Stability of black holes in Horndeski theory and beyond

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based on: [1712.04398]
[in preparation]

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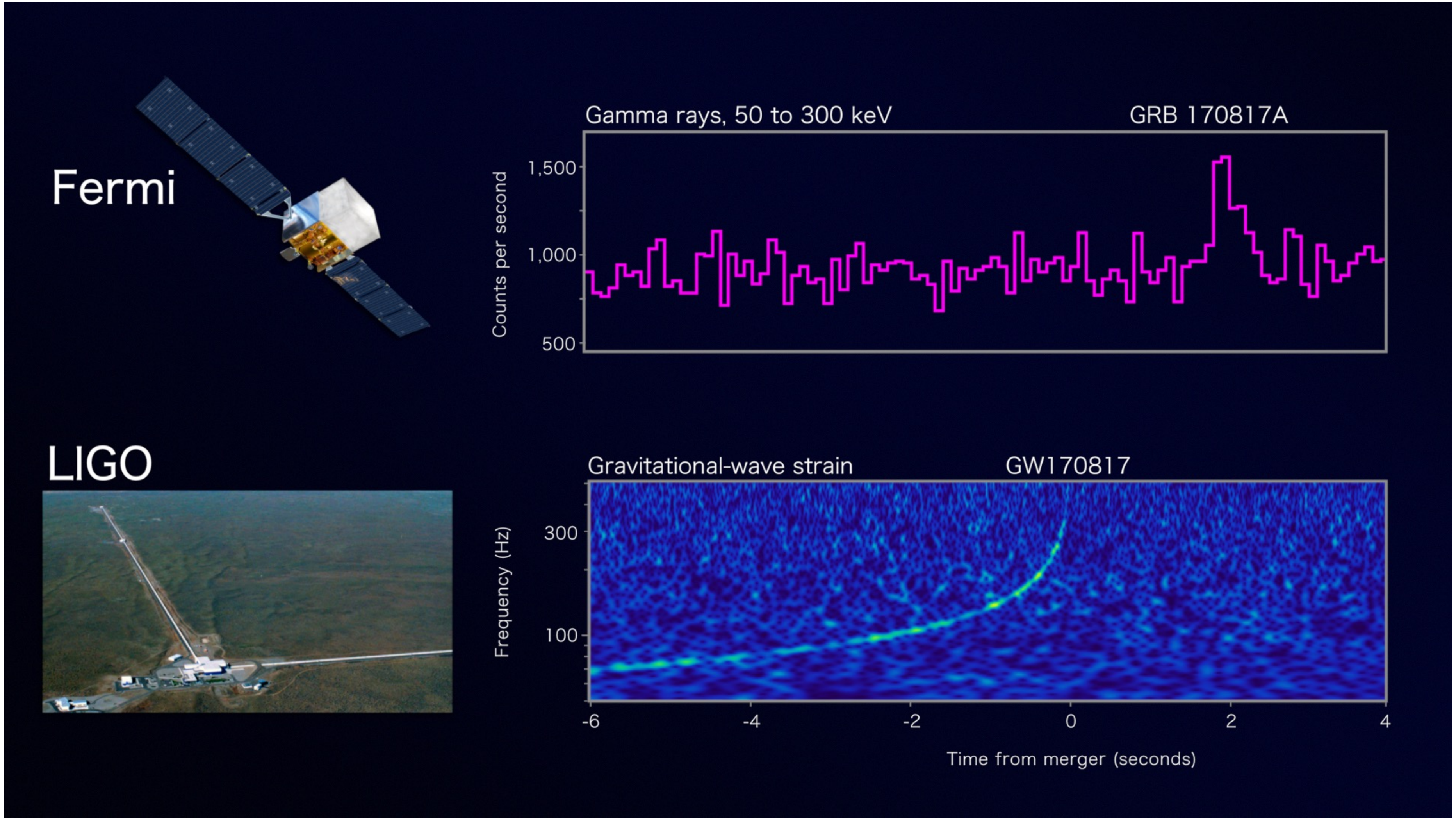
Motivation:

GW170817/GRB170817A



graviton speed = speed of light

Motivation:

Black holes found in [EB, Charmousis'13] are unstable

[Ogawa, Kobayashi, Suyama'15]

Similar arguments for instability are used in

[Takahashi, Suyama, Kobayashi'15]

[Takahashi, Suyama'17]

[Kase, Minamitsuji, Tsujikawa, Zhang'18]

[Maselli, Silva , Minamitsuji, Berti'16]

