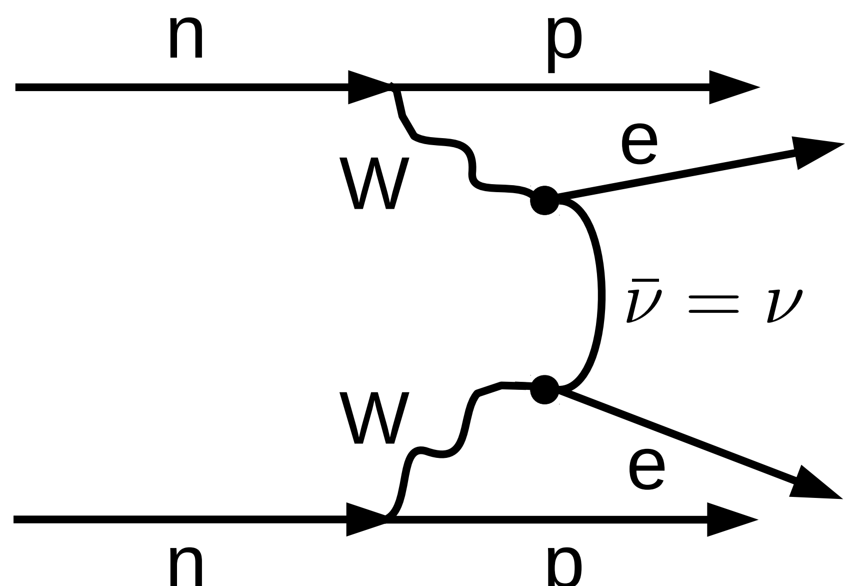
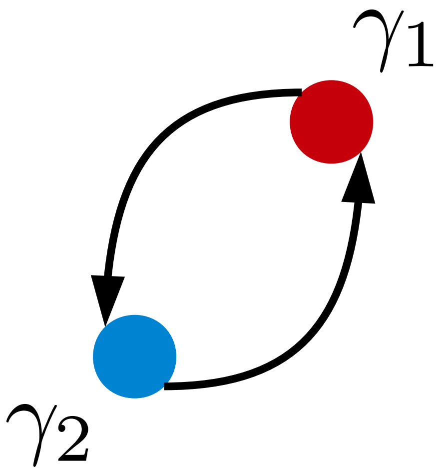


Majorana fermions @ zero energy

$$\gamma_s = \gamma_s^\dagger \qquad \{\gamma_s, \gamma_{s'}\} = \delta_{s,s'} \qquad 2\gamma_s^2 = 1$$

Ettore Majorana 1937



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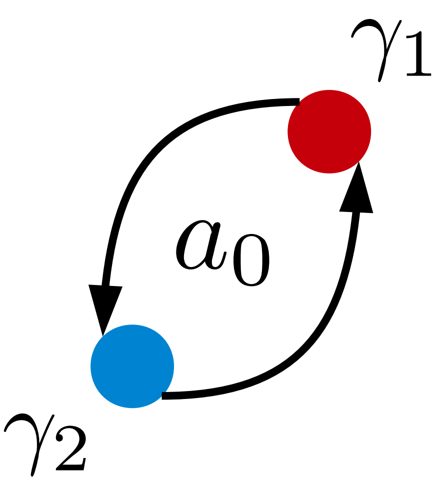
Ettore Majorana 1937

$$a_0 = \frac{\gamma_1 + i\gamma_2}{\sqrt{2}} \qquad a_0^2 = 0$$

Non-locality & protection

$$a_0^\dagger \left|0\right\rangle = \left|1\right\rangle$$



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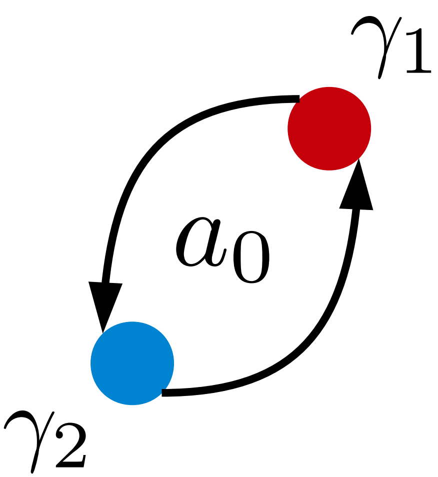

Braiding → Phase gate

$$\left|0\right\rangle \to e^{+i\pi/4} \left|0\right\rangle$$
$$\left|1\right\rangle \to e^{-i\pi/4} \left|1\right\rangle$$

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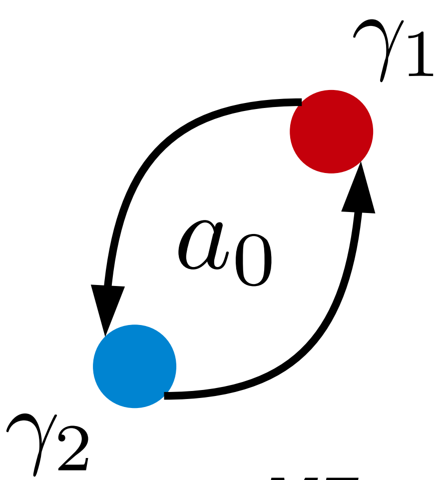

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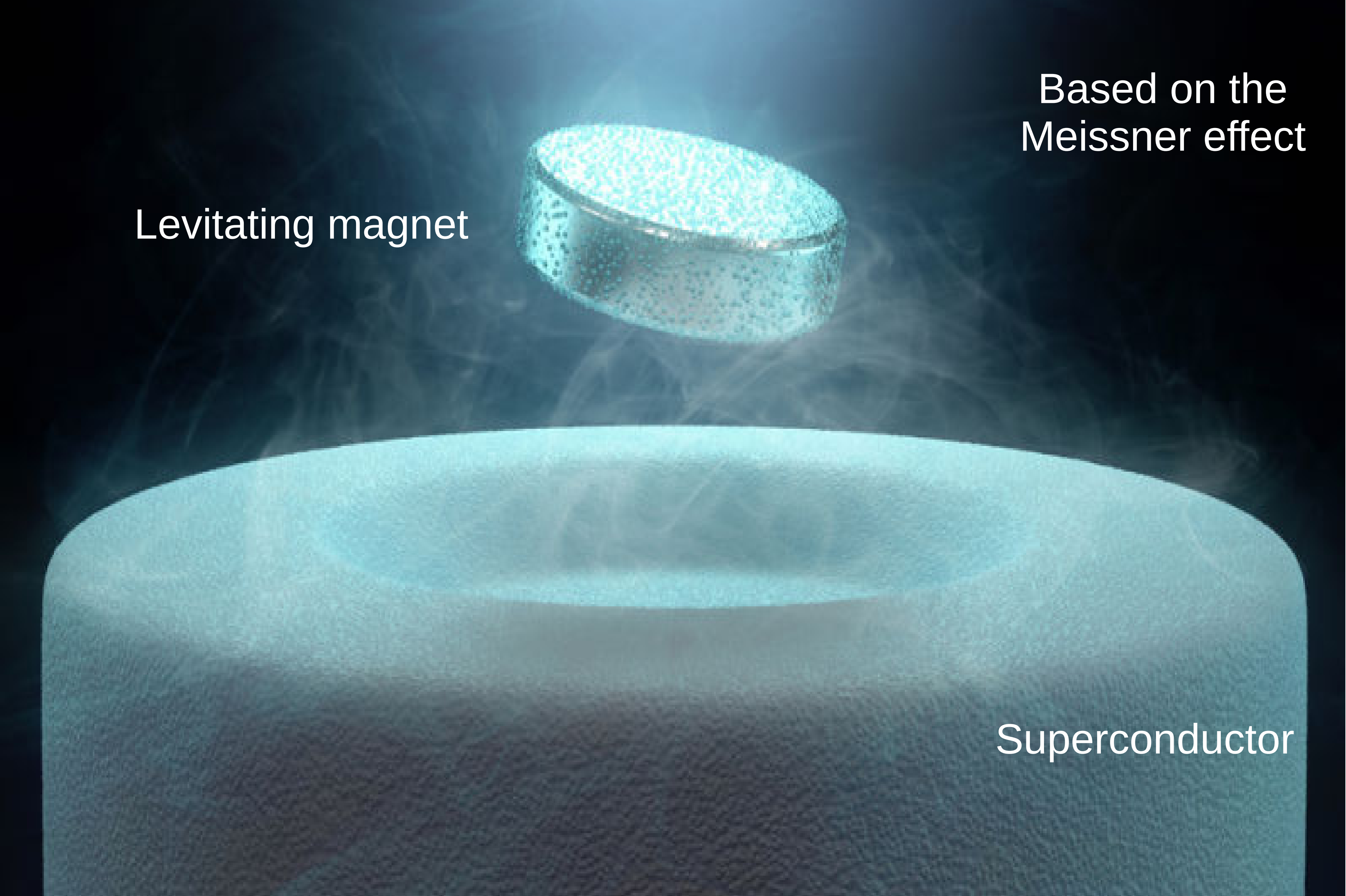
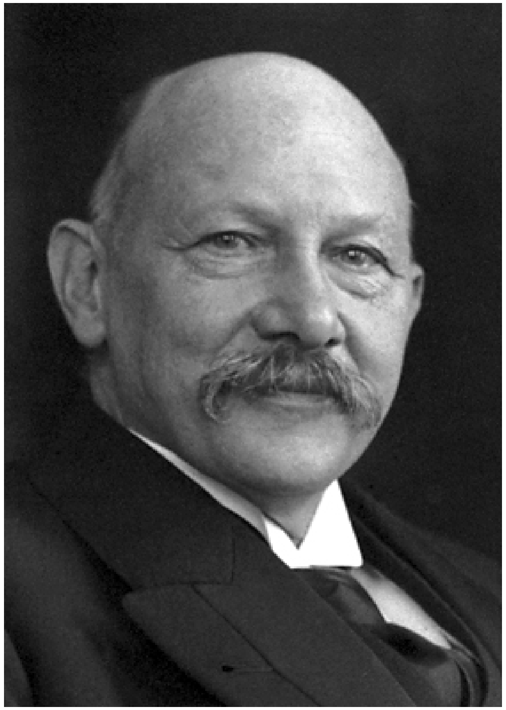
MF topological quantum computing is non-universal!

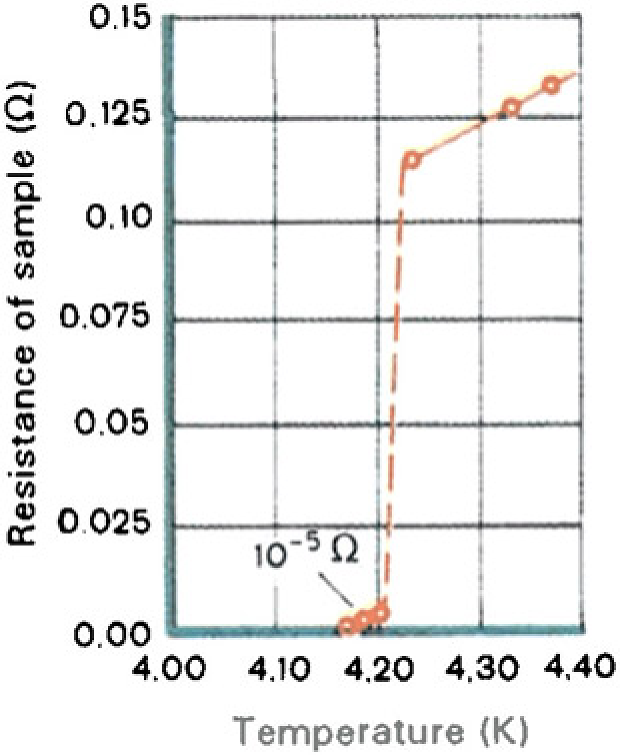
Superconductivity



Conventional superconductivity

Zero resistance below a critical temperature

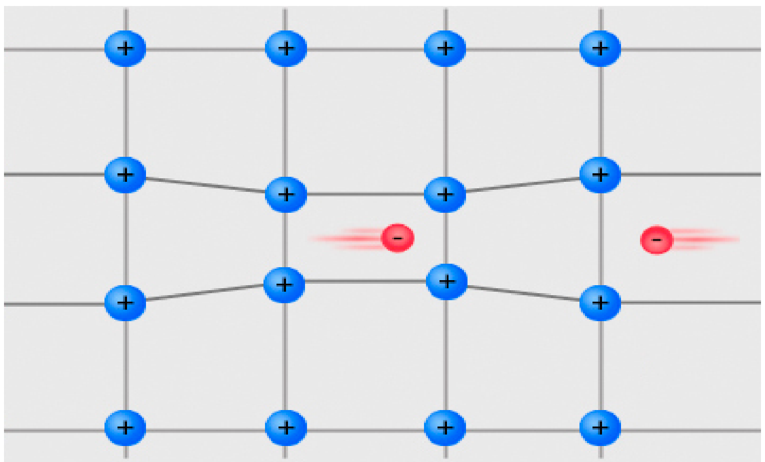


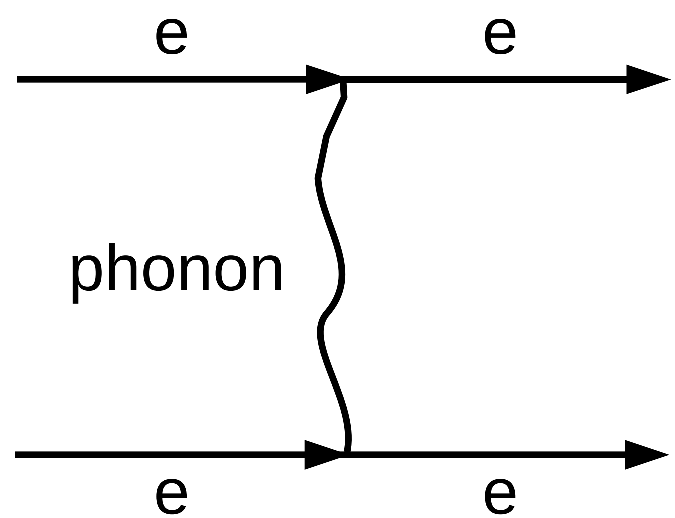


Kamerlingh Onnes 1911 in Hg

Conventional superconductivity

Bardeen – Cooper – Schrieffer theory 1957





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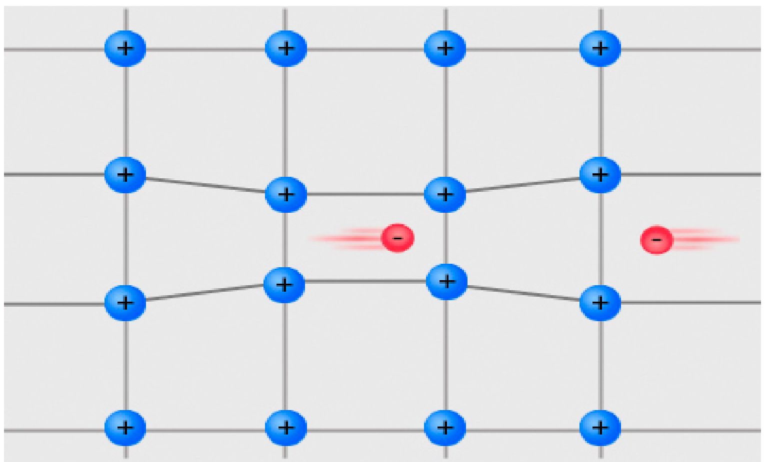
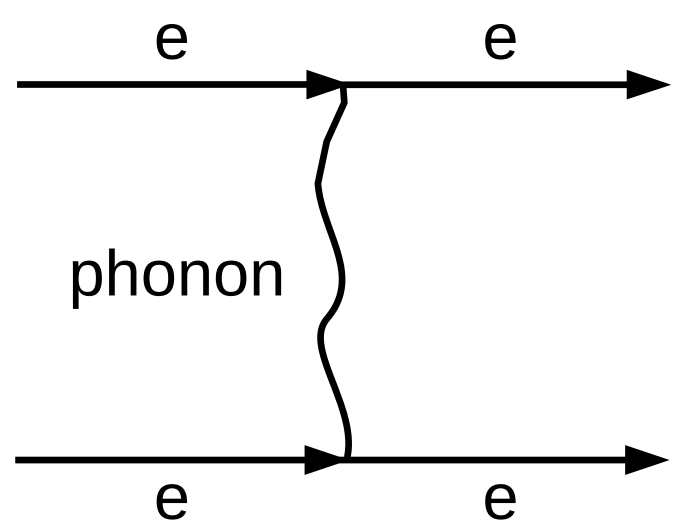
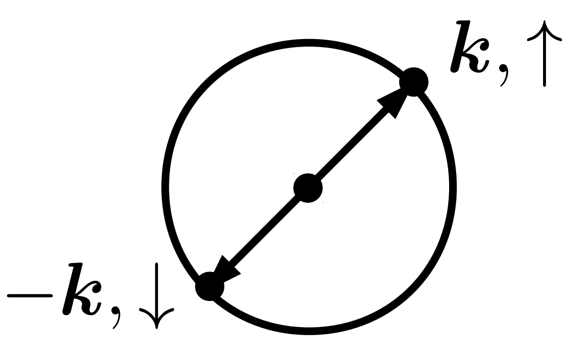




Effective attractive interaction for electrons near the Fermi level

$$\mathcal{V} = -g \sum_{\boldsymbol{k},\boldsymbol{k}'} \psi^\dagger_{\boldsymbol{k}\uparrow} \psi^\dagger_{-\boldsymbol{k}\downarrow} \psi_{-\boldsymbol{k}'\downarrow} \psi_{\boldsymbol{k}'\uparrow}$$



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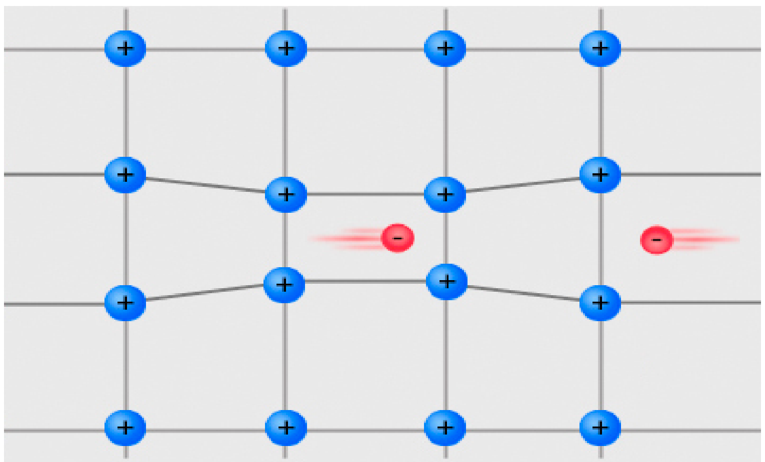

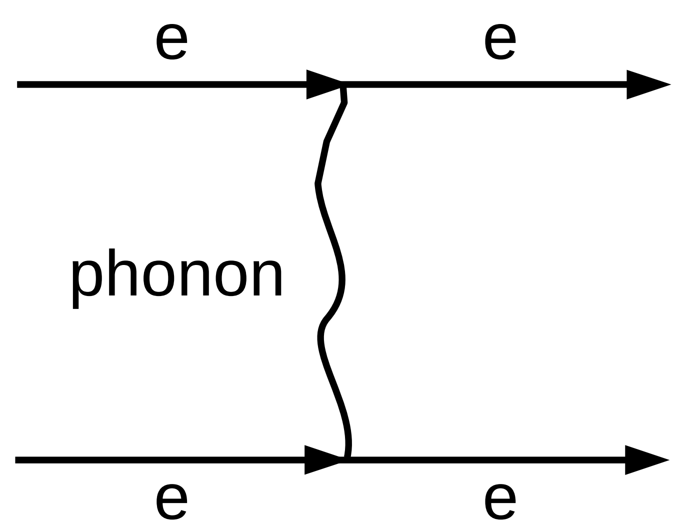

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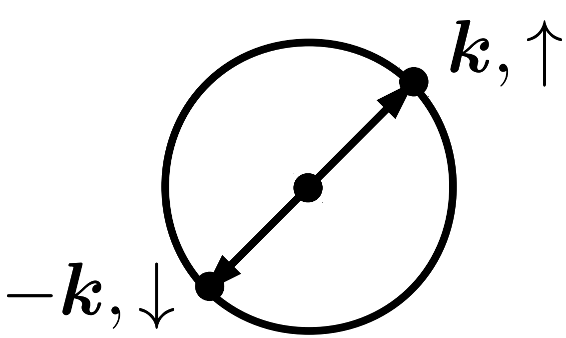

Spin-singlet Cooper pair $\langle \psi_{\boldsymbol{k}\uparrow} \psi_{-\boldsymbol{k}\downarrow} \rangle \neq 0$

Only the particle number modulo 2 - aka fermion parity - is conserved

Unconventional superconductivity

Additional symmetries are completely or partially broken

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Heavy fermions & High-Tc Cu- or Fe-based: $f_{\boldsymbol{k}} \sim \cos k_x \pm \cos k_y$

Unconventional superconductivity

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Spin-triplet Cooper pair $\langle f_{\boldsymbol{k}\alpha\beta} \psi_{\boldsymbol{k}\alpha} \psi_{-\boldsymbol{k}\beta} \rangle \neq 0$

Ruthenates, Organic superconductors & Bechgård salts, UGe2

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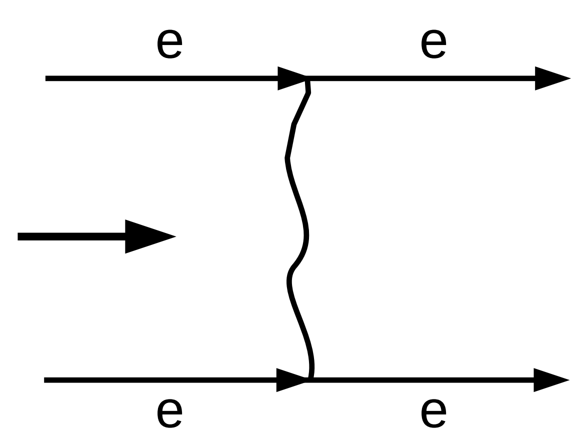
Different Cooper-pair glue

Collective excitations, e.g., (anti)ferromagnetic spin fluctuations



Mean-field decoupling in the Cooper channel

Assume the general case of a four-fermion interaction:

$$\mathcal{V} = -\frac{1}{2} \sum_{a,b,c,d} V_{abcd}\, \psi_a^\dagger\, \psi_b^\dagger\, \psi_c\, \psi_d \qquad V_{bacd} = V_{abdc} = -V_{abcd}$$

$$\psi_a^\dagger \psi_b^\dagger = < \psi_a^\dagger \psi_b^\dagger > + \underbrace{\psi_a^\dagger \psi_b^\dagger - < \psi_a^\dagger \psi_b^\dagger >}_{\text{deviation}}$$

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Apply mean-field theory→

$$\psi_a^\dagger \psi_b^\dagger \approx \langle \psi_a^\dagger \psi_b^\dagger \rangle + \underbrace{\psi_a^\dagger \psi_b^\dagger - \langle \psi_a^\dagger \psi_b^\dagger \rangle}_{\text{small}}$$

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$$\mathcal{V} \approx \frac{1}{2} \sum_{a,b,c,d} V_{abcd} < \psi_a^\dagger \psi_b^\dagger >< \psi_c \psi_d >$$

$$-\frac{1}{2} \sum_{a,b,c,d} V_{abcd} < \psi_c \psi_d > \psi_a^\dagger \psi_b^\dagger - \frac{1}{2} \sum_{a,b,c,d} V_{abcd} < \psi_a^\dagger \psi_b^\dagger > \psi_c \psi_d$$

Cooper pair order parameter: $$\Delta_{ab} = -\sum_{a,b,c,d} V_{abcd} < \psi_c \psi_d >$$

$$\Delta_{ba} = -\Delta_{ab}$$

Mean-field decoupling in the Cooper channel

Thus we obtain:

$$\mathcal{V} \approx \frac{1}{2} \sum_{a,b,c,d} \Delta^*_{ab} (V^{-1})_{abcd} \Delta_{dc} + \frac{1}{2} \sum_{a,b} \left(\psi^\dagger_a \Delta_{ab} \psi^\dagger_b + \psi_a \Delta^*_{ba} \psi_b \right)$$

with: $$\sum_{c,d} V_{abcd} (V^{-1})_{dckl} = \delta_{al} \delta_{bk}$$

Add free fermion Hamiltonian: $$\mathcal{H}_0 = \sum_{a,b} \psi^\dagger_a H_{0,ab} \psi_b$$

with:

$$H^*_{0,ab} = H_{0,ba}$$

Mean-field decoupling in the Cooper channel

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Obtain Bogoliubov – de Gennes Hamiltonian:

$$\mathcal{H}_{\mathrm{MF}} = \frac{1}{2} \sum_{a,b} \begin{pmatrix} \psi^\dagger_a & \psi_a \end{pmatrix} \underbrace{\begin{pmatrix} H_{0,ab} & \Delta_{ab} \\ -\Delta^*_{ab} & -H^*_{0,ab} \end{pmatrix}}_{\mathcal{H}_{\mathrm{BdG},ab}} \begin{pmatrix} \psi_b \\ \psi^\dagger_b \end{pmatrix}$$

Mean-field decoupling in the Cooper channel

electrons

$$\mathcal{H}_{\mathrm{BdG}} = \begin{pmatrix} H_0 & \Delta \\ -\Delta^* & -H_0^* \end{pmatrix}$$

holes

Pauli matrices: $\boldsymbol{\tau}$

$$= H_0^{\Re}\tau_z + H_0^{\Im} i\mathbf{1} + \Delta^{\Re} i\tau_y + \Delta^{\Im} i\tau_x$$

Charge-conjugation symmetry: $\Xi^\dagger \mathcal{H}_{\mathrm{BdG}} \Xi = -\mathcal{H}_{\mathrm{BdG}}$

Antiunitary operator: $\Xi = \tau_x \mathcal{K}$ ← complex conjugation

Mean-field decoupling in the Cooper channel

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The above implies that for every solution:

$$\mathcal{H}_{\mathrm{BdG}} \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix} = E_\nu \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix}$$ there is another one satisfying:

$$\mathcal{H}_{\mathrm{BdG}}\, \Xi \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix} = -E_{\nu^*} \Xi \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix}$$ with $\Xi \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix} = \begin{pmatrix} v_\nu^* \\ u_\nu^* \end{pmatrix}$

Spin- singlet vs triplet superconductivity

Assume the electron quantum numbers: wave vector & spin

$$\Delta^{\alpha\beta}_{\boldsymbol{k},-\boldsymbol{k}+\boldsymbol{q}} = -\sum_{\boldsymbol{k'}} V^{\alpha\beta\gamma\delta}_{\boldsymbol{k},\boldsymbol{k'},\boldsymbol{q}} < \psi_{-\boldsymbol{k'}+\boldsymbol{q},\gamma}\psi_{\boldsymbol{k'},\delta} >$$

$$\Delta^{\alpha\beta}_{\boldsymbol{k},-\boldsymbol{k}+\boldsymbol{q}} = -\Delta^{\beta\alpha}_{-\boldsymbol{k}+\boldsymbol{q},\boldsymbol{k}}$$

Finite center-of-mass momentum for the Cooper pair

Pair density wave

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Spin-singlet: $\Delta^{\uparrow\downarrow}_{\boldsymbol{k}} = -\Delta^{\downarrow\uparrow}_{-\boldsymbol{k}} = \Delta^{\uparrow\downarrow}_{-\boldsymbol{k}} \sim \Delta_{\boldsymbol{k}} i\sigma_y \qquad \Delta_{-\boldsymbol{k}} = \Delta_{\boldsymbol{k}}$

Spin-triplet: $\Delta^{\alpha\beta}_{\boldsymbol{k}} = -\Delta^{\beta\alpha}_{-\boldsymbol{k}} = -\Delta^{\alpha\beta}_{-\boldsymbol{k}} \sim \boldsymbol{d}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} i\sigma_y \qquad \boldsymbol{d}_{-\boldsymbol{k}} = -\boldsymbol{d}_{\boldsymbol{k}}$

Basic types of spin-triplet superconductivity

Helical spin-triplet superconductivity in 3D

Time-reversal symmetry $\widehat{\mathcal{T}} = i\sigma_y \widehat{\mathcal{K}}$

$$\mathcal{H}_{\mathrm{BdG}}(\hat{\boldsymbol{p}}) = \Delta \hat{\boldsymbol{p}} \cdot \tau_x \boldsymbol{\sigma} + \left(\frac{\hat{\boldsymbol{p}}^2}{2m} - E_F \right) \tau_z$$

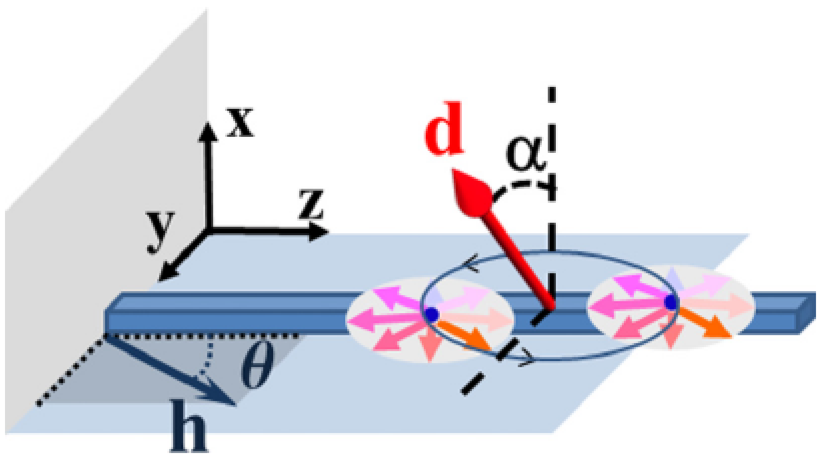
Akin to the massless Dirac Hamiltonian

Chiral spin-triplet superconductivity in 2D

$$\mathcal{H}_{\mathrm{BdG}}(\hat{\boldsymbol{p}}) = \Delta \left(\hat{p}_x \tau_y + \hat{p}_y \tau_x \right) + \left(\frac{\hat{\boldsymbol{p}}^2}{2m} - E_F \right) \tau_z$$

Broken time-reversal symmetry

1D p-wave superconductor in a magnetic field



Assume that: $\boldsymbol{d}_k = \boldsymbol{d} \sin k$

$$\mathcal{H}_{\mathrm{MF}} = \frac{1}{2} \sum_k \Psi_k^\dagger \begin{pmatrix} \varepsilon_k - \boldsymbol{h} \cdot \boldsymbol{\sigma} & \boldsymbol{d}_k \cdot \boldsymbol{\sigma} \\ \boldsymbol{d}_k^* \cdot \boldsymbol{\sigma} & -\varepsilon_k - \boldsymbol{h} \cdot \boldsymbol{\sigma} \end{pmatrix} \Psi_k$$

with: $\Psi_k^\dagger = \begin{pmatrix} \psi_{k\uparrow}^\dagger & \psi_{k\downarrow}^\dagger & \psi_{-k\downarrow} & -\psi_{-k\uparrow} \end{pmatrix}$

We neglect the spin-orbit coupling

1D p-wave superconductor in a magnetic field

$$\mathcal{H}_{\mathrm{MF}} = \frac{1}{2}\sum_k \Psi_k^\dagger \begin{pmatrix} \varepsilon_k - \boldsymbol{h}\cdot\boldsymbol{\sigma} & \boldsymbol{d}_k\cdot\boldsymbol{\sigma} \\ \boldsymbol{d}_k^*\cdot\boldsymbol{\sigma} & -\varepsilon_k - \boldsymbol{h}\cdot\boldsymbol{\sigma} \end{pmatrix} \Psi_k$$

$$E_k = \pm\sqrt{\varepsilon_k^2 + \boldsymbol{h}^2 + |\boldsymbol{d}_k|^2 \pm 2\sqrt{|\boldsymbol{d}_k\cdot\boldsymbol{h}|^2 + (\varepsilon_k\boldsymbol{h} - i\boldsymbol{d}_k\times\boldsymbol{d}_k^*/2)^2}}$$

$$i(\boldsymbol{d}\times\boldsymbol{d}^*)\cdot\boldsymbol{h}\,\varepsilon_k\sin^2 k$$

Sigrist and Ueda RMP 1991

Hyart, Wright, Rosenow PRB 2014

Mercaldo, Cuoco and ***PK*** PRB 2016

Magnetization:

$$\boldsymbol{M} \sim i\boldsymbol{d}\times\boldsymbol{d}^*$$

Landau approach @ zero temperature

The total energy of the system reads:

$$E = \frac{|\boldsymbol{d}|^2}{V} + \frac{1}{2}\sum_{\nu}^{occ} E_{k,\nu}(\boldsymbol{d}, \boldsymbol{d}^*)$$

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By expanding up to quartic order in the order parameters:

$$F(\boldsymbol{d}, \boldsymbol{d}^*) = \left(\frac{1}{V} - \chi_0\right)|\boldsymbol{d}|^2 + \beta|\boldsymbol{d}|^4 - \beta'|\boldsymbol{d} \times \boldsymbol{d}^*|^2$$

$$+ gi\,(\boldsymbol{d} \times \boldsymbol{d}^*) \cdot \boldsymbol{h} + g'|\boldsymbol{d} \cdot \boldsymbol{h}|^2$$

Hyart, Wright, Rosenow
PRB 2014

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$$+ gi\,(\boldsymbol{d} \times \boldsymbol{d}^*) \cdot \boldsymbol{h} + g'|\boldsymbol{d} \cdot \boldsymbol{h}|^2$$

Hyart, Wright, Rosenow
PRB 2014

The $\boldsymbol{d}$-vector reorganizes so to maximize: $|\boldsymbol{h} \times \boldsymbol{d}|$ and $|\boldsymbol{d} \times \boldsymbol{d}^*|$

Mercaldo, Cuoco and ***PK*** PRB 2016

Spinless limit

For a strong Zeeman field along the z axis one obtains:

$$\boldsymbol{d} = d(-i, 1, 0)$$

$$\boldsymbol{d} \cdot \boldsymbol{\sigma} i\sigma_y \sim i\frac{1+\sigma_z}{2} \rightarrow \psi^{\dagger}_{k\uparrow}\psi^{\dagger}_{-k\uparrow}$$

Equal spin-pairing state ~ previously discussed in superfluid ^{3}He

Spinless Kitaev chain

A.Y. Kitaev, Phys.-Usp. 2001

chemical potential **n.n. hopping** **p-wave SC**

$$\mathcal{H}_\uparrow = \sum_n \left(-\mu_\uparrow c^\dagger_{n\uparrow} c_{n\uparrow} + t_\uparrow c^\dagger_{n\uparrow} c_{n+1\uparrow} + \Delta_\uparrow c^\dagger_{n\uparrow} c^\dagger_{n+1\uparrow} + \mathrm{h.c.} \right)$$

$c_{n\uparrow}$

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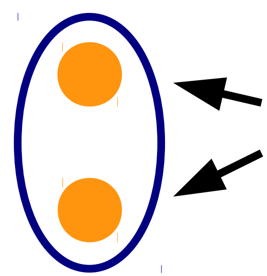
$c_{n\uparrow}$

$$c_{n\uparrow} = \frac{\gamma_{n\uparrow} + i\tilde{\gamma}_{n\uparrow}}{\sqrt{2}} = \begin{array}{c} \gamma_{n\uparrow} \\ \tilde{\gamma}_{n\uparrow} \end{array}$$



MFs

Reality condition: $\gamma^\dagger_{n\uparrow} = \gamma_{n\uparrow}$ **and** $\tilde{\gamma}^\dagger_{n\uparrow} = \tilde{\gamma}_{n\uparrow}$

Spinless Kitaev chain

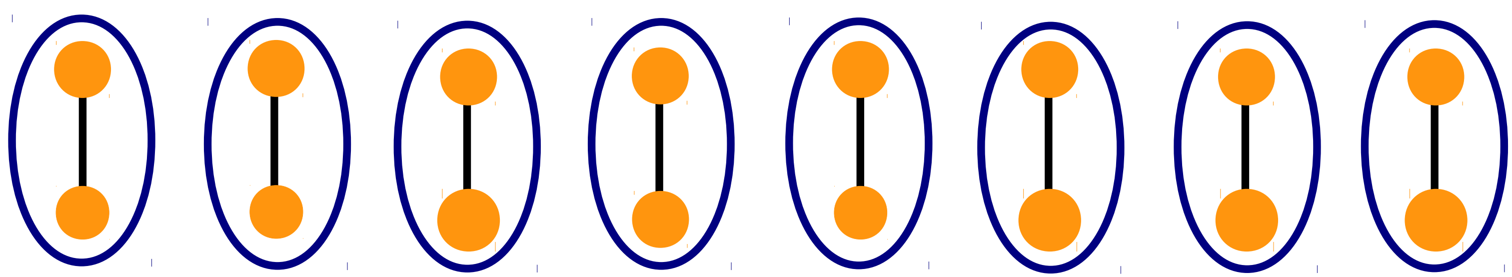
Topologically trivial phase

$$\mathcal{H}_{\uparrow} = \sum_n \left(-\mu_{\uparrow} c^{\dagger}_{n\uparrow} c_{n\uparrow} + \cancel{t_{\uparrow} c^{\dagger}_{n\uparrow} c_{n+1\uparrow}} + \cancel{\Delta_{\uparrow} c^{\dagger}_{n\uparrow} c^{\dagger}_{n+1\uparrow}} + \text{h.c.} \right)$$

$c_{n\uparrow}$

$\gamma_{n\uparrow}$

$\tilde{\gamma}_{n\uparrow}$

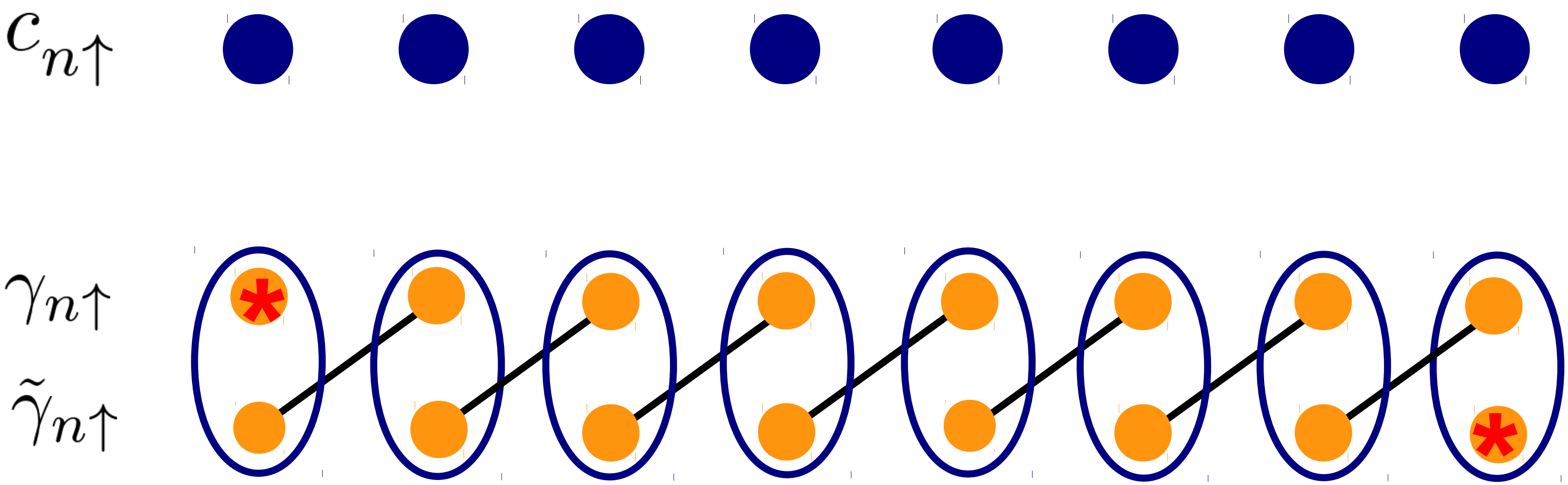


Spinless Kitaev chain

Topologically non-trivial phase

$$t_\uparrow = \Delta_\uparrow$$

$$\mathcal{H}_\uparrow = \sum_n \left(\cancel{-\mu_\uparrow c^\dagger_{n\uparrow} c_{n\uparrow}} + t_\uparrow c^\dagger_{n\uparrow} c_{n+1\uparrow} + \Delta_\uparrow c^\dagger_{n\uparrow} c^\dagger_{n+1\uparrow} + \mathrm{h.c.} \right)$$



Kitaev's unpaired MFs *

Spinless Kitaev chain: General case

chemical potential **n.n. hopping** **p-wave SC**

$$\mathcal{H}_\uparrow = \sum_n \left(-\mu_\uparrow c^\dagger_{n\uparrow} c_{n\uparrow} + t_\uparrow c^\dagger_{n\uparrow} c_{n+1\uparrow} + \Delta_\uparrow c^\dagger_{n\uparrow} c^\dagger_{n+1\uparrow} + \mathrm{h.c.} \right)$$

$c_{n\uparrow}$ 

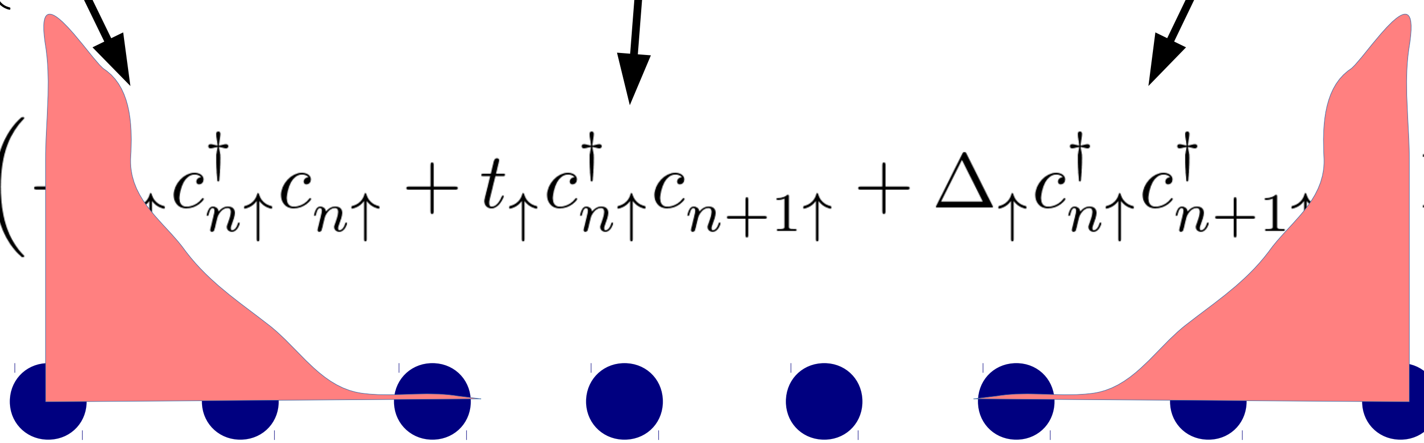
Solve Bogoliubov – de Gennes equation:

$$\begin{pmatrix} c_{n\uparrow} \\ c^\dagger_{n\uparrow} \end{pmatrix} = \sum_s^{E_s \neq 0} \begin{pmatrix} u_{ns} & v^*_{ns} \\ v_{ns} & u^*_{ns} \end{pmatrix} \begin{pmatrix} a_{s\uparrow} \\ a^\dagger_{s\uparrow} \end{pmatrix} + \sum_s^{E_s = 0} \begin{pmatrix} u_{ns} \\ v_{ns} \end{pmatrix} \gamma_{s\uparrow}$$