Outline

- Perturbations in galileons
- Perturbations of black hole solutions in Horndeski theory
- Hamiltonian vs stability
- Stable black holes after GW170817

Perturbations in galileons

Horndeski theory

Most general galileon shift-symmetric action: [Horndeski'1974,Deffayet et al'09]

$$\mathcal{L}_2 = K\left(X, \varphi\right)$$
$$\mathcal{L}_3 = G_3\left(X, \varphi\right)\Box\varphi$$
$$\mathcal{L}_4 = G_4(X, \varphi)\, R + G_{4,X}(X, \varphi)\left[\left(\Box\varphi\right)^2 - \left(\nabla\nabla\varphi\right)^2\right],$$
$$\mathcal{L}_5 = G_{5,X}\left(X, \varphi\right)\left[\left(\Box\varphi\right)^3 - 3\Box\varphi\left(\nabla\nabla\varphi\right)^2 + 2\left(\nabla\nabla\varphi\right)^3\right] - 6G_5\left(X, \varphi\right)\, G_{\mu\nu}\nabla^{\mu}\nabla^{\nu}\varphi$$

where $X \equiv (\partial_\lambda\varphi)^2$

EOMs are of second order

Kinetic mixing

- Standard kinetic term:

$$\sqrt{-g}\, g^{\mu\nu}\, \partial_\mu\phi\partial_\nu\phi \quad \text{only mixing of } g^{\mu\nu} \text{ and } \partial\phi \text{ — no kinetic mixing}$$

- k-essence:

$$\sqrt{-g}\, g^{\mu\nu}\, K(X) : \quad \text{only mixing of } g^{\mu\nu} \text{ and } \partial\phi \text{ — no kinetic mixing}$$

- G_3 galileon:

$$X\Box\phi \;\supset\; \Gamma\partial\phi \;\sim\; \partial g\partial\phi : \quad \text{kinetic mixing}$$

Perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$
$$\phi = \bar{\phi} + \pi$$

- Standard kinetic term, k-essence:

$$\mathcal{L}^{(2)} \sim \underbrace{(\partial h)^2 + (\partial\delta\phi)^2}_{\text{kinetic terms}} + \underbrace{h^2 + (\delta\phi)^2 + h(\delta\phi)}_{\text{lower order terms}}$$

- Higher-order galileons:

$$\mathcal{L}^{(2)} \sim \underbrace{(\partial h)^2 + (\partial\delta\phi)^2 + (\partial\delta\phi)\partial h}_{\text{kinetic terms}} + h^2 + (\delta\phi)^2 + h(\delta\phi)$$

Perturbation in G3 model

[EB,Esposito-Farese'12]

$$S = M_P^2 \int d^4x \sqrt{-g} \left\{ \frac{R}{2} - \eta(\partial_\mu \phi)^2 - \gamma \Box\phi (\partial_\mu \phi)^2 \right\}$$

Perturbation Lagrangian:

$$\frac{1}{M_P^2} \frac{\mathcal{L}_2^{\text{kinetic}}}{\sqrt{-g}} =$$

$$-\frac{1}{4} \nabla_\mu h_{\alpha\beta} P^{\alpha\beta\gamma\delta} \nabla^\mu h_{\gamma\delta} + \frac{1}{4} \left(h^\lambda_{\nu;\lambda} - \frac{1}{2} h_{,\nu} \right)^2 - \eta \left(\partial_\mu \pi \right)^2$$

$$-\gamma \left[2\Box\phi \left(\partial_\mu \pi \right)^2 - 2\nabla_\mu \partial_\nu \phi \, \partial^\mu \pi \partial^\nu \pi \underbrace{+ \partial_\mu \phi \partial_\nu \phi \, \partial_\lambda \pi \nabla^\lambda h^{\mu\nu} - 2\partial^\mu \phi \partial^\nu \phi \, \partial_\mu \pi \left(h^\lambda_{\nu;\lambda} - \frac{1}{2} h_{,\nu} \right)}_{\text{mixing terms}} \right]$$

Perturbation in G3 model

Change of variables

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{4k_3}{M^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\lambda\phi\right)^2\right]\pi$$

$$\frac{1}{M_P^2}\frac{\mathcal{L}_2^{\text{kinetic}}}{\sqrt{-g}} = -\frac{1}{4}\nabla_\mu h_{\alpha\beta}P^{\alpha\beta\gamma\delta}\nabla^\mu h_{\gamma\delta} + \frac{1}{4}\left(h^\lambda_{\nu;\lambda} - \frac{1}{2}h_{,\nu}\right)^2 - \mathcal{S}^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$$

$$\mathcal{S}^{\mu\nu} \equiv g^{\mu\nu}\left[\eta + 2\gamma\Box\varphi - \gamma^2\left(\partial_\lambda\varphi\right)^4\right] - 2\gamma\nabla^\mu\partial^\nu\varphi + 4\gamma^2\left(\partial_\lambda\varphi\right)^2\partial^\mu\varphi\partial^\nu\varphi$$

Effective metric felt by the scalar perturbations π

Perturbation in "John" model

$$\mathcal{L}_J = G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

The mixing is more complicated,

$$\mathcal{L}^{(2)}_{J,mix} \sim \partial_\lambda\pi\nabla^\lambda h^{\mu\nu} + \partial_\mu\pi\left(h^\lambda_{\nu;\lambda} - \frac{1}{2}h_{,\nu}\right) + \partial_\lambda\pi\nabla^\lambda h + \partial_\mu\pi\nabla_\nu h^{\mu\nu}$$

$h_{\mu\nu} \to$?

It is not clear how to demix perturbations in this case

Let us use symmetry of the background solution

Perturbations of black hole solutions in Horndeski theory

Black hole solution

[Babichev, Charmousis'13]

$$S_{\mathrm{J}}[g_{\mu\nu},\phi] = \int \sqrt{-g}\,\mathrm{d}^4x\,\left[\zeta(R - 2\Lambda_{\mathrm{bare}}) + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \eta\,\phi_\lambda^2\right]$$

In terms of standard Horndeski notations:

$$G_4 = \zeta - \frac{\beta}{2}\phi_\lambda^2, \qquad G_2 = -2\zeta\Lambda_{\mathrm{bare}} - \eta\phi_\lambda^2.$$

Black hole solution

[Christos talk]

Self-tuning Schwarzschild-de Sitter solution:

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2,$$

$$A(r) = 1 - \frac{2Gm}{r} - \frac{\Lambda_{\rm eff}}{3}\,r^2,$$

$$\phi = q\left[t \pm \int \frac{\sqrt{1-A(r)}}{A(r)}dr\right],$$

$$q^2 = \frac{\eta + \beta\,\Lambda_{\rm bare}}{\eta\,\beta}\,\zeta \quad \Lambda_{\rm eff} = -\frac{\eta}{\beta},$$

Also stealth solution when $\eta = \Lambda_{\rm bare} = 0$:

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2, \quad A(r) = 1 - \frac{2Gm}{r}$$

$$\phi = q\left[t \pm \int \frac{\sqrt{1-A(r)}}{A(r)}\,dr\right]$$

Perturbations of black holes

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$
$$\phi = \bar{\phi} + \pi$$

Expansion in spherical harmonics—even parity (polar) and odd-parity (axial) modes, which do not interact with each other:

1. Odd-parity modes only contain spin-2 polarizations; [Ogawa, Kobayashi, Suyama'15]
2. Even-parity modes: $l = 0$ contains pure i.e. scalar, $l = 1$ is dipole, $l \geq 2$ contain both scalar and spin-2 polarizations

Perturbations of black holes

Spherically symmetric perturbations (Regge and Wheeler formalism):

$$h_{\mu\nu} = \begin{pmatrix} A(r)H_0(t,r) & H_1(t,r) & 0 & 0 \\ H_1(t,r) & H_2(t,r)/B(r) & 0 & 0 \\ 0 & 0 & K(t,r)r^2 & 0 \\ 0 & 0 & 0 & K(t,r)r^2\sin^2\theta \end{pmatrix}$$

Second order action in terms of π, H_0, H_1 H_2, K,

$$\delta_{\rm s}^{(2)} S_{\rm J} = \int {\rm d}t{\rm d}r 4\pi r^2 \mathcal{L}_{\rm s}^{(2)}$$

Perturbations of black holes

$$\mathcal{L}_{\rm s}^{(2)} = \mathcal{P}^2 + \mathcal{A}y^2 + \mathcal{B}xy + \mathcal{C}x^2$$

$$\mathcal{P} = \dot{x} - y' + \tilde{a}_1 x + \tilde{a}_2 y$$

variation wrt y gives: $2\mathcal{P}' + 2\mathcal{A}y + \mathcal{B}x = 0$

Constraint on y (nonlocal in space): in principle one can find y in terms of x by solving ODE with known boundary conditions.