Spinless Kitaev chain: General case

chemical potential **n.n. hopping** **p-wave SC**

$$\mathcal{H}_\uparrow = \sum_n \left(-\mu_\uparrow c^\dagger_{n\uparrow} c_{n\uparrow} + t_\uparrow c^\dagger_{n\uparrow} c_{n+1\uparrow} + \Delta_\uparrow c^\dagger_{n\uparrow} c^\dagger_{n+1\uparrow} + \text{h.c.} \right)$$



Solve Bogoliubov – de Gennes equation:

Majorana fermions

$$\begin{pmatrix} c_{n\uparrow} \\ c^\dagger_{n\uparrow} \end{pmatrix} = \sum_s^{E_s \neq 0} \begin{pmatrix} u_{ns} & v^*_{ns} \\ v_{ns} & u^*_{ns} \end{pmatrix} \begin{pmatrix} a_{s\uparrow} \\ a^\dagger_{s\uparrow} \end{pmatrix} + \sum_s^{E_s = 0} \begin{pmatrix} u_{ns} \\ v_{ns} \end{pmatrix} \gamma_{s\uparrow}$$

Ground-state: two-fold degenerate: $\gamma_{1,2\uparrow}$ $|2t_\uparrow| > |\mu_\uparrow|$

Degenerate ground state

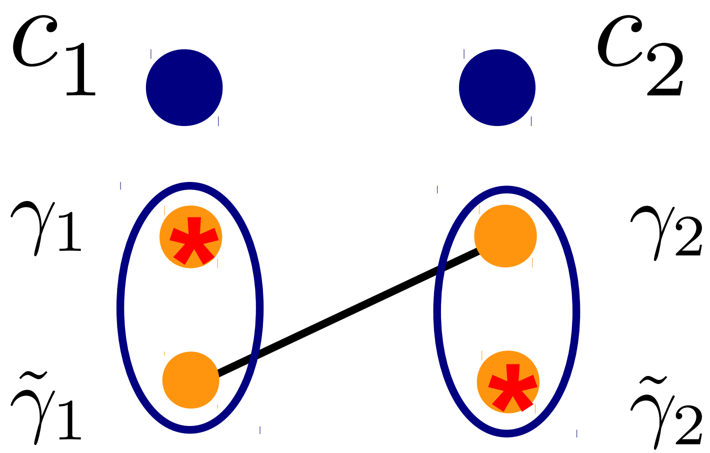
$$a_s \left|0\right\rangle = 0$$

$$\left|0\right\rangle = a_0 \prod_{s}^{E_s \neq 0} a_s \left|\mathrm{vac}\right\rangle$$

Degenerate ground state

$$a_s\,|0\rangle = 0 \qquad\qquad |0\rangle = a_0 \prod_s^{E_s \neq 0} a_s\,|\mathrm{vac}\rangle$$

@ the sweet spot:



Degenerate ground state

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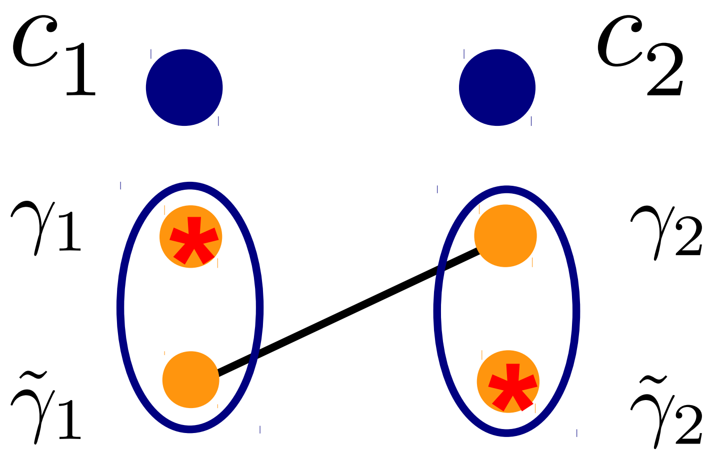

$$a_E = \frac{\gamma_2 - i\tilde{\gamma}_1}{\sqrt{2}} \qquad a_0 = \frac{\gamma_1 + i\tilde{\gamma}_2}{\sqrt{2}}$$

Degenerate ground state and braiding

$$|0\rangle = \frac{1 + c_1^\dagger c_2^\dagger}{\sqrt{2}} |\mathrm{vac}\rangle \qquad |1\rangle = a_0^\dagger |0\rangle = \frac{c_1^\dagger + c_2^\dagger}{\sqrt{2}} |\mathrm{vac}\rangle$$

$$\widehat{U}^\dagger \begin{pmatrix} \gamma_1 \\ \tilde{\gamma}_2 \end{pmatrix} \widehat{U} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \tilde{\gamma}_2 \end{pmatrix}$$

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Similar to a rotation matrix: $\mathcal{R}_{\hat{\boldsymbol{n}}}^{\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\{\gamma_1, \tilde{\gamma}_2\} = 0 \qquad (\sqrt{2}\gamma_1)^2 = (\sqrt{2}\tilde{\gamma}_2)^2 = 1$$

$$\{\sigma_x, \sigma_y\} = 0 \qquad \sigma_{x,y}^2 = 1$$

Braiding operator

$$[\widehat{\mathcal{R}}_{\hat{\boldsymbol{z}}}^{\pi/2}]^\dagger \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} \widehat{\mathcal{R}}_{\hat{\boldsymbol{z}}}^{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix}$$

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$$\widehat{\mathcal{R}}_{\hat{\boldsymbol{z}}}^{\theta} = e^{-i\theta S_z/\hbar} = e^{-i\theta\sigma_z/2} \qquad \widehat{\mathcal{R}}_{\hat{\boldsymbol{z}}}^{\pi/2} = e^{-i\pi\sigma_z/4}$$

$$\sigma_z = -i\sigma_x\sigma_y \equiv -i2\gamma_1\tilde{\gamma}_2 = -(a_0^{\dagger}a_0 - a_0a_0^{\dagger})$$

Braiding operator

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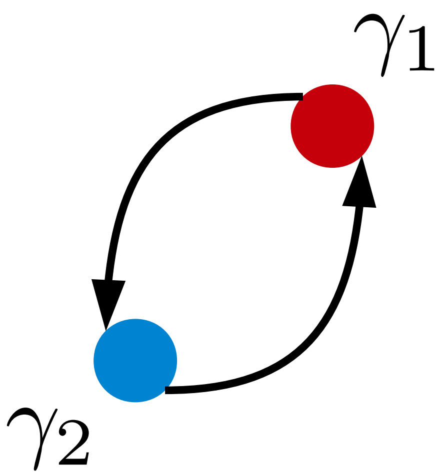
$$\sigma_z = -i\sigma_x\sigma_y \equiv -i2\gamma_1\tilde{\gamma}_2 = -(a_0^\dagger a_0 - a_0 a_0^\dagger)$$

$$(a_0^\dagger a_0 - a_0 a_0^\dagger)\,|1\rangle = +1\,|1\rangle$$
$$(a_0^\dagger a_0 - a_0 a_0^\dagger)\,|0\rangle = -1\,|0\rangle$$

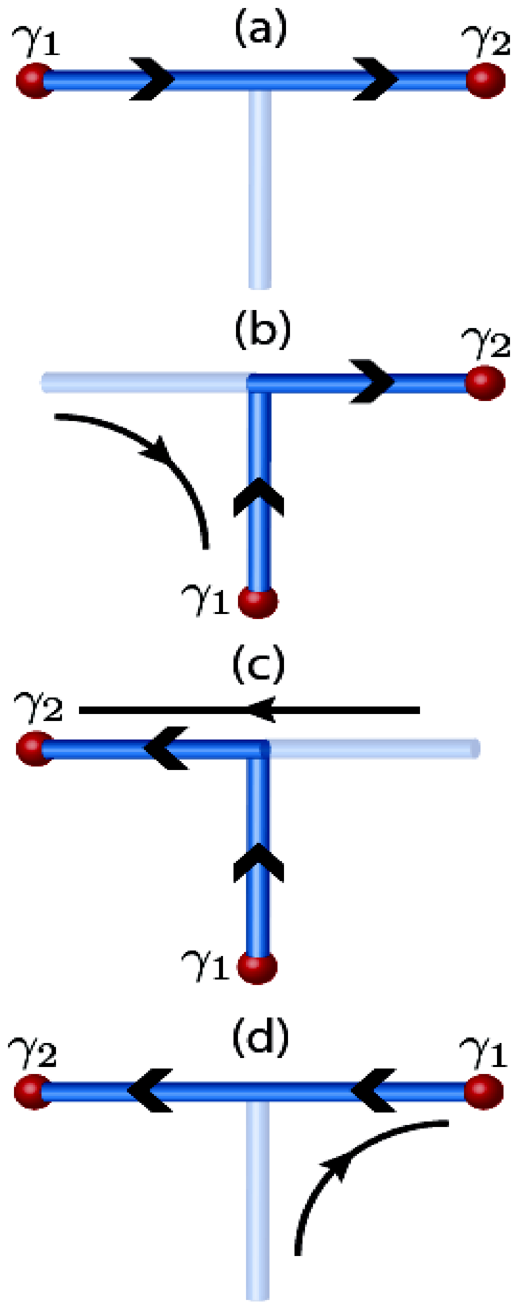
Fermion parity operator

Majorana fermion braiding



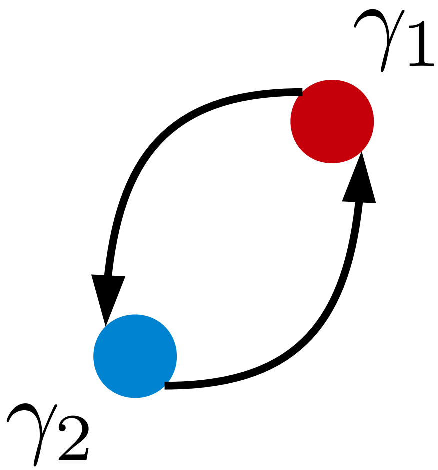
T-junction

Alicea *et al.*,
Nat. Phys. 2011



$$|0\rangle \rightarrow e^{+i\pi/4}\,|0\rangle$$
$$|1\rangle \rightarrow e^{-i\pi/4}\,|1\rangle$$

Majorana fermion braiding



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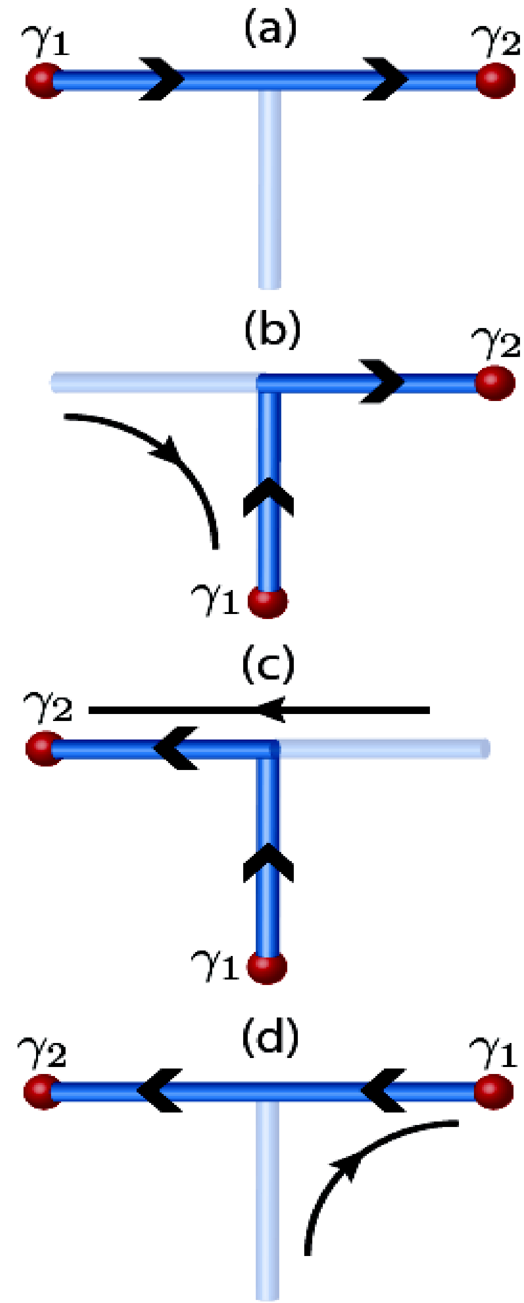
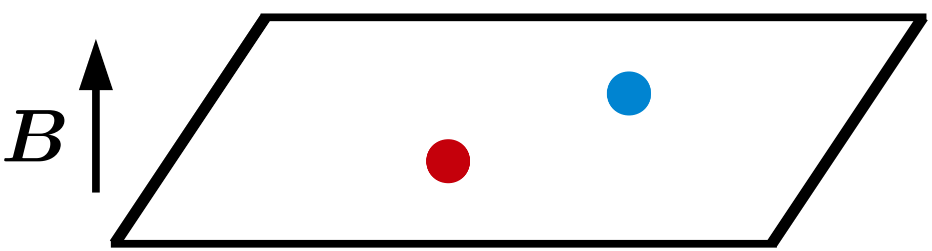
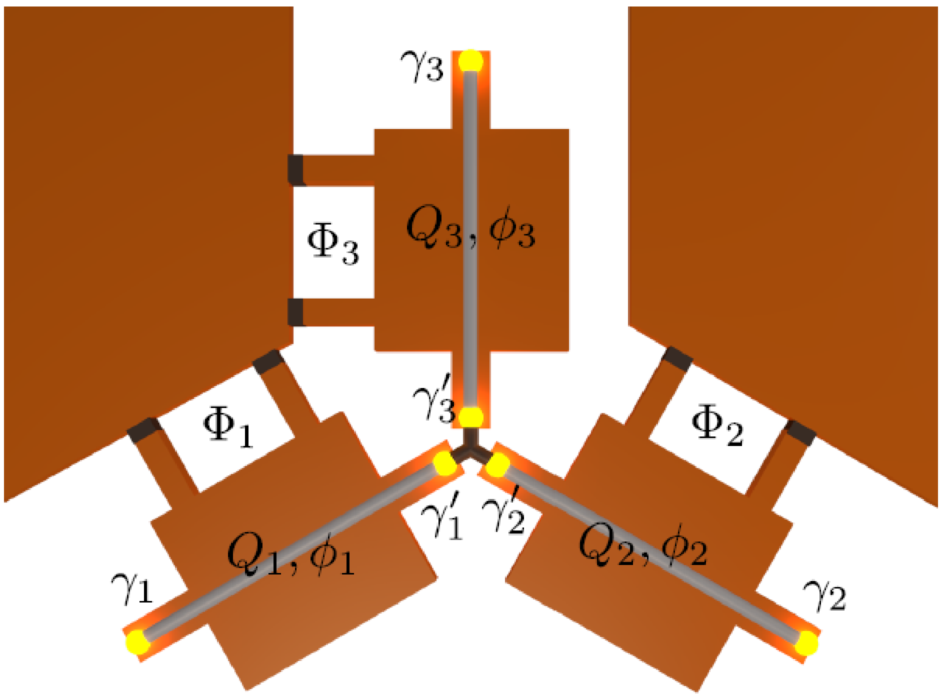


$p_x + ip_y$ superconductor



Majorana fermions become trapped
in superconducting vortices

Majorana fermion braiding: effective



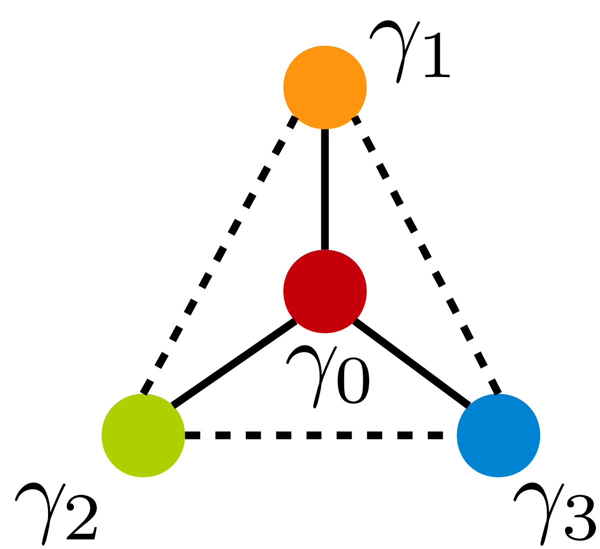
Y-junction

van Heck *et al.*, NJP 2012

Inter-nanowire
Majorana Josephson couplings

$$\mathcal{H}_{12} = J \cos\left(\frac{\varphi_1 - \varphi_2}{2}\right) i\gamma_1\gamma_2$$

Intra-nanowire
Majorana Capacitive couplings



Universal quantum gates

All qubit operations can be efficiently approximated by the set:

Hadamard gate:

$$\mathrm{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Phase gate:

$$\mathrm{Braiding} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

Controlled NOT:

$$\mathrm{cNOT} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Magic gate:

$$\pi/8 \ \mathrm{Gate} = \begin{pmatrix} e^{i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{pmatrix}$$

Majorana fermion braiding: non-universality

PHYSICAL REVIEW X 6, 031019 (2016)

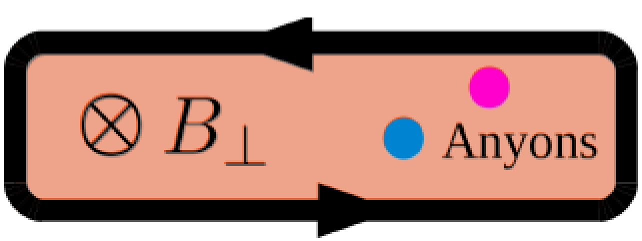
Universal Geometric Path to a Robust Majorana Magic Gate

Torsten Karzig,[1,2,3] Yuval Oreg,[4] Gil Refael,[1,2] and Michael H. Freedman[3,5]

The magic gate is not accessible in a protected manner

Other types of non-Abelian anyons are required with the simplest being the so-called Fibonacci anyons

Fractional quantum Hall system



ν; FQH State ($\mathbb{Z}_k$); Anyons:
1/3; Laughlin state ; Abelian
5/2;Moore-Read($\mathbb{Z}_2$);Majorana
12/5;Read-Rezayi($\mathbb{Z}_3$);Fibonacci

Periodic table of topological systems

	Symmetries:			d							
Class	$\widehat{\Theta}^2$	$\widehat{\Xi}^2$	$\widehat{\Pi}^2$	1	2	3	4	5	6	7	8
A	$\hat{0}$	$\hat{0}$	$\hat{0}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$
AIII	$\hat{0}$	$\hat{0}$	$\hat{1}$	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–
AI	$\hat{1}$	$\hat{0}$	$\hat{0}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\hat{1}$	$\hat{1}$	$\hat{1}$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\hat{0}$	$\hat{1}$	$\hat{0}$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$
DIII	$-\hat{1}$	$\hat{1}$	$\hat{1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–
AII	$-\hat{1}$	$\hat{0}$	$\hat{0}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$
CII	$-\hat{1}$	$-\hat{1}$	$\hat{1}$	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–
C	$\hat{0}$	$-\hat{1}$	$\hat{0}$	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–
CI	$\hat{1}$	$-\hat{1}$	$\hat{1}$	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–

A. Altland and M. R. Zirnbauer PRB 1997; Kitaev AIP Conf Proc 2009;

Ryu *et al.* PRB 2008 and NJP 2010; Hasan and Kane RMP 2010

Topological classification principles

$$\mathcal{H} = \frac{1}{2}\int dr \widehat{\Psi}^\dagger(r) \widehat{\mathcal{H}}(\hat{p}, r) \widehat{\Psi}(r)$$

BdG Hamiltonian

$$\widehat{\Psi}^\dagger(r) = \left(\psi_\uparrow^\dagger(r)\,, \psi_\downarrow^\dagger(r)\,, \psi_\uparrow(r)\,, \psi_\downarrow(r)\right)$$

3 allowed symmetries per irreducible Hamiltonian subspace

Generalized time-reversal symmetry (antiunitary):

$$[\widehat{\mathcal{H}}(\hat{p}, r), \Theta] = 0 \Rightarrow \Theta^{-1}\widehat{\mathcal{H}}(\hat{p}, r)\Theta = +\widehat{\mathcal{H}}(\hat{p}, r)$$