Instead focus on higher order terms:

$$\dot{x}' - y'' = 0 \quad \Rightarrow \quad \dot{x} = y'$$

$$x = \chi'$$

$$\mathcal{L}_{\rm s;\,Kin}^{(2)} = -\frac{1}{2}\left(\mathcal{S}^{tt}\dot{\chi}^2 + 2\mathcal{S}^{tr}\dot{\chi}\chi' + \mathcal{S}^{rr}\chi'^2\right)$$

Stability?

$$\mathcal{L}_2 = -\frac{1}{2}\left(\mathcal{S}^{tt}\dot{\chi}^2 + 2\mathcal{S}^{tr}\dot{\chi}\chi' + \mathcal{S}^{rr}\chi'^2\right) \quad \Leftrightarrow \quad \mathcal{L}_2 = -\frac{1}{2}\mathcal{S}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$$

$\mathcal{S}^{\mu\nu}$ is a function of background

Need check for:

1. Hyperbolicity (no Laplace instability)
2. No ghosts

1. Hyperbolicity:

$$D \equiv \mathcal{S}^{00}\mathcal{S}^{11} - \left(\mathcal{S}^{01}\right)^2 < 0 \quad \Rightarrow \text{the cone is defined}$$

2. No ghost:

"Energy of particles >0"

Calculate Hamiltonian and check if it is bounded from below

Stability?

$$S_{\mathrm{J}}[g_{\mu\nu},\phi] = \int \sqrt{-g}\,\mathrm{d}^4x\,[\zeta R + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi]$$

$$\eta = \Lambda_{\mathrm{bare}} = 0$$

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2, \quad A(r) = 1 - \frac{2Gm}{r}$$

$$\phi = q\left[t \pm \int \frac{\sqrt{1-A(r)}}{A(r)}\,dr\right]$$

Perturbations for stealth solution are given by parabolic equation:
pathological behaviour

Hamiltonian vs stability

Hamiltonian

$$\mathcal{L}_2 = -\frac{1}{2}\mathcal{S}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$$

$$p \equiv \frac{\partial \mathcal{L}_2}{\partial \dot\chi} = -\mathcal{S}^{00}\dot\chi - \mathcal{S}^{0i}\partial_i\chi, \quad \text{canonical momentum}$$

$$\mathcal{H}_2 = p\,\dot\chi - \mathcal{L}_2 = -\frac{1}{2\mathcal{S}^{00}}\left(p + \mathcal{S}^{0i}\partial_i\chi\right)^2 + \frac{1}{2}\mathcal{S}^{ij}\partial_i\chi\partial_j\chi$$

In 1+1 we have:

$$\mathcal{H}_2 = p\,\dot\chi - \mathcal{L}_2 = -\frac{1}{2\mathcal{S}^{00}}\left(p + \mathcal{S}^{01}\chi'\right)^2 + \frac{1}{2}\mathcal{S}^{11}\chi'^2$$

Does unbounded from below Hamiltonian necessarily imply instability?

NO

Hamiltonian: example

$$\mathcal{L} = \frac{1}{2}\dot{\chi}^2 - \frac{c_s^2}{2}\chi'^2$$

$$\text{Relativistic boost } c = 1 :$$

$$\tilde{t} = \frac{t + vx}{\sqrt{1 - v^2}}, \quad \tilde{x} = \frac{x + vt}{\sqrt{1 - v^2}}$$

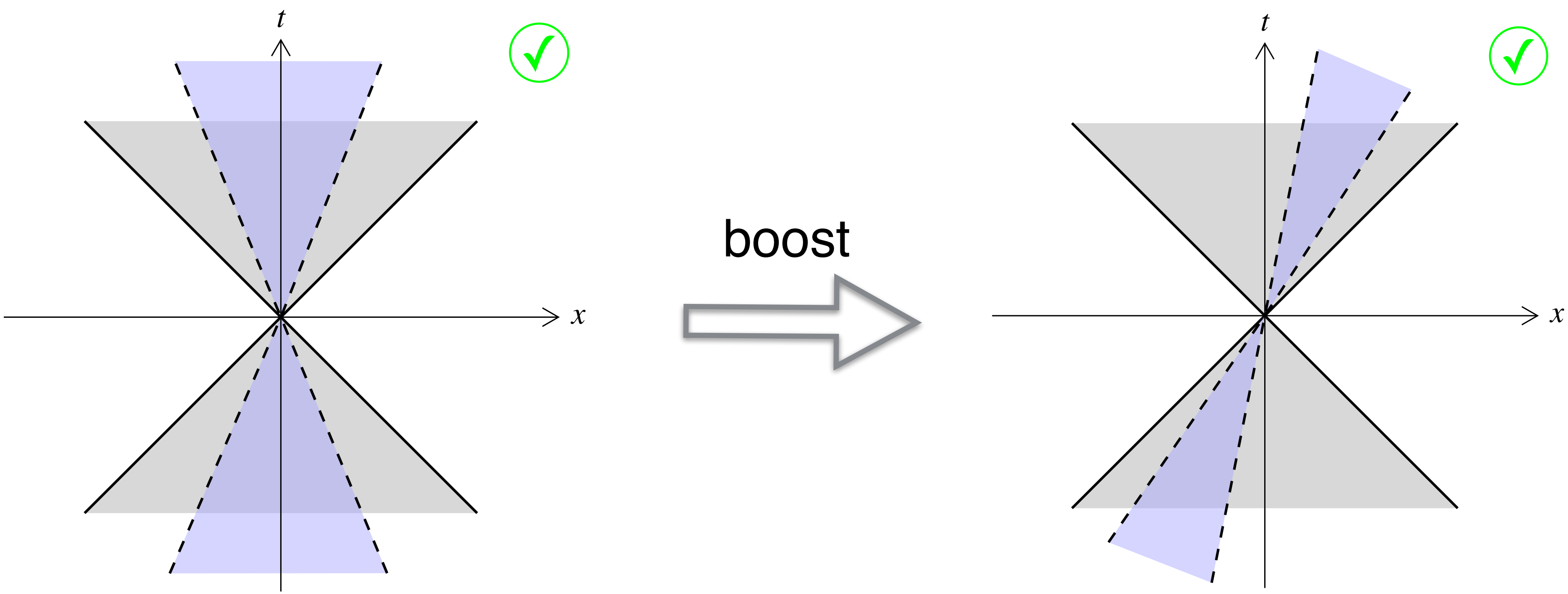
$$\mathcal{L} \to \frac{1}{1 - v^2}\left[\frac{1}{2}(1 - c_s^2 v^2)\dot{\chi}^2 + (1 - c_s^2) v \dot{\chi}\chi' - \frac{1}{2}(c_s^2 - v^2)\chi'^2\right]$$

Compute Hamiltonian: $\mathcal{H}_2 = \frac{1}{2}(\ldots)^2 + \frac{1}{2}(c_s^2 - v^2)\pi'^2$

$\mathcal{H}_2 < 0 \text{ for } |v| > c_s$

However the system is clearly stable

Stable configurations:



$$\mathcal{H}_2 = \frac{1}{2}(\ldots)^2 + \frac{1}{2}c_s^2 \pi'^2 \qquad\qquad \mathcal{H}_2 = \frac{1}{2}(\ldots)^2 + \frac{1}{2}(c_s^2 - v^2)\pi'^2$$

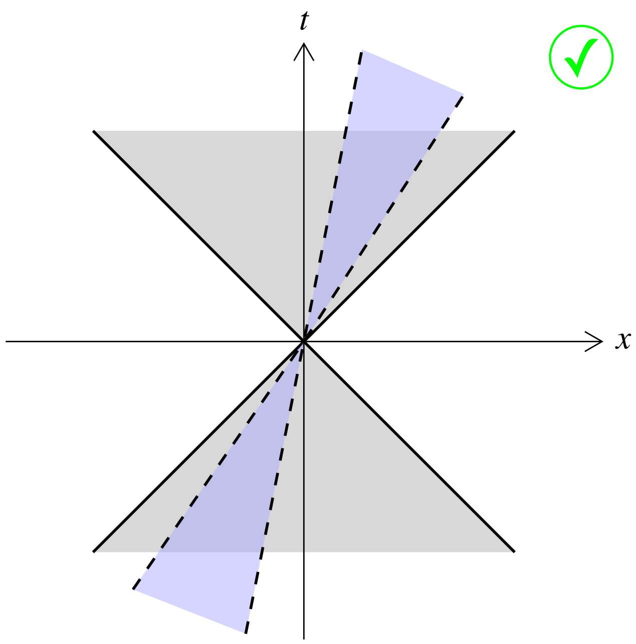
In both cases the system is stable

Stability vs Hamiltonian

Hyperbolicity (existence of propagation cone, characteristics):
$D \equiv \mathcal{S}^{00}\mathcal{S}^{11} - \left(\mathcal{S}^{01}\right)^2 < 0.$