Generalized charge-conjugation symmetry (antiunitary):

$$\{\widehat{\mathcal{H}}(\hat{p}, r), \Xi\} = 0 \Rightarrow \Xi^{-1}\widehat{\mathcal{H}}(\hat{p}, r)\Xi = -\widehat{\mathcal{H}}(\hat{p}, r)$$

Chiral symmetry (unitary): $\{\widehat{\mathcal{H}}(\hat{p}, r), \Theta\Xi\} = \{\widehat{\mathcal{H}}(\hat{p}, r), \Pi\} = 0$

Bulk spinless Kitaev-chain model

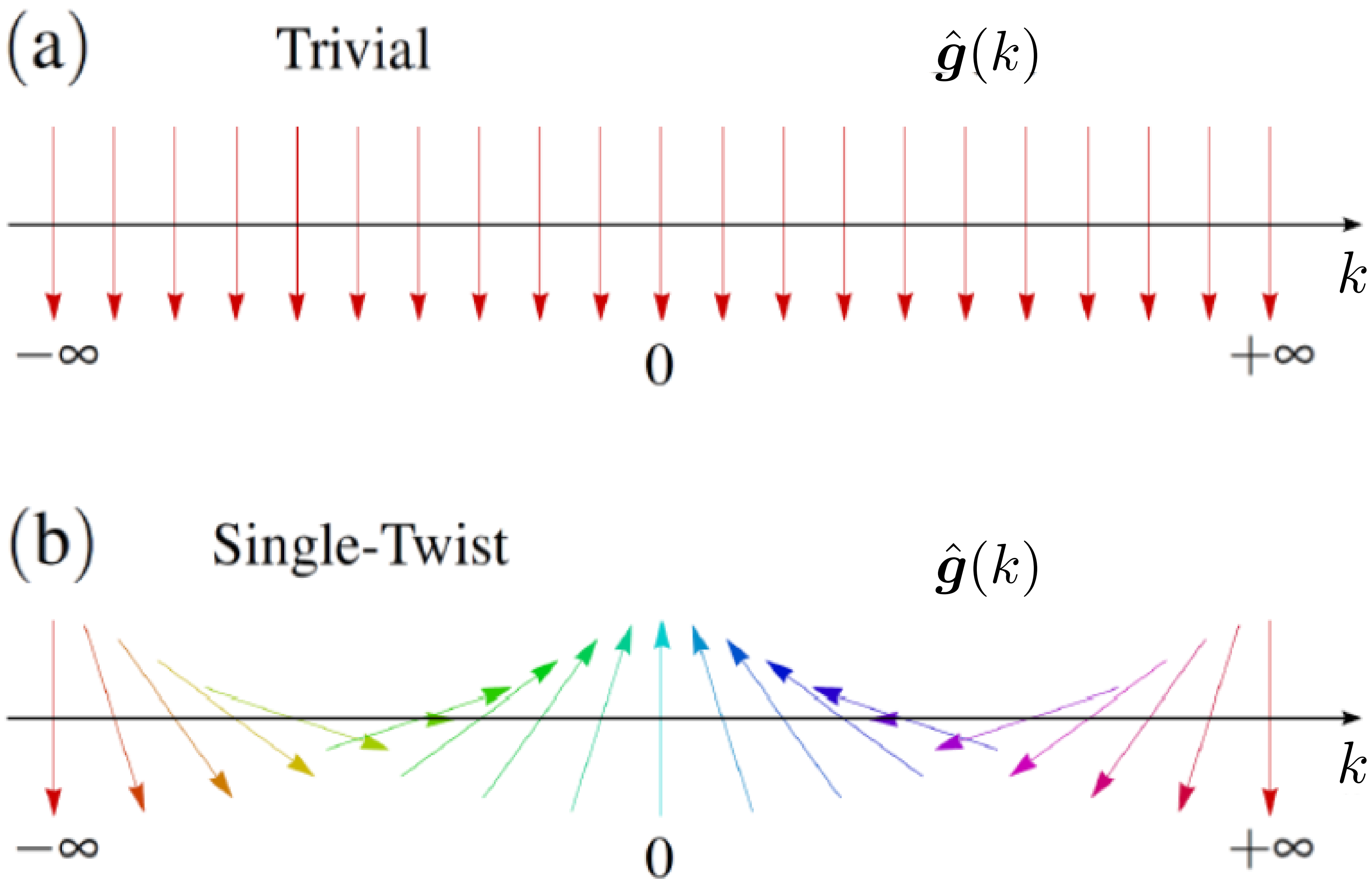
$$\mathcal{H}_{\uparrow} = \frac{1}{2}\sum_{k}^{B.Z.} \begin{pmatrix} c_{k\uparrow}^{\dagger} & c_{-k\uparrow} \end{pmatrix} \begin{pmatrix} +2t_{\uparrow}\cos(ka) - \mu_{\uparrow} & \Delta_{\uparrow}\sin(ka) \\ \Delta_{\uparrow}\sin(ka) & -2t_{\uparrow}\cos(ka) + \mu_{\uparrow} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\uparrow}^{\dagger} \end{pmatrix}$$

$$\mathcal{H}_{\uparrow}(k) = \boldsymbol{g}(k) \cdot \boldsymbol{\tau}$$

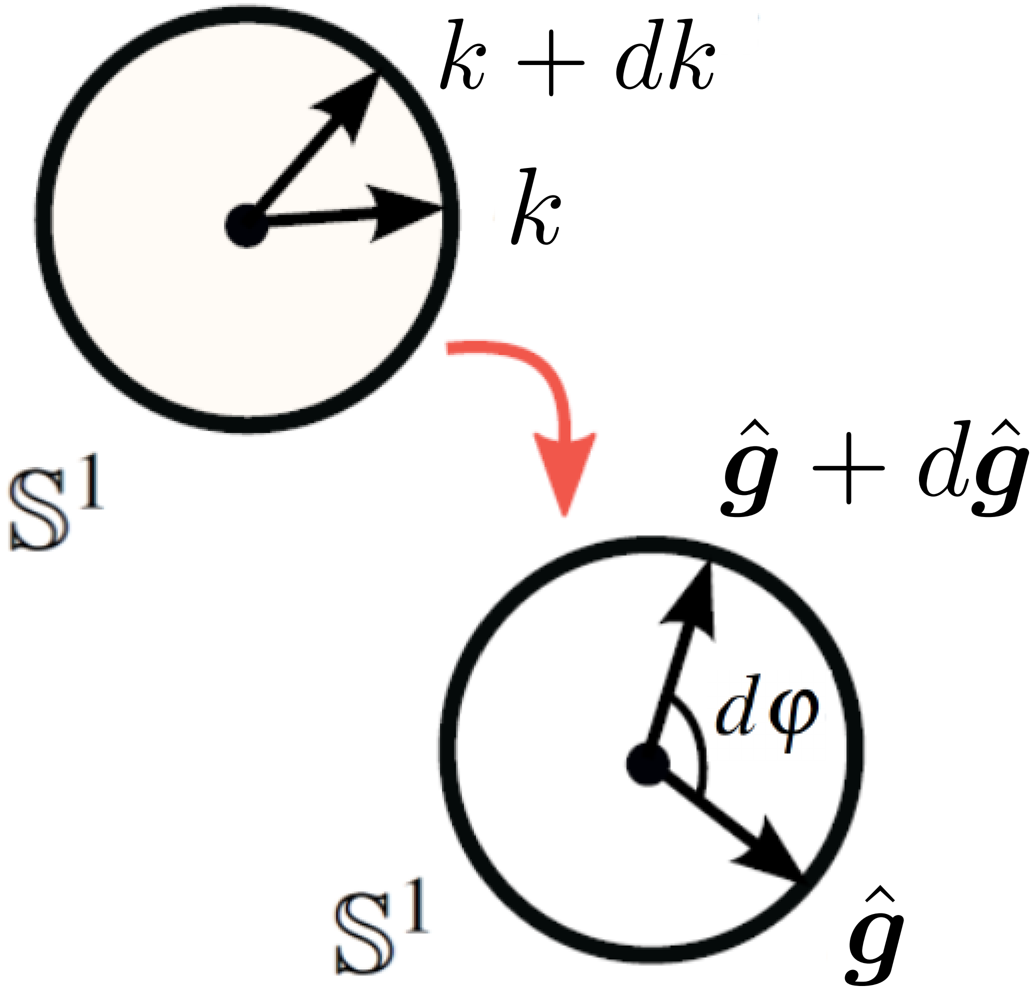
$$\boldsymbol{g}(k) = (\Delta_{\uparrow}\sin(ka), 0, 2t_{\uparrow}\cos(ka) - \mu_{\uparrow})$$ BDI class

Winding number $$w = \frac{1}{2\pi}\int dk \left(\hat{\boldsymbol{g}}(k) \times \frac{\partial \hat{\boldsymbol{g}}(k)}{\partial k}\right)_y$$

Bulk spinless Kitaev-chain model



Bulk spinless Kitaev-chain model



$$w = \frac{1}{2\pi} \int d\varphi$$

$$w \in \mathbb{Z}$$

Bulk spinless Kitaev-chain model

Main contributions from: $k = 0, \dfrac{\pi}{a}$

exactly when the bulk energy gap closes

$$E(k=0) = +2t_\uparrow - \mu_\uparrow = 0$$

$$E(k=\frac{\pi}{a}) = -2t_\uparrow - \mu_\uparrow = 0$$

For a single MF only one gap closing

$$w = \frac{\mathrm{sgn}(2t_\uparrow + \mu_\uparrow) + \mathrm{sgn}(2t_\uparrow - \mu_\uparrow)}{2} = \frac{1 + \mathrm{sgn}(2t_\uparrow - \mu_\uparrow)}{2}$$

$k = \dfrac{\pi}{a}$

$k = 0$

Are there any alternative topological SCs?

Where should we look for them ?

Periodic table of topological systems

	Symmetries:			*d*							
Class	$\widehat{\Theta}^2$	$\widehat{\Xi}^2$	$\widehat{\Pi}^2$	1	2	3	4	5	6	7	8
A	$\hat{0}$	$\hat{0}$	$\hat{0}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$
AIII	$\hat{0}$	$\hat{0}$	$\hat{1}$	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–
AI	$\hat{1}$	$\hat{0}$	$\hat{0}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\hat{1}$	$\hat{1}$	$\hat{1}$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\hat{0}$	$\hat{1}$	$\hat{0}$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$
DIII	$-\hat{1}$	$\hat{1}$	$\hat{1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–
AII	$-\hat{1}$	$\hat{0}$	$\hat{0}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$
CII	$-\hat{1}$	$-\hat{1}$	$\hat{1}$	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–
C	$\hat{0}$	$-\hat{1}$	$\hat{0}$	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–
CI	$\hat{1}$	$-\hat{1}$	$\hat{1}$	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–

Periodic table of topological systems

	Symmetries:			d							
Class	$\widehat{\Theta}^2$	$\widehat{\Xi}^2$	$\widehat{\Pi}^2$	1	2	3	4	5	6	7	8
A	$\hat{0}$	$\hat{0}$	$\hat{0}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$
AIII	$\hat{0}$	$\hat{0}$	$\hat{1}$	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–	$\mathbb{Z}$	–
AI	$\hat{1}$	$\hat{0}$	$\hat{0}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\hat{1}$	$\hat{1}$	$\hat{1}$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\hat{0}$	$\hat{1}$	$\hat{0}$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$
DIII	$-\hat{1}$	$\hat{1}$	$\hat{1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$	–
AII	$-\hat{1}$	$\hat{0}$	$\hat{0}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–	$2\mathbb{Z}$
CII	$-\hat{1}$	$-\hat{1}$	$\hat{1}$	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–	–
C	$\hat{0}$	$-\hat{1}$	$\hat{0}$	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–
CI	$\hat{1}$	$-\hat{1}$	$\hat{1}$	–	–	$2\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–

Symmetry classes supporting MFs

$$\gamma_\alpha = \int dr \left[u^*_{\uparrow,\alpha}(r)\psi_\uparrow(r) + u^*_{\downarrow,\alpha}(r)\psi_\downarrow(r) + u_{\uparrow,\alpha}(r)\psi^\dagger_\uparrow(r) + u_{\downarrow,\alpha}(r)\psi^\dagger_\downarrow(r) \right]$$

Class	Θ	Ξ	Π	1d	2d	3d
BDI	$+\mathrm{I}$	$+\mathrm{I}$	I	$\mathbb{Z}$	0	0
D	0	$+\mathrm{I}$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	$-\mathrm{I}$	$+\mathrm{I}$	I	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Here we focus on 2D, quasi-1D and 1D systems

Microscopic model for engineered TSCs

$$\mathcal{H} = \int d\boldsymbol{r}\ \hat{\psi}^\dagger(\boldsymbol{r}) \left[\frac{\hat{\boldsymbol{p}}^2}{2m} - \mu + V(\boldsymbol{r}) - \boldsymbol{M}(\boldsymbol{r}) \cdot \boldsymbol{\sigma}\right] \hat{\psi}(\boldsymbol{r})$$

$$+ \int d\boldsymbol{r}\ \ \hat{\psi}^\dagger(\boldsymbol{r}) \frac{\{v(\boldsymbol{r}), \hat{p}_x\sigma_y - \hat{p}_y\sigma_x\}}{2} \hat{\psi}(\boldsymbol{r})$$

$$+ \int d\boldsymbol{r} \left[\psi^\dagger_\uparrow(\boldsymbol{r})\Delta(\boldsymbol{r})\psi^\dagger_\downarrow(\boldsymbol{r}) + \psi_\downarrow(\boldsymbol{r})\Delta^*(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})\right]$$

$$\hat{\psi}^\dagger(\boldsymbol{r}) = (\psi^\dagger_\uparrow(\boldsymbol{r})\psi^\dagger_\downarrow(\boldsymbol{r}))$$

Classification of engineered TSCs

Case	$v(\boldsymbol{r})$	$\boldsymbol{M}(\boldsymbol{r})$	$\Delta(\boldsymbol{r})$	2D	quasi-1D	1D
I	✓	×	$\mathcal{K}=I$	DIII	DIII	*no MFs*
II	✓	×	$\mathcal{K}=0$	D	D	*no MFs*
III	×	$\mathcal{K}=I$	$\mathcal{K}=I$	*BDI*	BDI	BDI
IV	×	$\mathcal{K}=\{0,I,0\}$	$\mathcal{K}=\{I,0,0\}$	D	D	D
V	✓	$\mathcal{K}=I$	$\mathcal{K}=I$	D	D	BDI
VI	✓	$\mathcal{K}=\{0,\mathrm{I},0\}$	$\mathcal{K}=\{\mathrm{I},0,0\}$	D	D	D

PK, NJP 2013

Complex conjugation operator

$$\mathcal{H} = \int d\boldsymbol{r}\, \hat{\psi}^\dagger(\boldsymbol{r}) \left[\frac{\hat{\boldsymbol{p}}^2}{2m} - \mu + V(\boldsymbol{r}) - \boldsymbol{M}(\boldsymbol{r}) \cdot \boldsymbol{\sigma} + \frac{\{v(\boldsymbol{r}), \hat{p}_x\sigma_y - \hat{p}_y\sigma_x\}}{2} \right] \hat{\psi}(\boldsymbol{r})$$
$$+ \int d\boldsymbol{r} \left[\psi^\dagger_\uparrow(\boldsymbol{r})\Delta(\boldsymbol{r})\psi^\dagger_\downarrow(\boldsymbol{r}) + \psi_\downarrow(\boldsymbol{r})\Delta^*(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r}) \right]$$

Classification of engineered TSCs

Case	$v(\boldsymbol{r})$	$\boldsymbol{M}(\boldsymbol{r})$	$\Delta(\boldsymbol{r})$	2D	quasi-1D	1D
I	✓	×	$\mathcal{K}=I$	DIII	DIII	*no MFs*
II	✓	×	$\mathcal{K}=0$	D	D	*no MFs*
III	×	$\mathcal{K}=I$	$\mathcal{K}=I$	*BDI*	BDI	BDI
IV	×	$\mathcal{K}=\{0,I,0\}$	$\mathcal{K}=\{I,0,0\}$	D	D	D
V	✓	$\mathcal{K}=I$	$\mathcal{K}=I$	D	D	BDI
VI	✓	$\mathcal{K}=\{0,\mathrm{I},0\}$	$\mathcal{K}=\{\mathrm{I},0,0\}$	D	D	D

PK, NJP 2013

Chiral symmetry: $\{\mathcal{H},\Pi\}=0$

Time-reversal symmetry: $[\mathcal{H},\Theta]=0$

Charge-conjugation symmetry: $\{\mathcal{H},\Xi\}=0$

Classification of engineered TSCs

Case	$v(\boldsymbol{r})$	$\boldsymbol{M}(\boldsymbol{r})$	$\Delta(\boldsymbol{r})$	2D	quasi-1D	1D
I	✓	×	$\mathcal{K}=I$	DIII	DIII	*no MFs*
II	✓	×	$\mathcal{K}=0$	D	D	*no MFs*
III	×	$\mathcal{K}=I$	$\mathcal{K}=I$	*BDI*	BDI	BDI
IV	×	$\mathcal{K}=\{0,I,0\}$	$\mathcal{K}=\{I,0,0\}$	D	D	D
V	✓	$\mathcal{K}=I$	$\mathcal{K}=I$	D	D	$\boxed{\text{BDI}}$
VI	✓	$\mathcal{K}=\{0,\mathrm{I},0\}$	$\mathcal{K}=\{\mathrm{I},0,0\}$	D	D	D

PK, NJP 2013

Chiral symmetry: $\{\mathcal{H},\Pi\}=0$ $\quad \Pi=\tau_y\sigma_y$

Time-reversal symmetry: $[\mathcal{H},\Theta]=0$ $\quad$ Broken

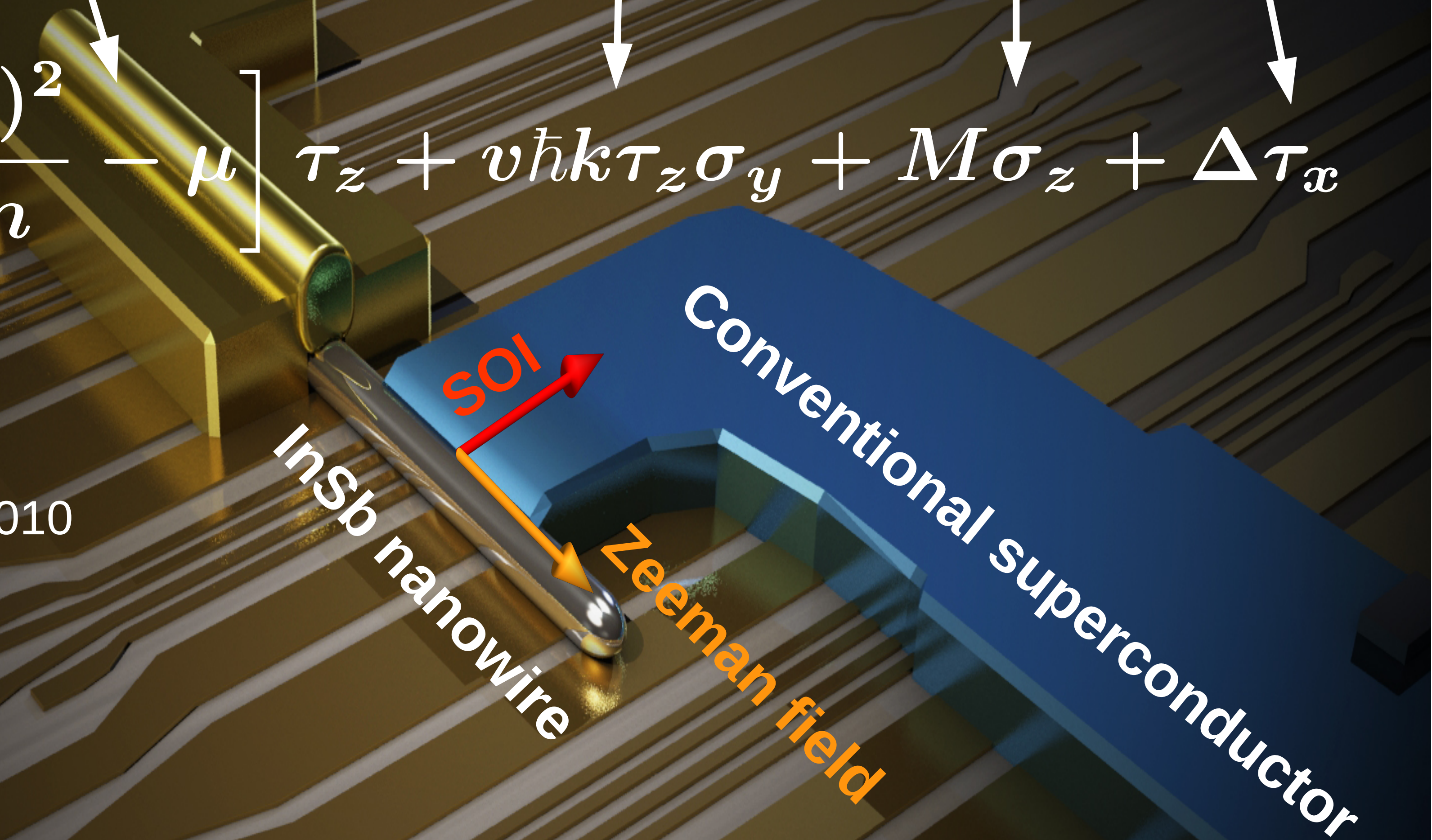
Charge-conjugation symmetry: $\{\mathcal{H},\Xi\}=0$ $\quad \Xi=\tau_y\sigma_y\mathcal{K}$

$$\mathcal{H}=\left(\frac{\hat{p}_x^2}{2m}-\mu\right)\tau_z+v\hat{p}_x\tau_z\sigma_y+M\sigma_z+\Delta\tau_x$$