The inverse is:

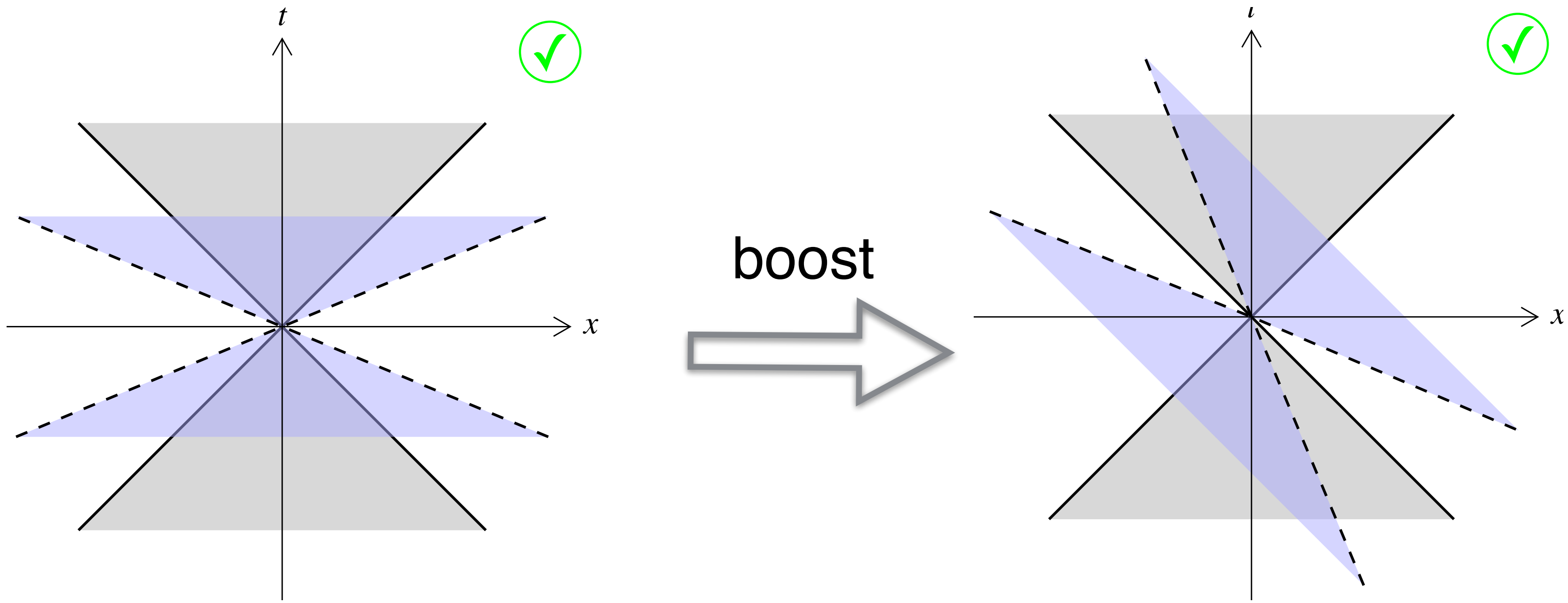
$$\mathcal{S}^{-1}_{\mu\nu} = \begin{pmatrix} \mathcal{S}^{11} & -\mathcal{S}^{01} \\ -\mathcal{S}^{01} & \mathcal{S}^{00} \end{pmatrix} / D. \qquad dS^2 = \mathcal{S}^{-1}_{\mu\nu} dx^\mu dx^\nu \quad \text{defines effective metric}$$



$$\mathcal{H}_2 = -\frac{1}{2\mathcal{S}^{00}}\left(p + \mathcal{S}^{01}\chi'\right)^2 + \frac{1}{2}\mathcal{S}^{11}\chi'^2$$

The time axis is *outside* the blue cone,
meaning $\mathcal{S}^{-1}_{00} dtdt > 0$ and therefore $\mathcal{S}^{11} < 0$

Stable configurations:



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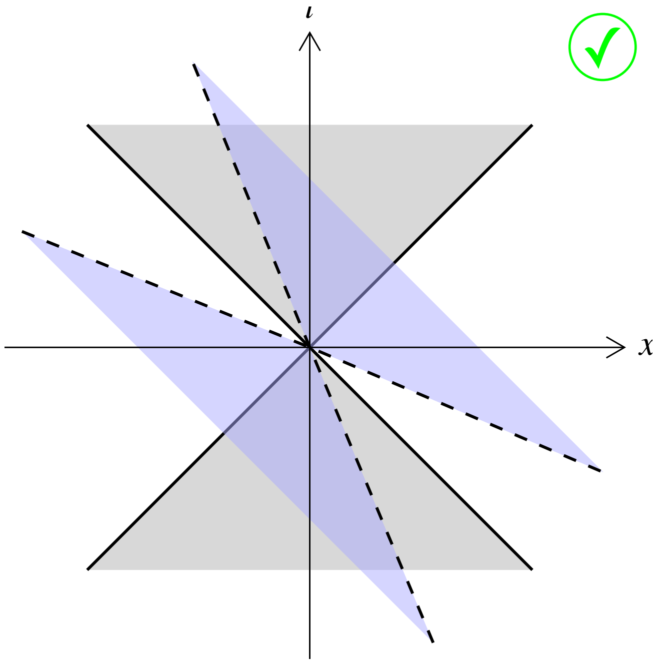
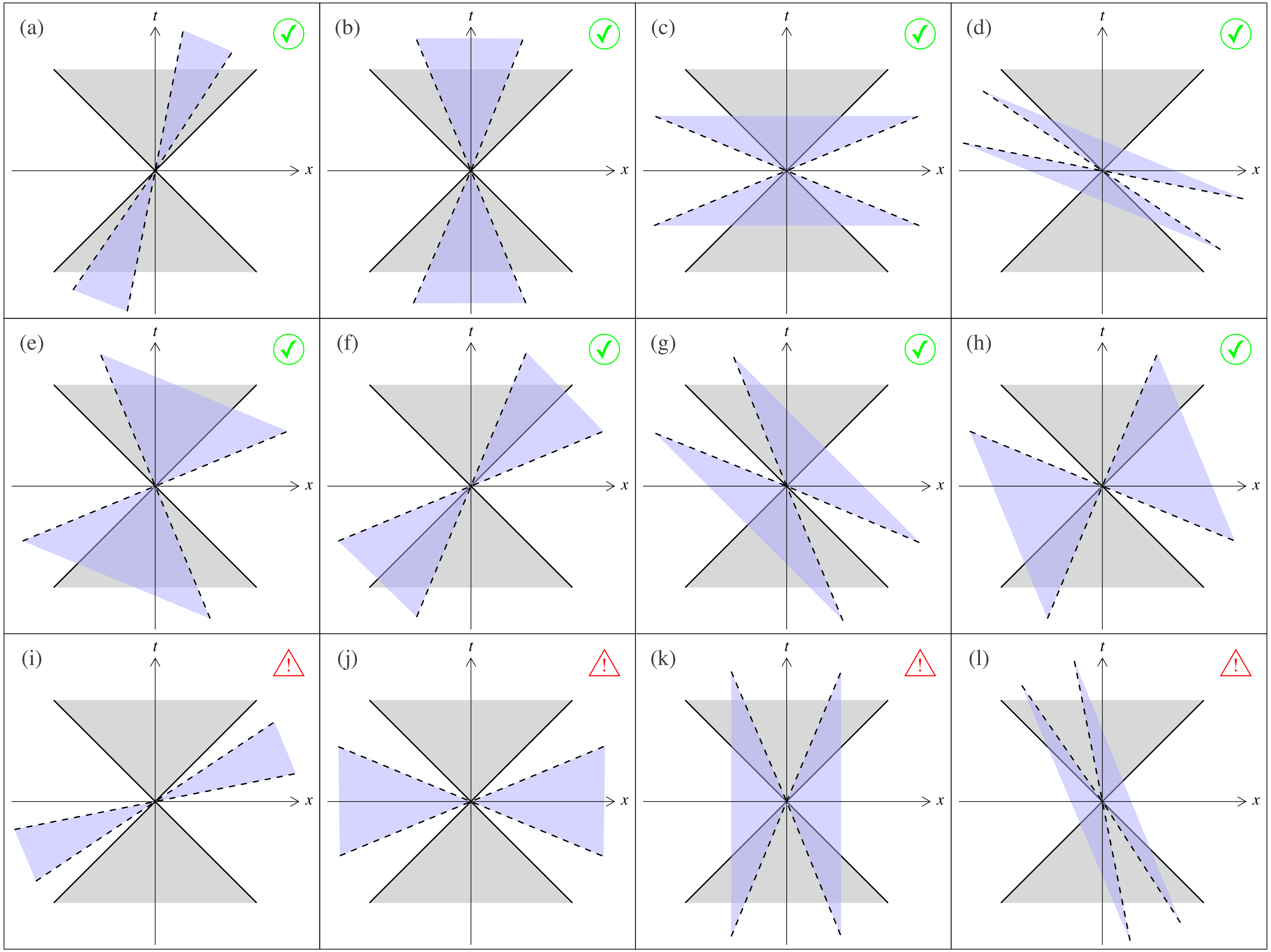


$$\mathcal{H}_2 = -\frac{1}{2\mathcal{S}^{00}}\left(p + \mathcal{S}^{01}\chi'\right)^2 + \frac{1}{2}\mathcal{S}^{11}\chi'^2$$