Topological superconductivity in nanowires

Kinetic term+chemical potential **Spin-Orbit coupling** **Zeeman field** **SC gap**

$$\mathcal{H}_k = \left[\frac{(\hbar k)^2}{2m} - \mu\right]\tau_z + v\hbar k \tau_z \sigma_y + M\sigma_z + \Delta\tau_x$$

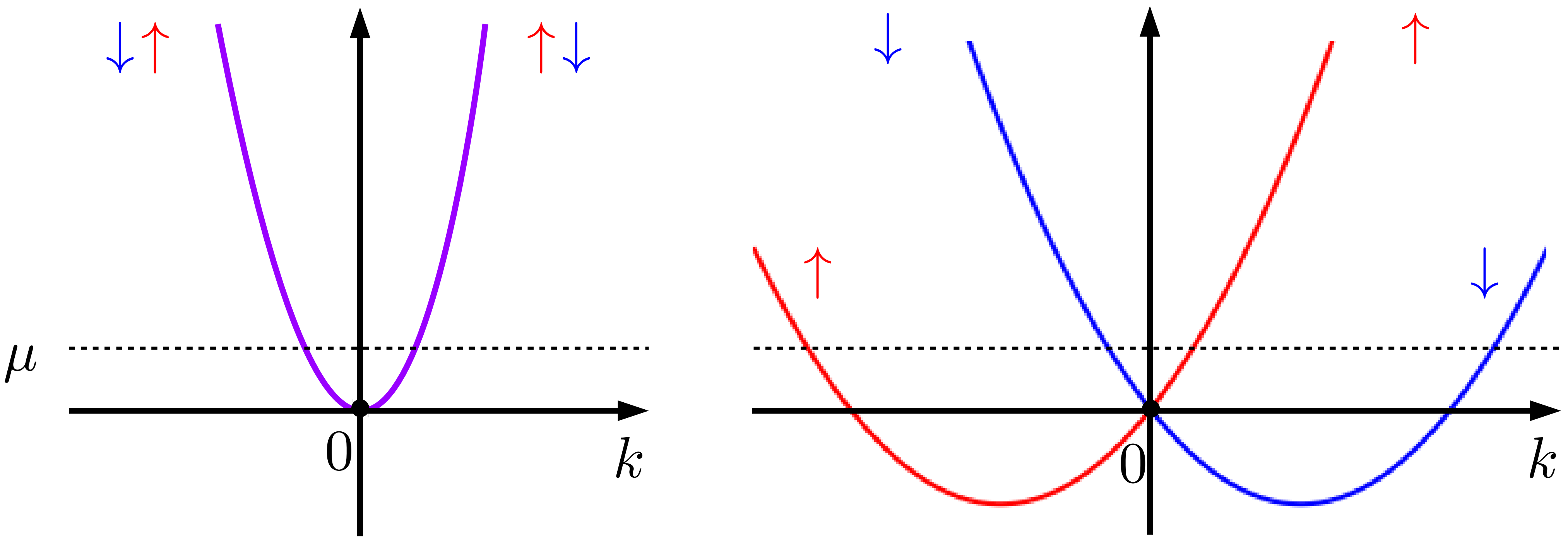


Lutchyn *et al.* and
Oreg *et al.* PRL 2010

$$\Psi_k^\dagger = \begin{pmatrix} \psi_{k\uparrow}^\dagger & \psi_{k\downarrow}^\dagger & \psi_{-k\downarrow} & -\psi_{-k\uparrow} \end{pmatrix}$$

Delft experiment
Mourik *et al.*, Science 2012

1D nanowire: spin-orbit coupling (SOC)

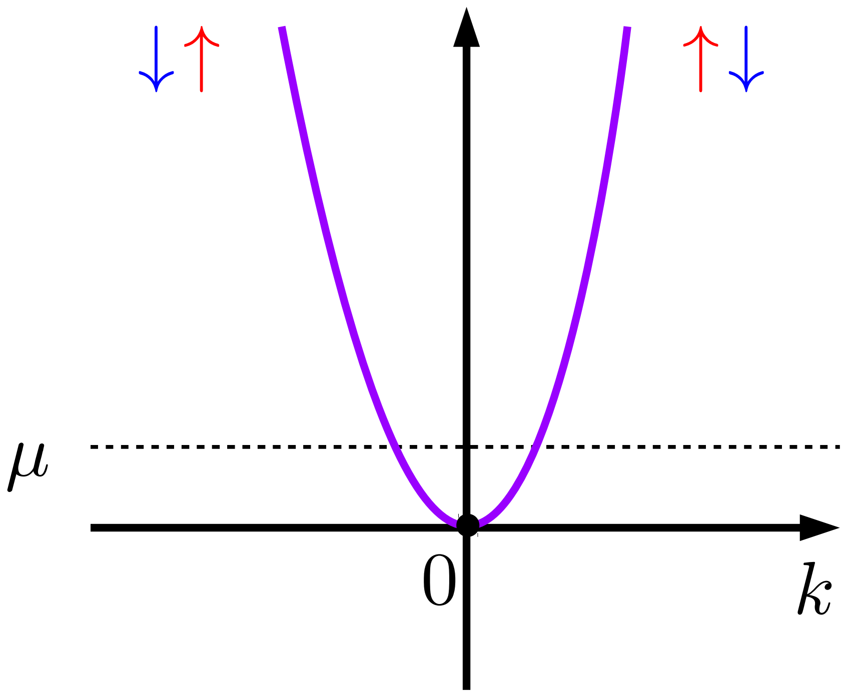


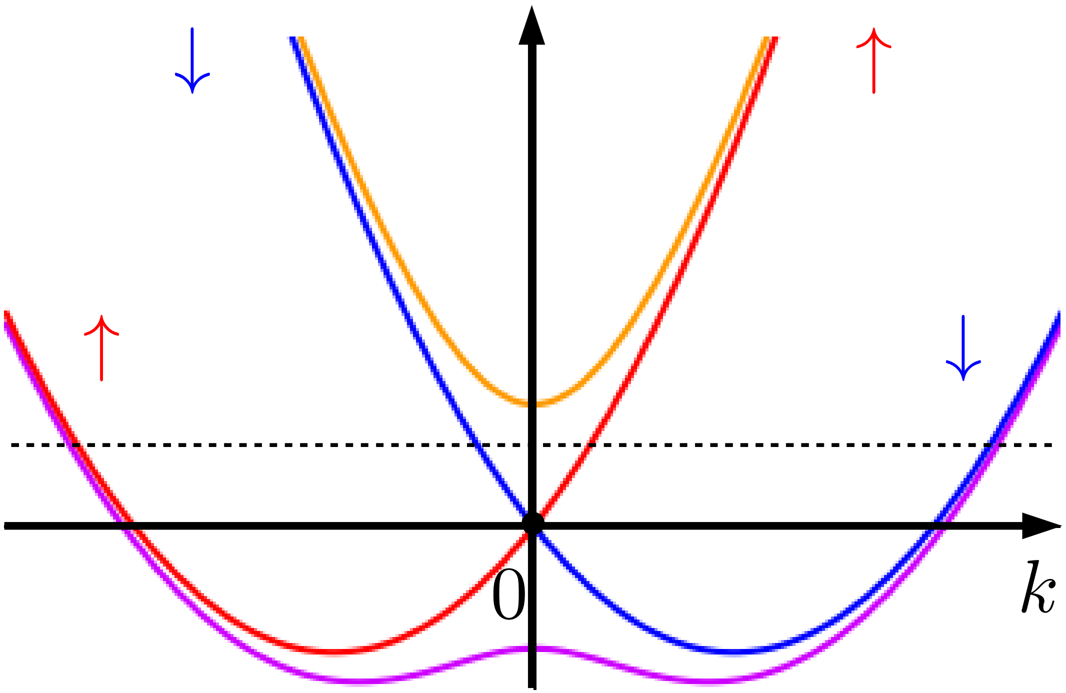
$$\frac{(\hbar k)^2}{2m}$$

$$\frac{(\hbar k)^2}{2m} + v\hbar k\sigma_y$$

Broken inversion symmetry:
electric field

1D nanowire: SOC and transverse field



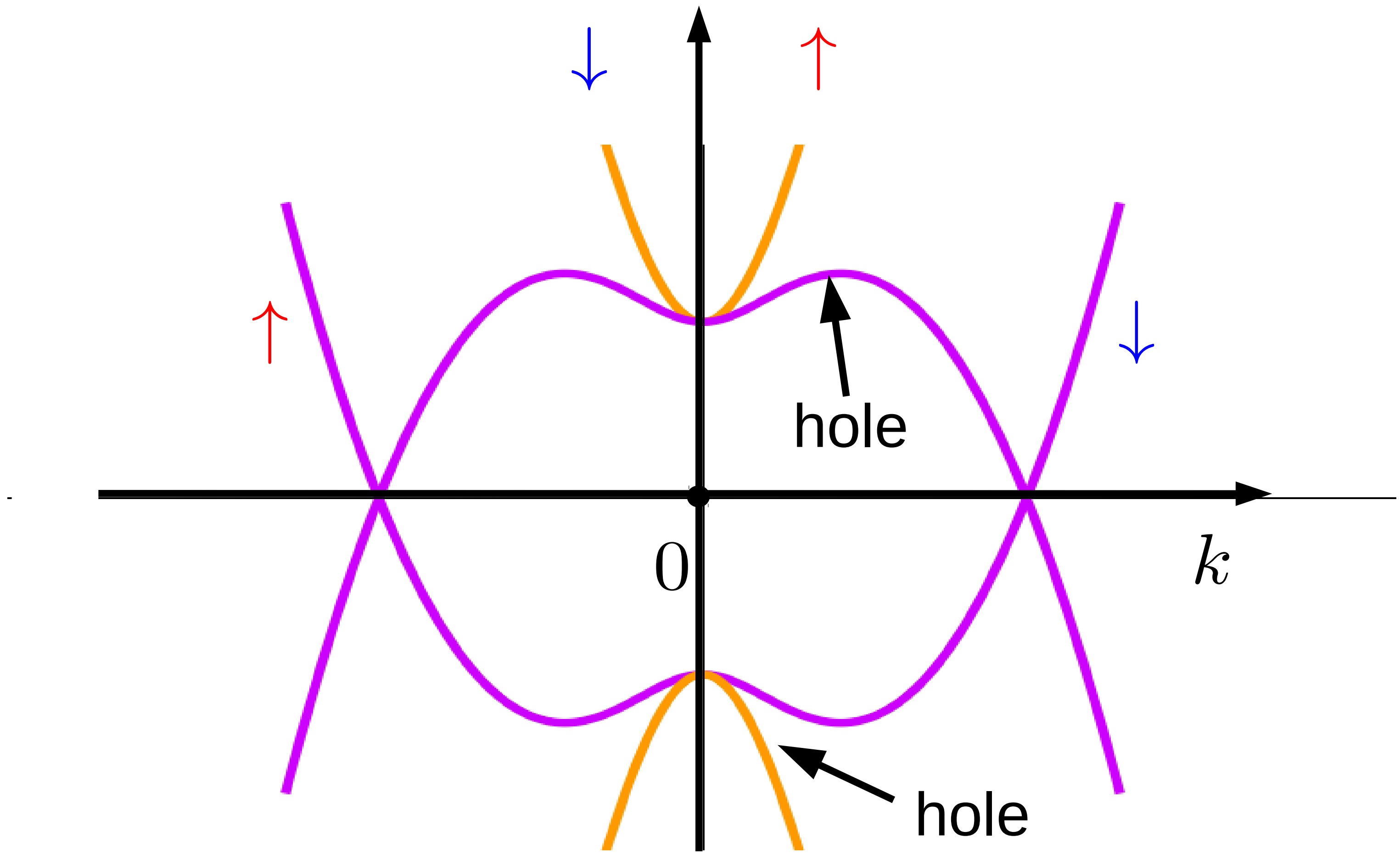


$$\frac{(\hbar k)^2}{2m}$$

$$\frac{(\hbar k)^2}{2m} + v\hbar k\sigma_y + M\sigma_z$$

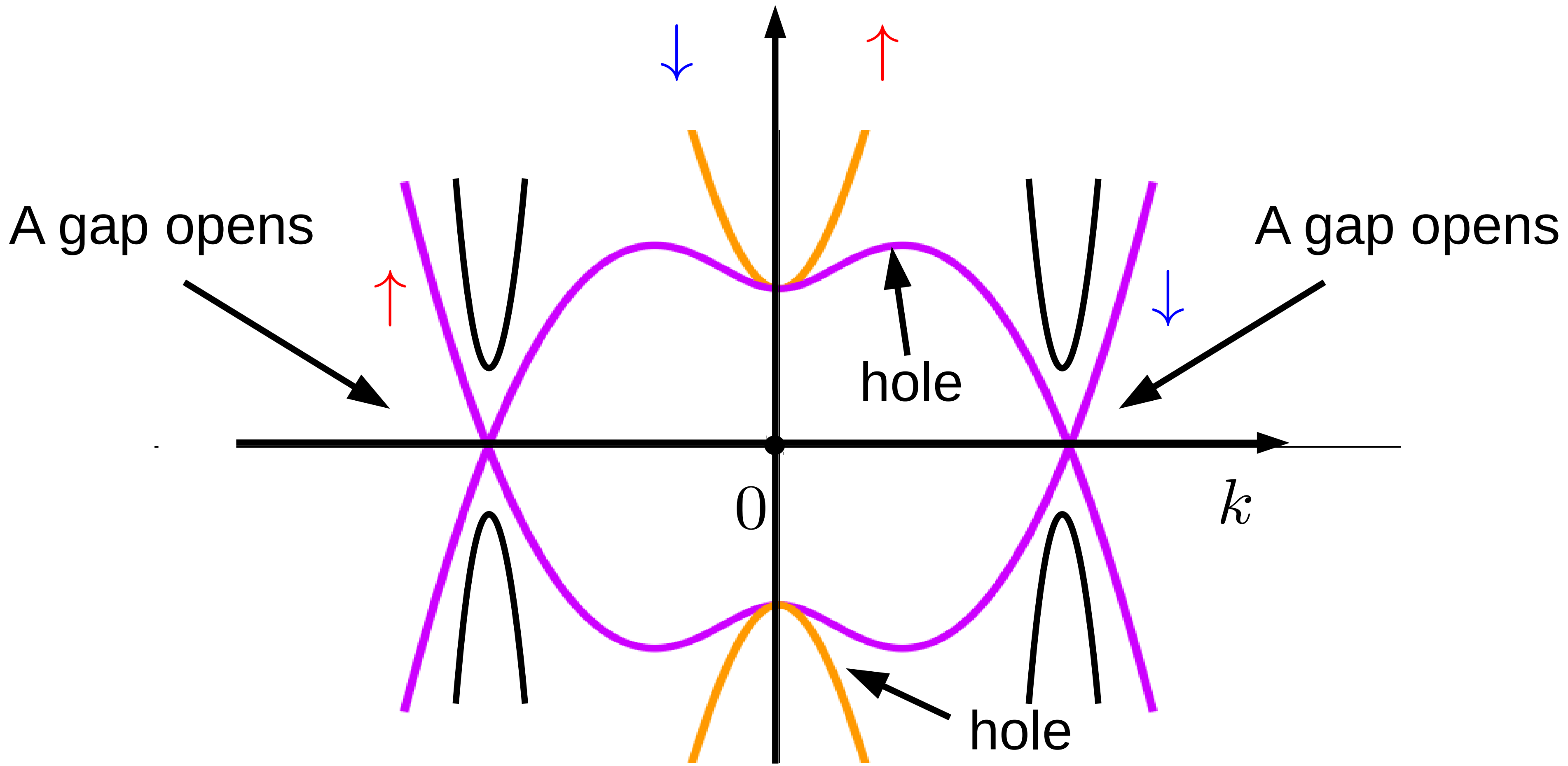
Zeeman field lifts
Kramers degeneracy

1D nanowire: adding a superconducting gap



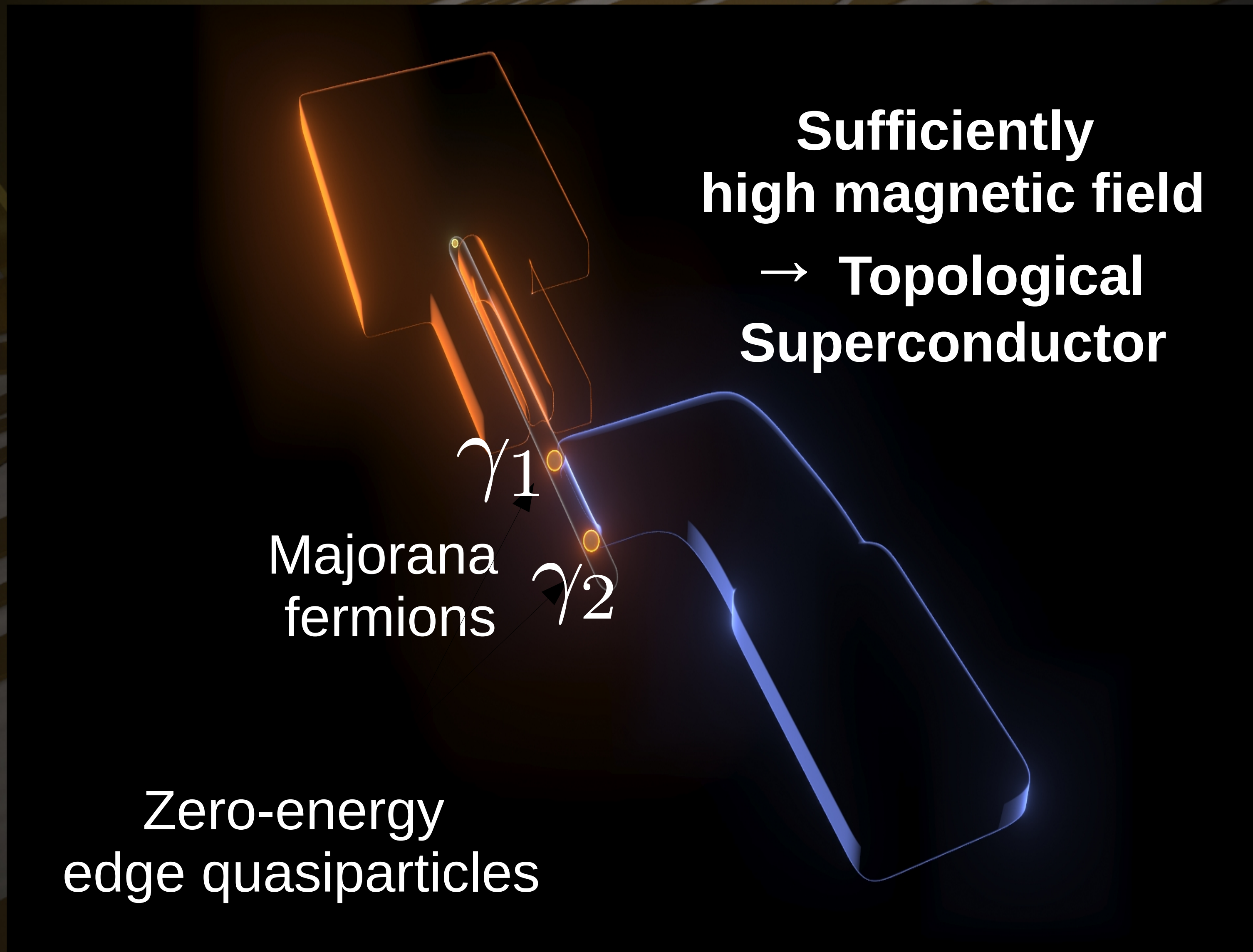
$\mu = 0$

1D nanowire: adding a superconducting gap



$\mu = 0$

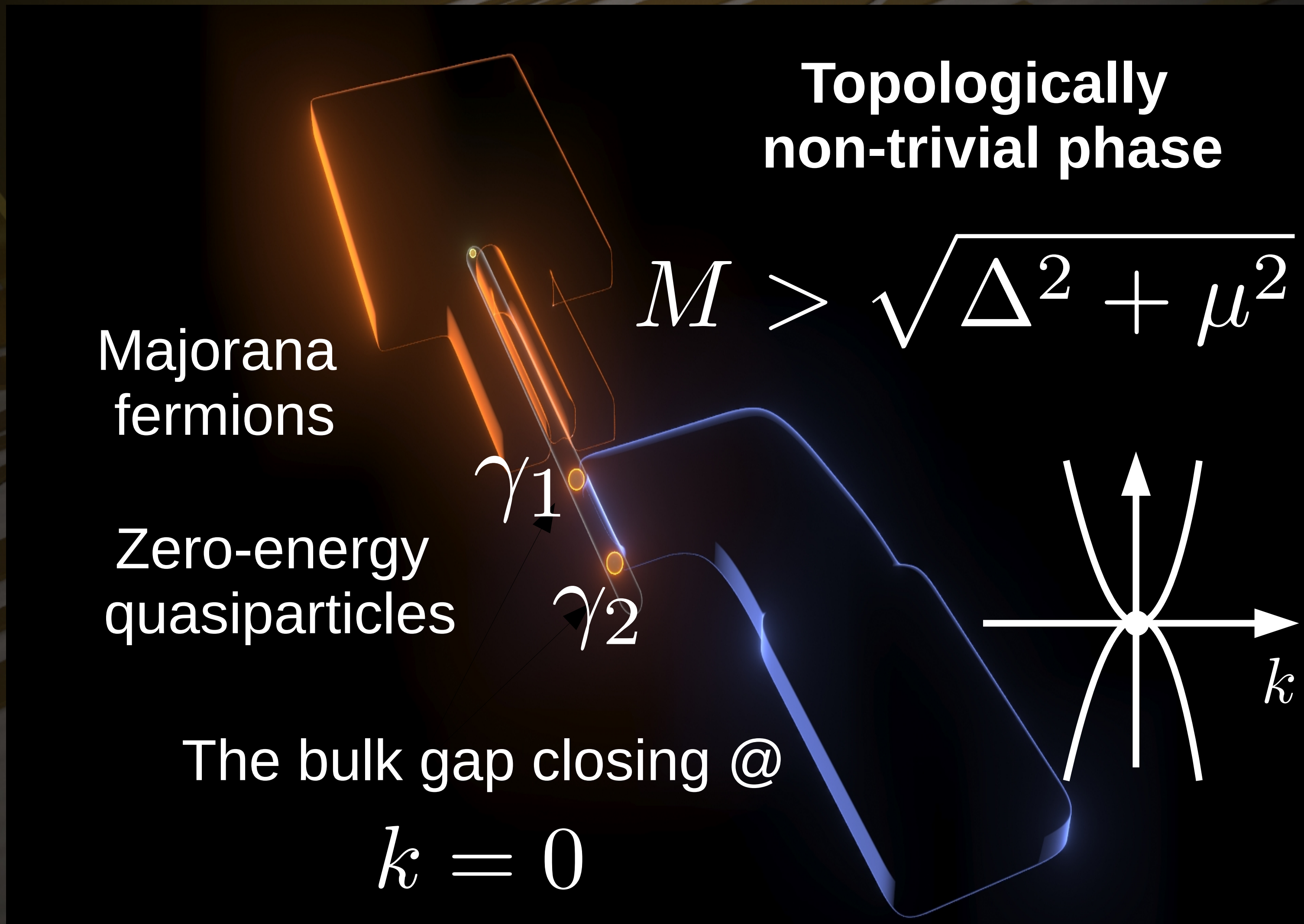
Topological superconductivity in nanowires



Delft experiment

Mourik *et al.*, Science 2012

Topological superconductivity in nanowires



$$M > \sqrt{\Delta^2 + \mu^2}$$

The bulk gap closing @

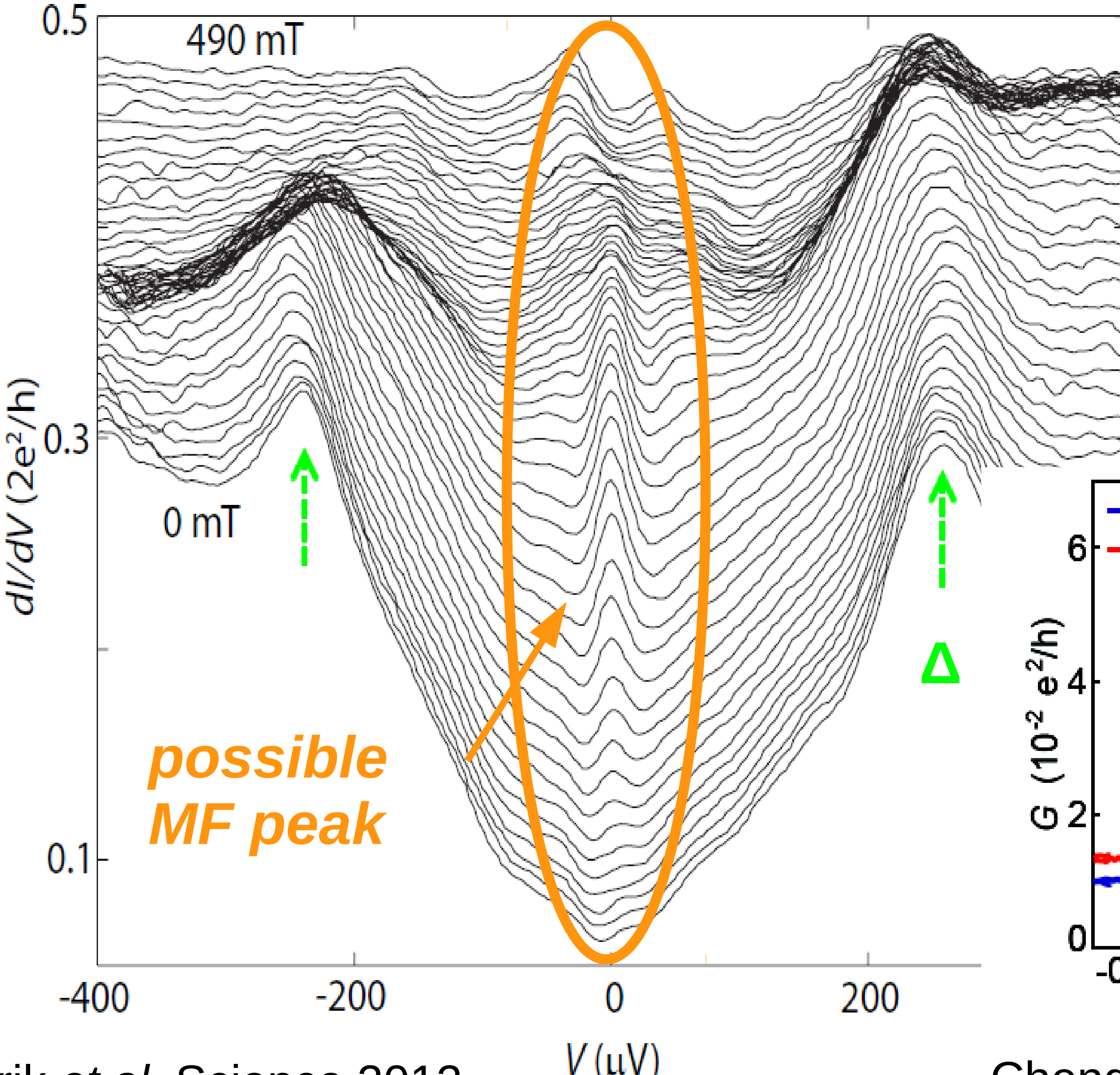
$$k = 0$$

drives the topological phase transition

Delft experiment

Mourik *et al.*, Science 2012

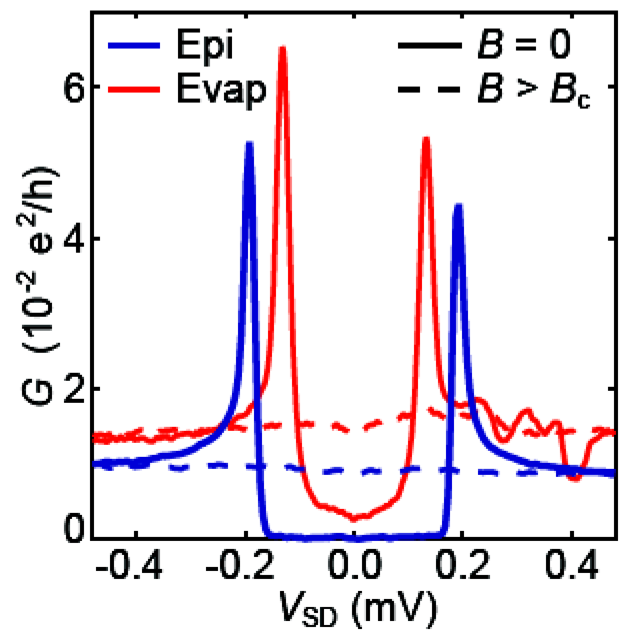
MF signatures in zero-bias anomaly peak



Mourik *et al.* Science 2012

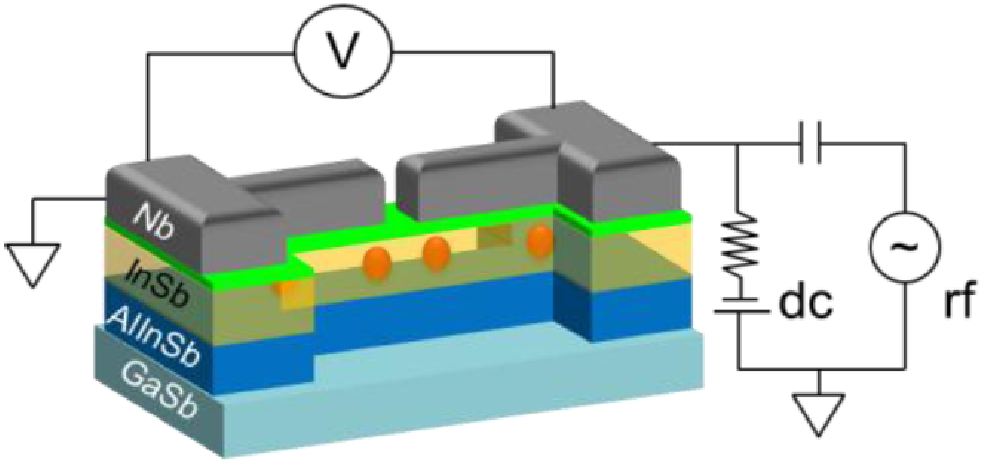
Zero-energy conductance peak should be quantized to 1!

Epitaxial hybrids



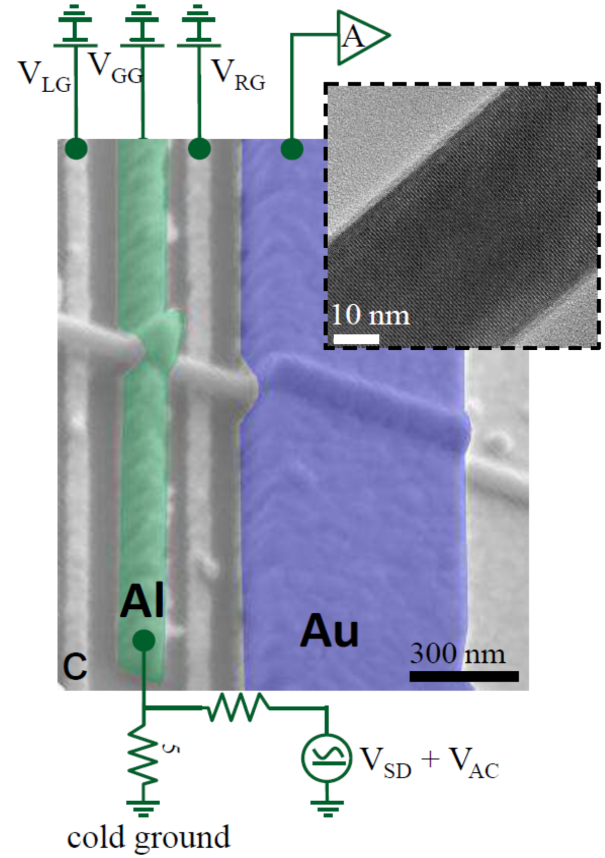
Cheng *et al.* Nat. Nanotech. 2015

Other nanowire experiments

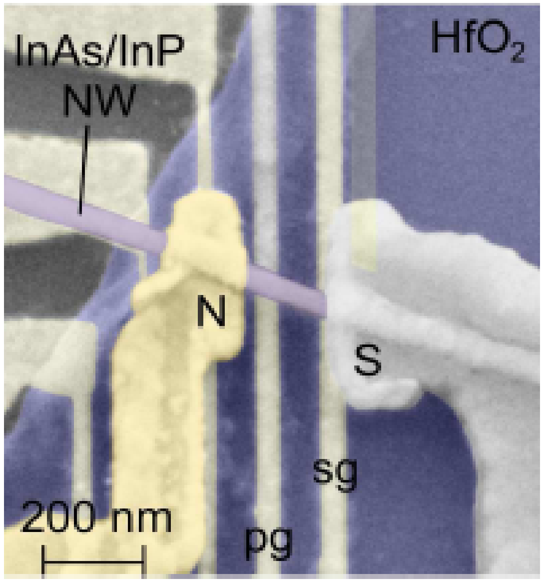


Rokhinson, Liu and Furdyna, Nat. Phys. 2012

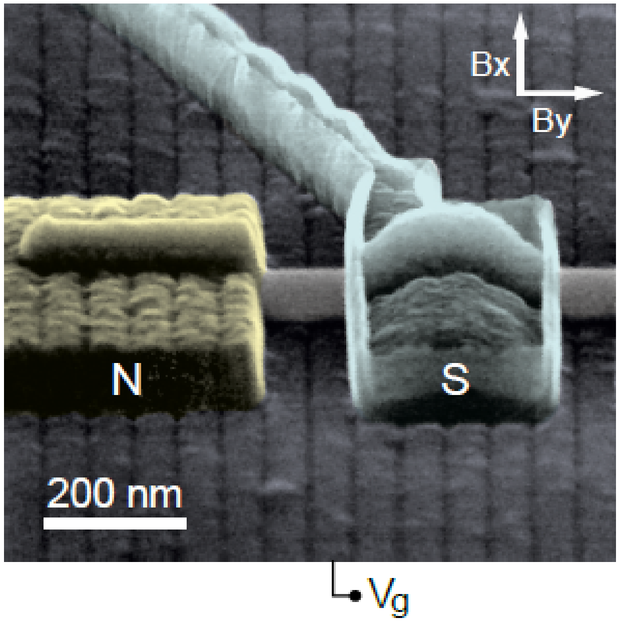
MFs or not MFs?



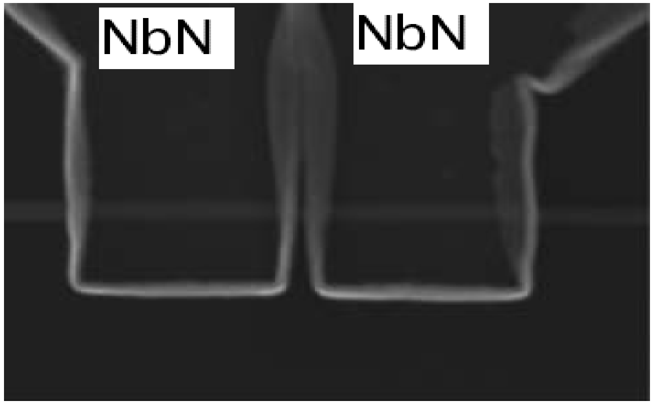
Das *et al.* Nat. Phys. 2012



Lee *et al.*, Nat. Nanotech. 2014

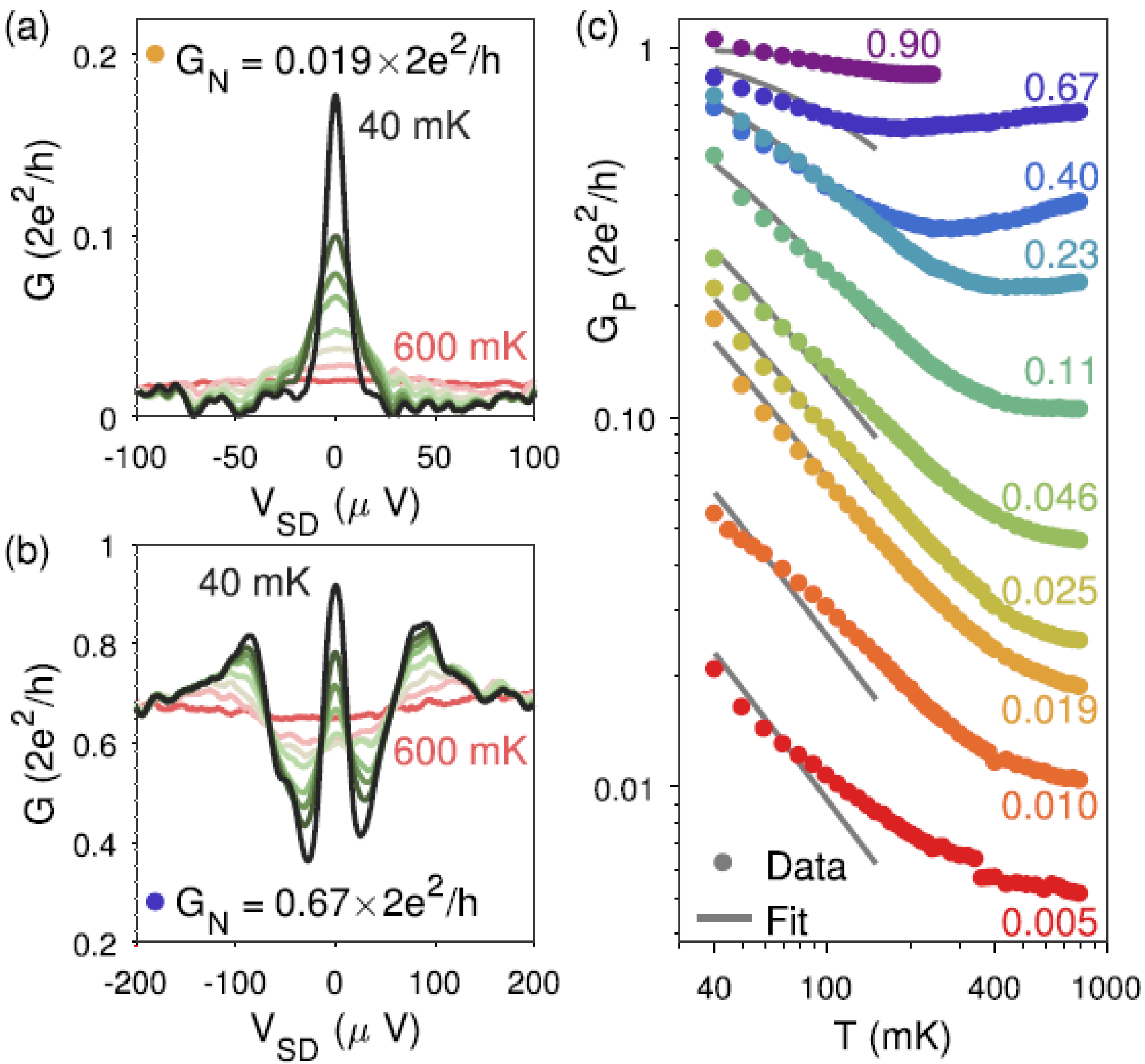


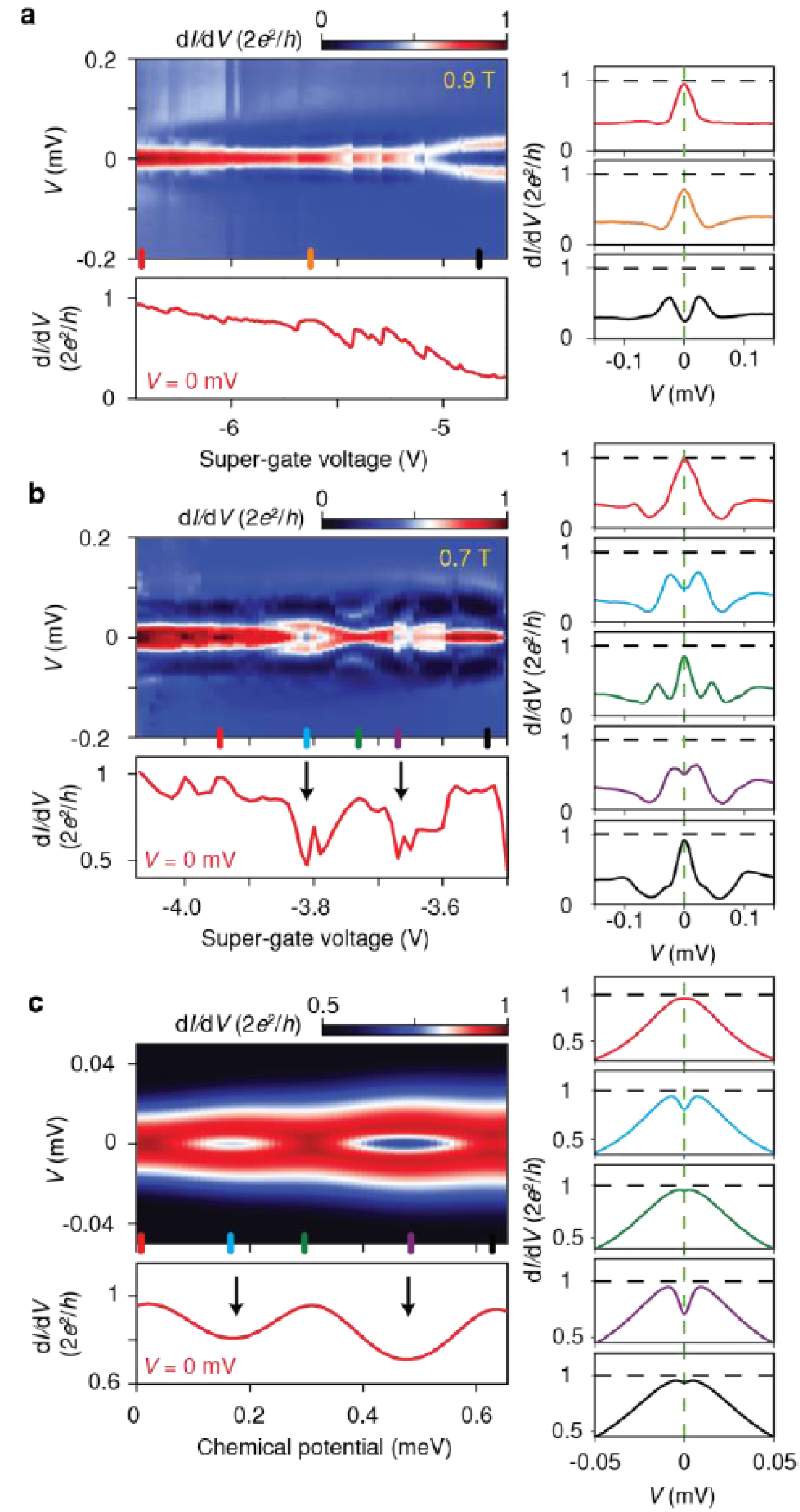
Churchill *et al.*, PRB 2013



A. D. K. Finck *et al.*, PRL 2013

Conductance quantization





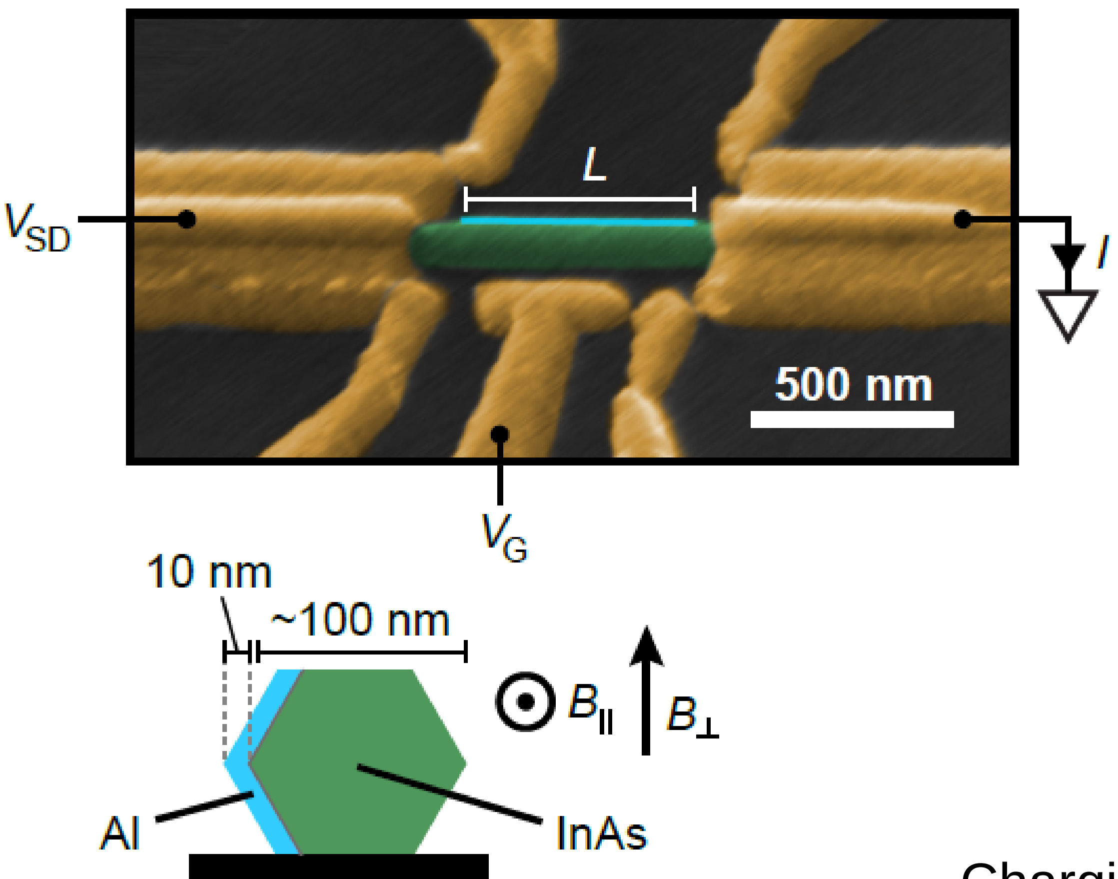
Nichele *et al.*, PRL 2017

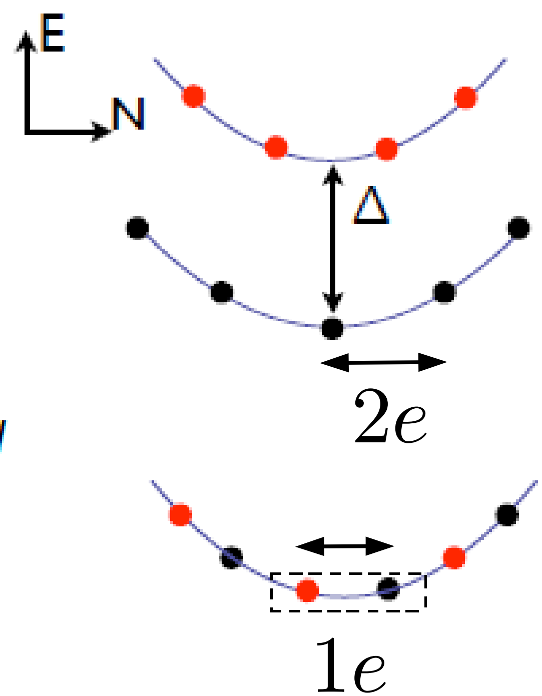
&

Zhang *et al.*, Nature 2018

Detecting MFs in topological SC islands

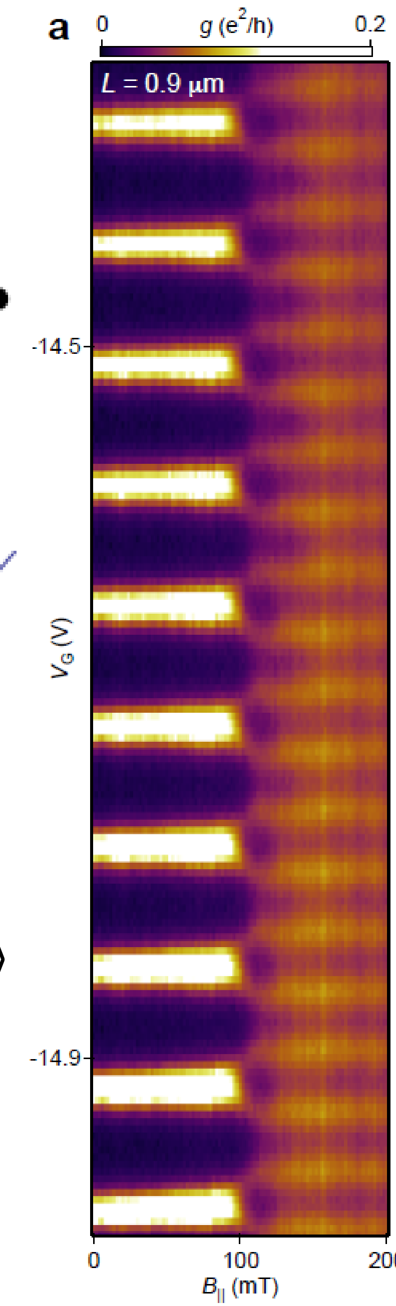
Albrecht *et al.*, Nature 2016





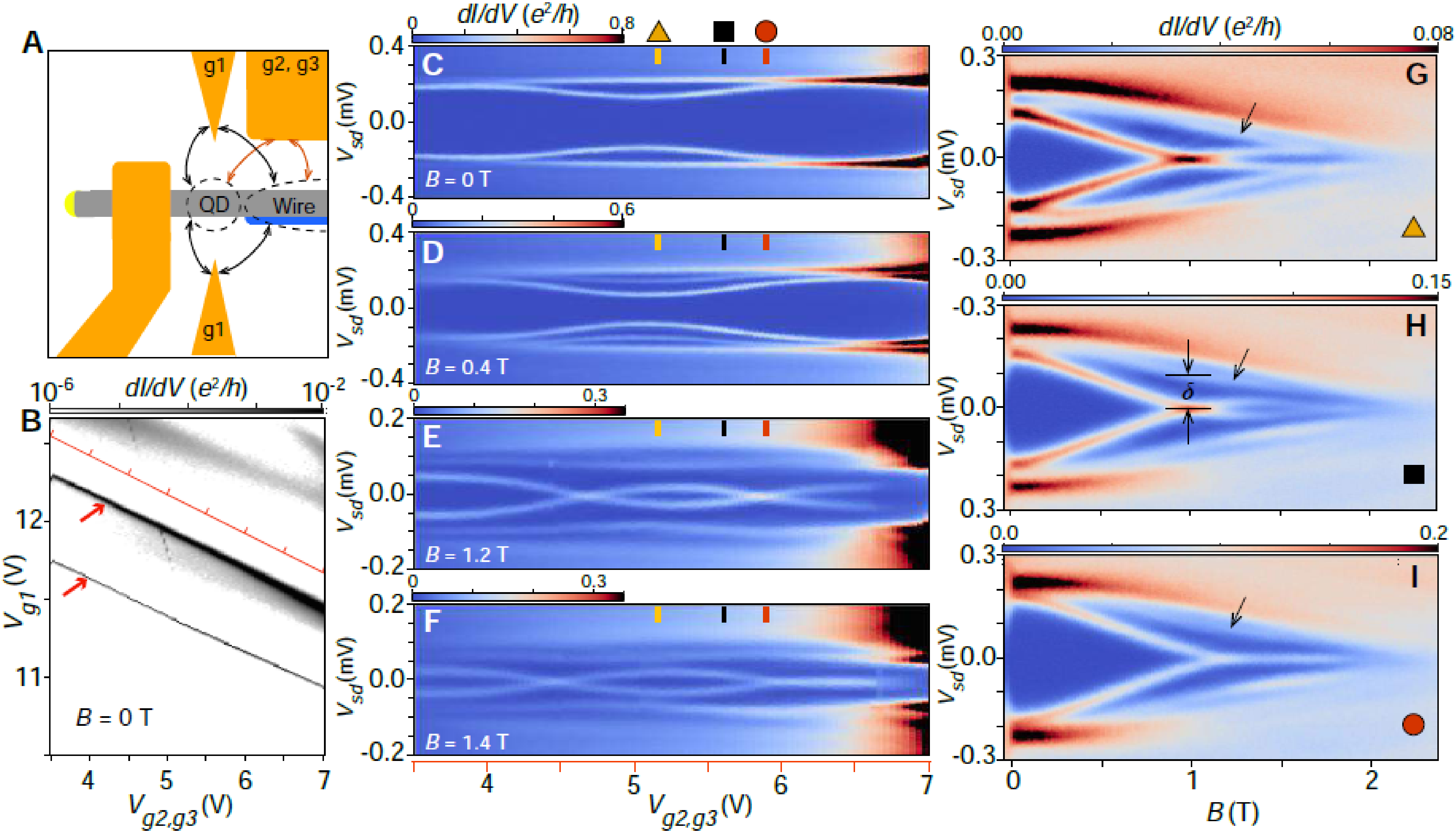
$$d_0^{\dagger}\,|0\rangle = |1\rangle$$

Charging effects
Fu PRL 2010

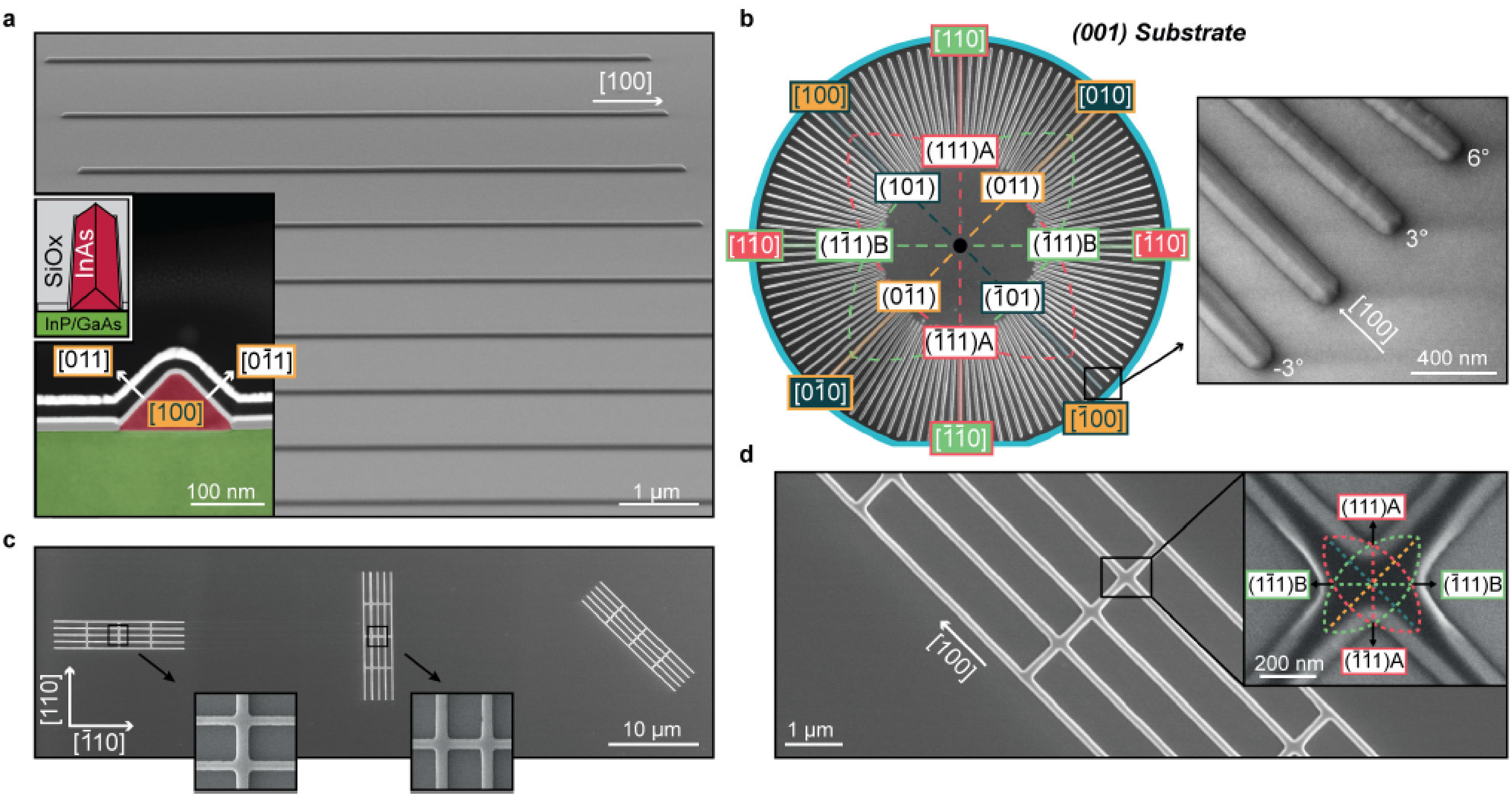


Detecting MFs in a nanowire-Qdot system

M-T Deng *et al.*, Science 2016



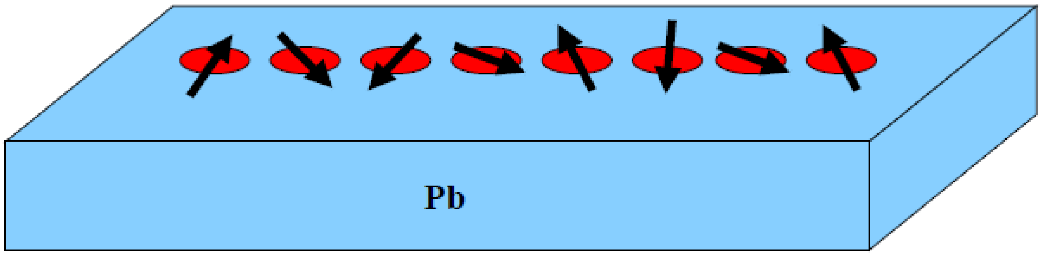
Scalable devices and networks



Krizek *et al.*, Phys. Rev. Mat. 2018.

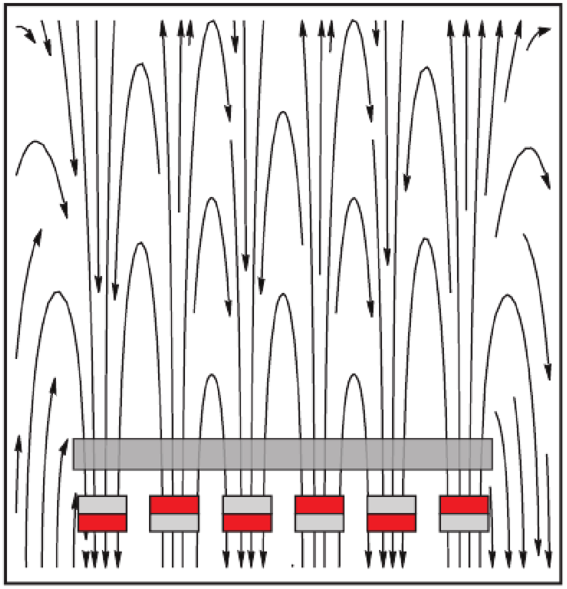
A twist: MFs in magnetic atomic chains

random magnetic impurities



Choy *et al.*, PRB 2011

nanomagnets



Kjaergaard *et al.*, PRB 2012

Bulk SC+helical magnetism

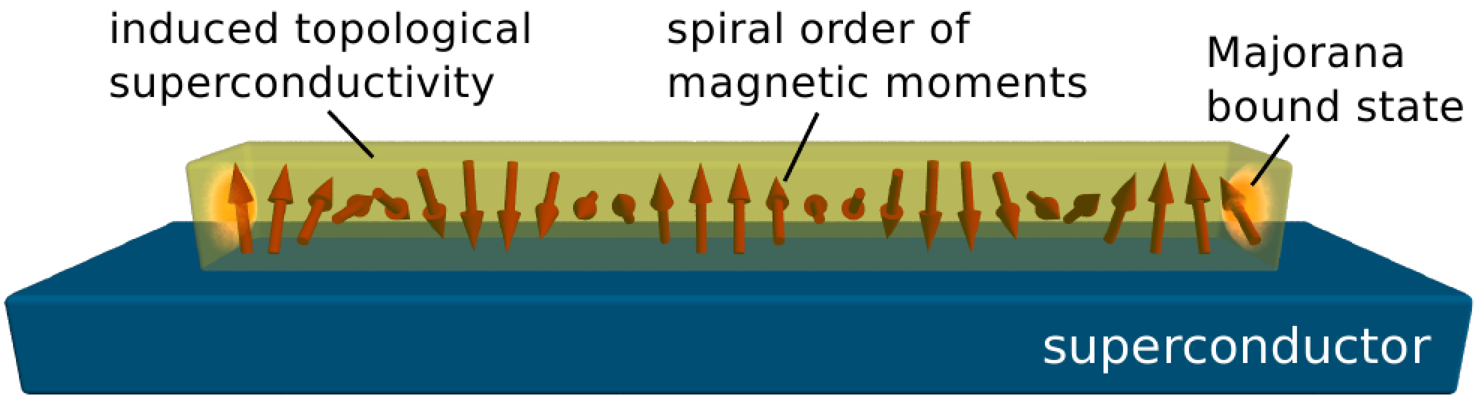
Martin & Morpurgo, PRB 2012

Proximity of SC+helical magnetism

Nadj-Perge *et al.*, PRB 2013
Nakosai *et al.*, PRB 2013
Braunecker *et al.*, PRL 2013
Klinovaja *et al.*, PRL 2013
Vazifeh & Franz, PRL 2013

Pientka *et al.*, PRB 2013, 2014
Kim *et al.*, PRB 2014
Pöyhönen *et al.*, PRB 2014
Li *et al.*, Nat. Commun. 2016
Röntynen & Ojanen, PRB 2014

Helical magnetism and synthetic SOC

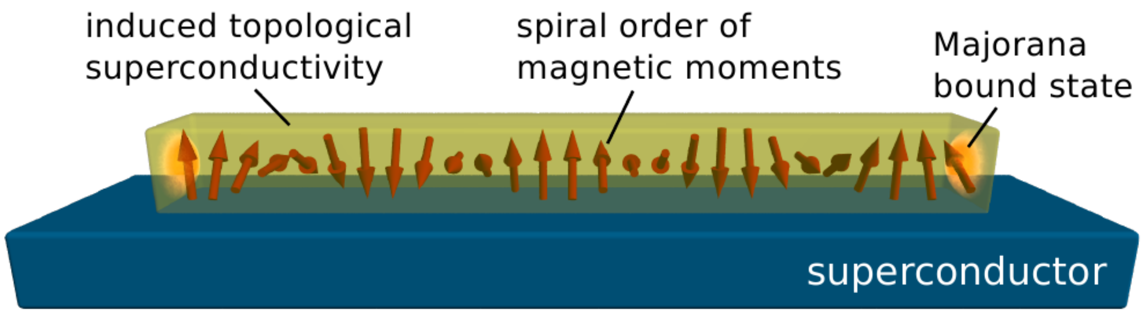


Braunecker *et al.*, PRL 2013

Electrons feel a helical Zeeman field

$$M\cos(Qx)\sigma_z - M\sin(Qx)\sigma_x = Me^{iQx\sigma_y/2}\sigma_z e^{-iQx\sigma_y/2}$$

Helical magnetism and synthetic SOC



Braunecker *et al.*, PRL 2013

Electrons feel a helical Zeeman field

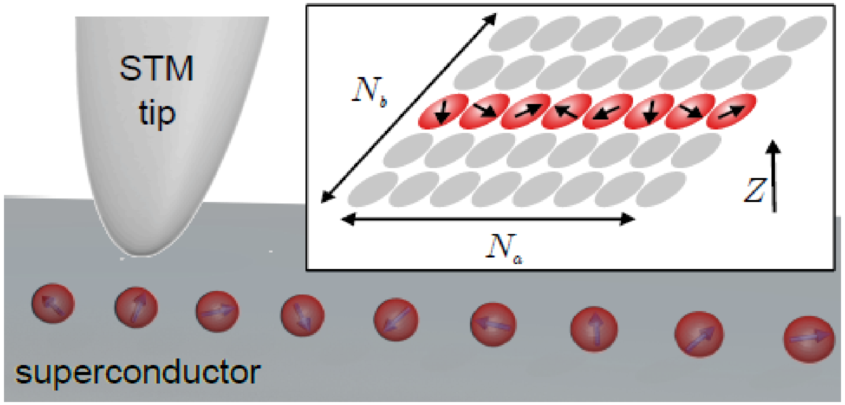
$$M\cos(Qx)\sigma_z - M\sin(Qx)\sigma_x = Me^{iQx\sigma_y/2}\sigma_z e^{-iQx\sigma_y/2}$$

Effective spin-orbit coupling in momentum space

$$\frac{(\hat{p}_x + \hbar Q\sigma_y/2)^2}{2m} - \mu = \frac{\hat{p}_x^2}{2m} - \underline{\left[\mu - \frac{(\hbar Q)^2}{8m}\right]} + \frac{\hbar Q}{2m}\hat{p}_x\sigma_y$$

Topological Criterion: $|M| > \sqrt{\left[\mu - (\hbar Q)^2/(8m)\right]^2 + \Delta_{\mathrm{FS}}^2}$

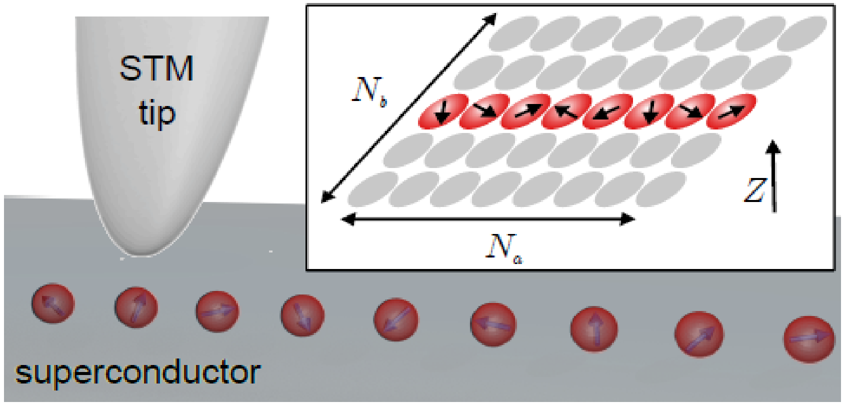
MFs in STM-probed magnetic adatom chains



Nadj-Perge *et al.*, PRB 2013

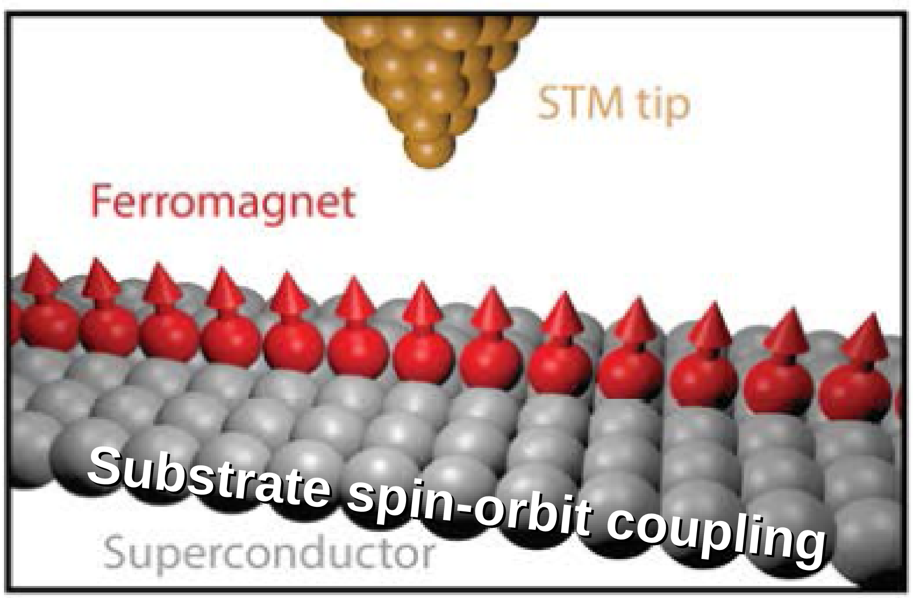
STM technology provides spatial resolution of MF wavefunctions

MFs in STM-probed magnetic adatom chains

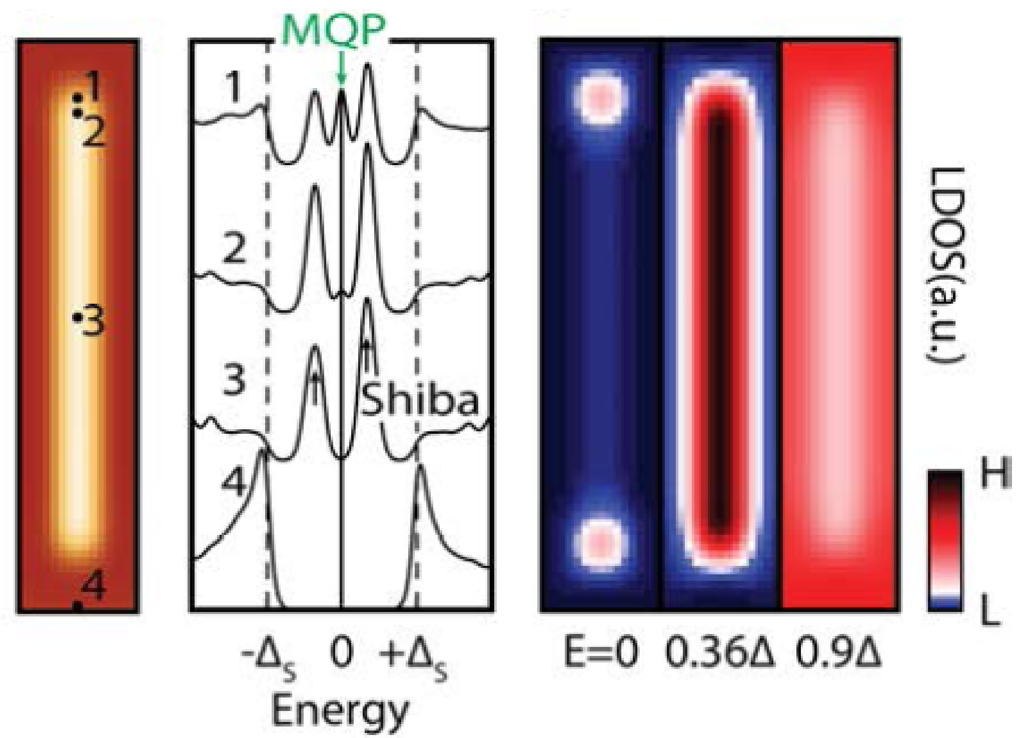


Nadj-Perge *et al.*, PRB 2013

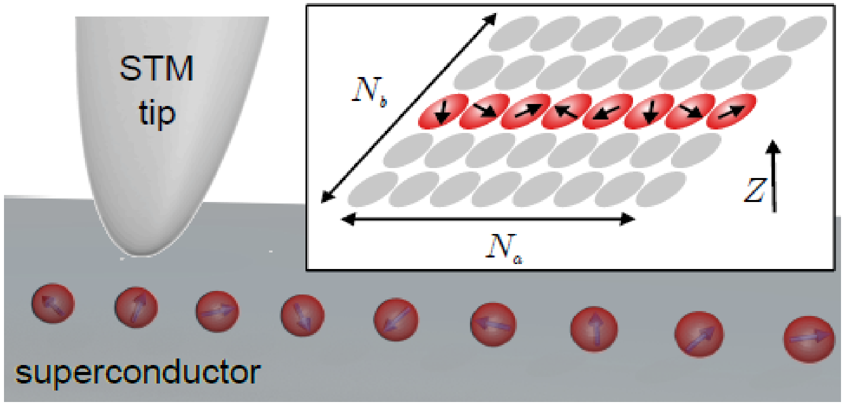
STM technology provides spatial resolution of MF wavefunctions



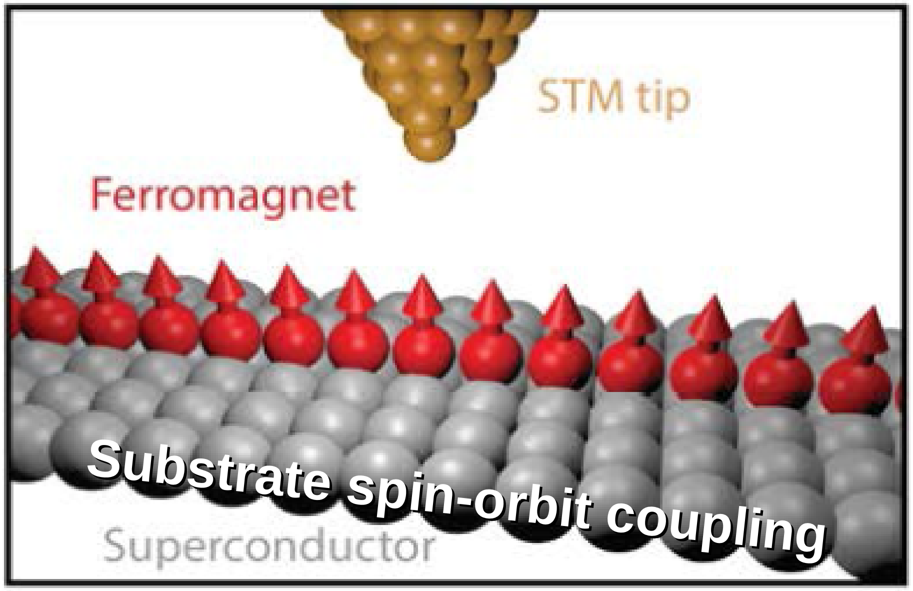
Nadj-Perge *et al.* Science 2014



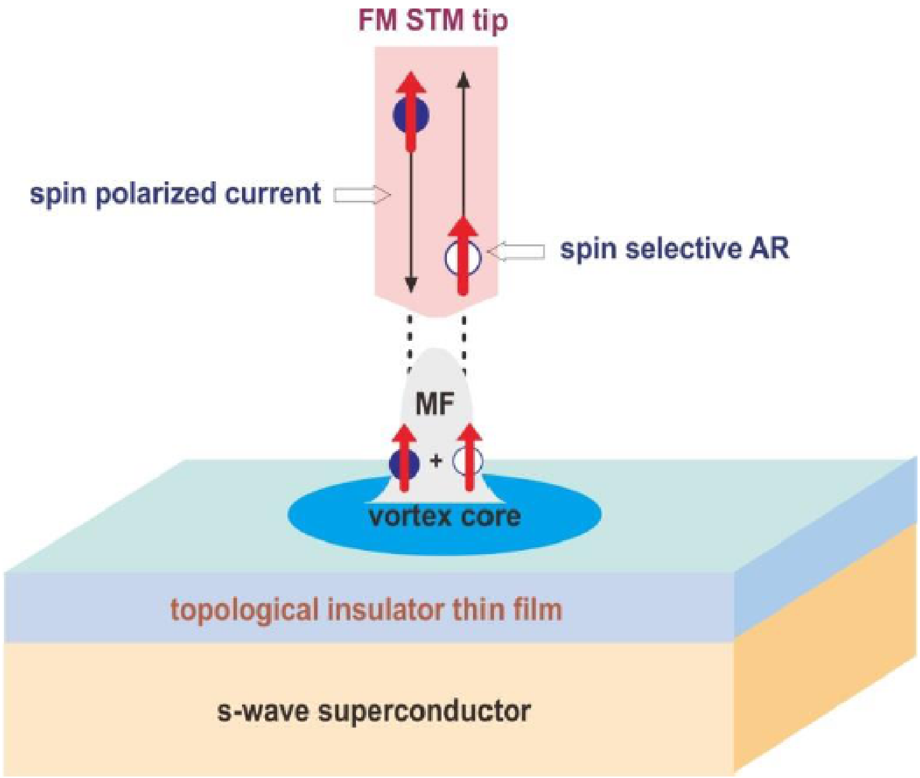
MFs in STM-probed magnetic adatom chains



Nadj-Perge *et al.*, PRB 2013

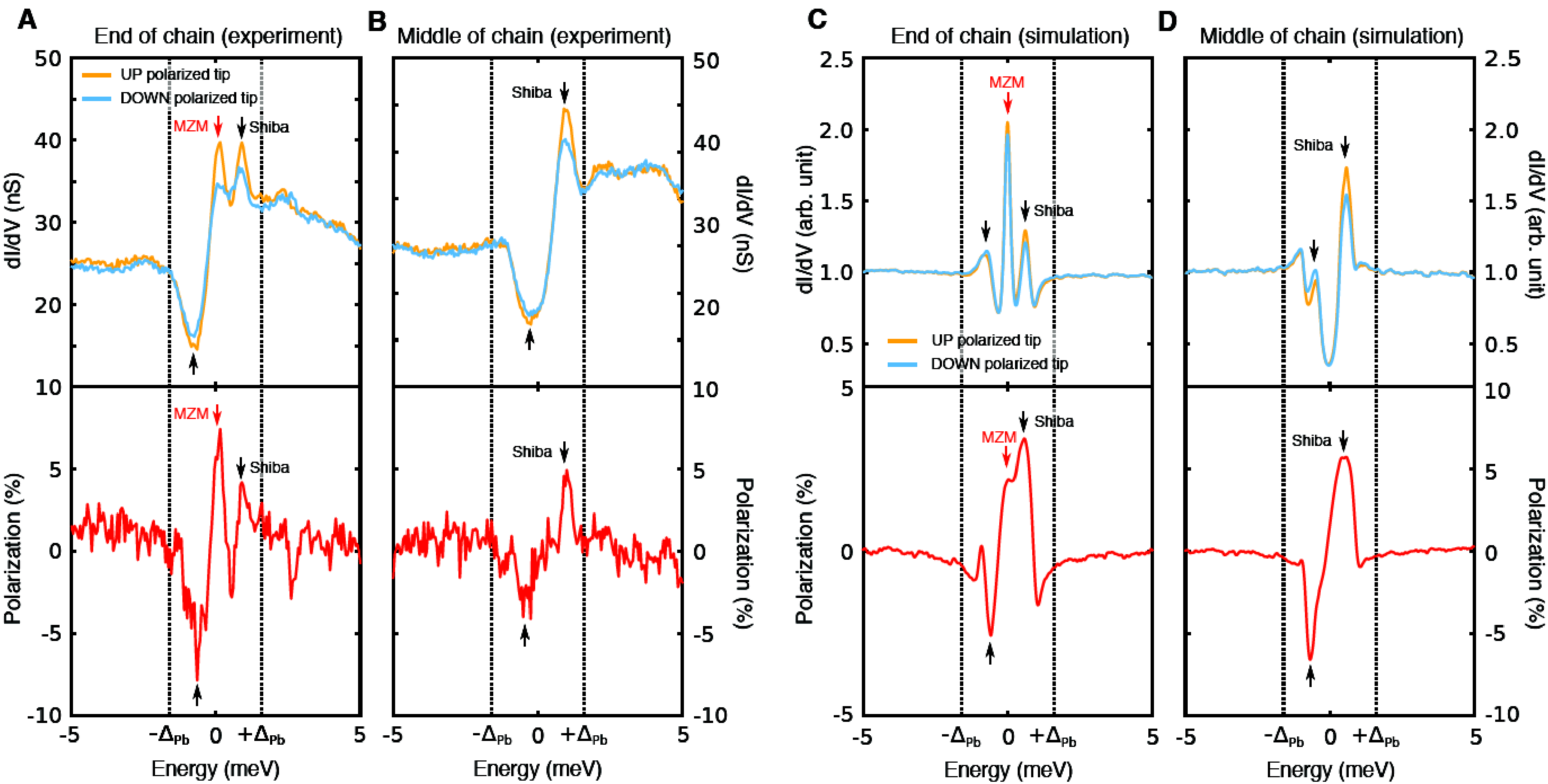


Nadj-Perge *et al.* Science 2014



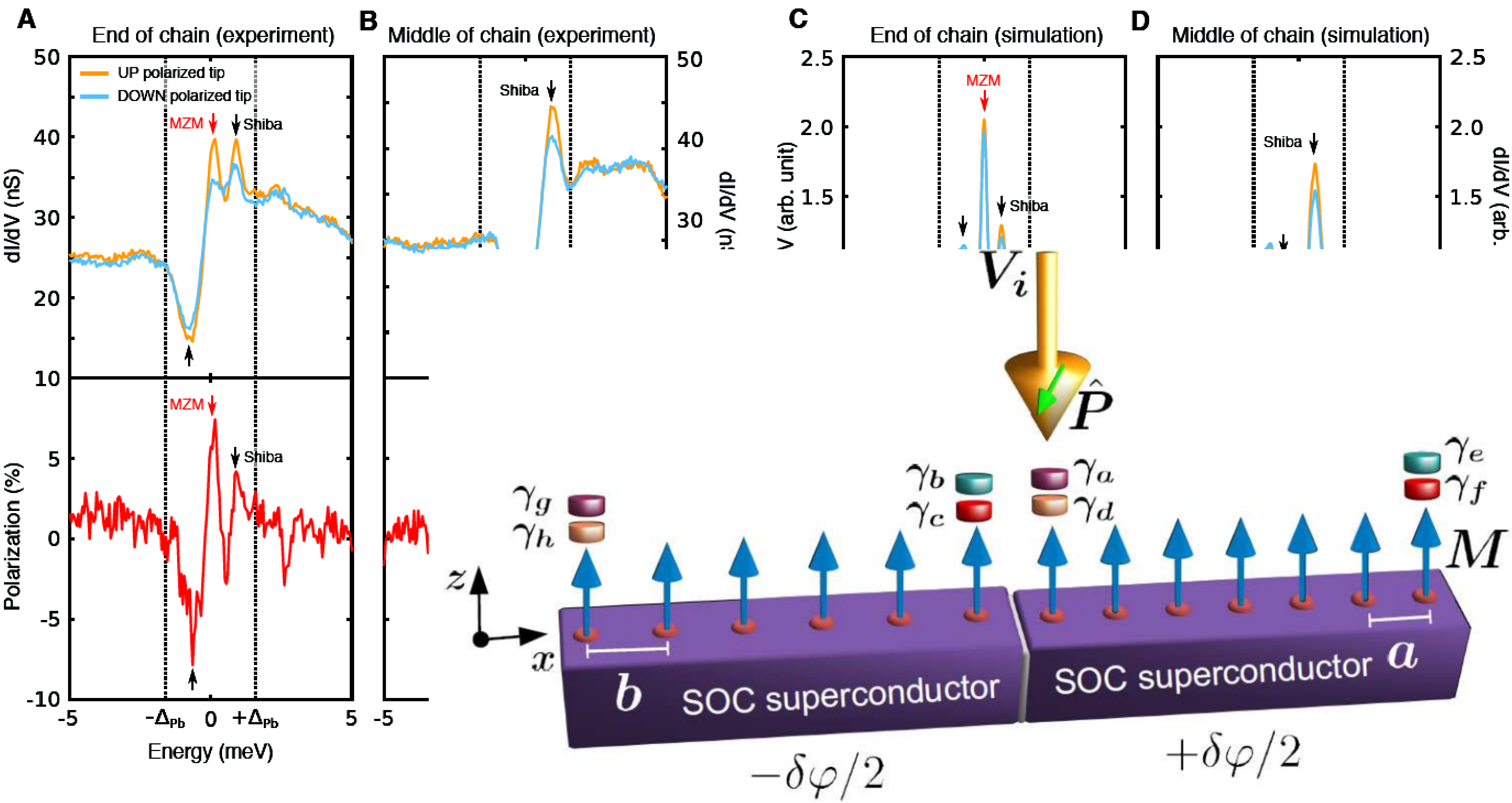
Sun *et al.*, PRL 2016

Recent spin-resolved measurements



Jeon *et al.*, Science 2017

Recent spin-resolved measurements



Jeon *et al.*, Science 2017

PK *et al.*, Physica E 2015

Conclusions and Outlook

- Find functional p-wave superconductors

- Or intrinsic topological magnetic superconductors due to textures

- Parafermions, Fibonacci anyons and universal Q-computing

- Topological phases and interactions beyond mean-field theory

- Majorana manipulations and braiding in nanowire systems

- Floquet and non-Hermitian topological systems

Thank you !!!