The x-axis is *inside* the blue cone,
meaning $\mathcal{S}^{-1}_{11} dx dx < 0$ and therefore $\mathcal{S}^{00} > 0$

Stability vs Hamiltonian



Stability vs Hamiltonian

- When total Hamiltonian density is bounded by below, then the lowest energy state is necessarily stable.
- Inverse is not true: A Hamiltonian density which is unbounded from below does not always imply instability.
- Sometimes the unbounded Hamiltonian appears due to the “bad” choice of coordinate

! The Hamiltonian is not a scalar with respect to coordinate transformations

instead of Hamiltonian?

$$\mathcal{L}_2 = -\frac{1}{2}\mathcal{S}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$$

Four conserved Noether currents: $T^\nu_\mu \equiv \frac{\delta\mathcal{L}_2}{\delta(\partial_\nu\chi)}\partial_\mu\chi - \delta^\nu_\mu\mathcal{L}_2$

Current conservation: $\partial_\nu T^\nu_\mu = 0 \Leftrightarrow \partial_0 T^0_\mu + \partial_i T^i_\mu = 0$

The energy density T^0_0 coincides with Hamiltonian density (on-shell): it is not diff invariant.

Instead of T'^0_0 one may consider an invariant (under boosts) quantity:

$$T'^0_0 - vT'^0_x$$

Coincides with Hamiltonian in unboosted frame

Stable black holes after GW170817

Constraints from GW170817

[EB,Charmousis,Esposito-Farese,Lehebel'17]

$$\mathcal{L} = \zeta\,(R - 2\Lambda_{\rm bare}) - \eta\,\varphi_\lambda^2 + \beta\,G^{\mu\nu}\varphi_\mu\varphi_\nu \;\; + \mathcal{L}_{\rm matter}[g]$$

$$c_{\rm grav} \neq c_{\rm light}$$

Horndeski with minimal coupling to matter

Constraints from GW170817

[EB,Charmousis,Esposito-Farese,Lehebel'17]

$$\mathcal{L} = \zeta\,(R - 2\Lambda_{\rm bare}) - \eta\,\varphi_\lambda^2 + \beta\,G^{\mu\nu}\varphi_\mu\varphi_\nu \quad + \mathcal{L}_{\rm matter}[\tilde{g}]$$

$c_{\rm grav} = c_{\rm light}$ on (almost) homogeneous backgrounds

Horndeski with nonminimal (disformal) coupling to matter

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{\beta}{\zeta + \frac{\beta}{2}\,\varphi_\lambda^2}\,\varphi_\mu\varphi_\nu$$

$$\tilde{g} \to g$$

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}\,(R - 2\Lambda_{\rm bare}) - \eta\varphi_\lambda^2 + \beta G^{\mu\nu}\varphi_\mu\varphi_\nu - \frac{\beta}{X}\,\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\,\varphi_\mu\,\varphi_\alpha\,\varphi_{\nu\beta}\,\varphi_{\rho\gamma} + \mathcal{L}_{\rm matter}[g]$$

$c_{\rm grav} = c_{\rm light}$ on (almost) homogeneous backgrounds

beyond Horndeski with minimal coupling to matter

Stability of black holes

$$S_{\rm J}[g_{\mu\nu},\varphi] = \int \sqrt{-g}\,{\rm d}^4x\,\left[\zeta(R-2\Lambda_{\rm bare}) + \beta G^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \eta\,\varphi_\lambda^2\right]$$

Matter couples to

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{\beta}{\zeta + \frac{\beta}{2}\varphi_\lambda^2}\partial_\mu\varphi\partial_\nu\varphi$$

$$S_{\rm J}[g_{\mu\nu},\varphi] + S_{\rm m}[\tilde{g}_{\mu\nu},\Psi]$$

Stability of black holes

1. We concentrated on scalar mode (even *l*=0 mode).
2. The odd-parity modes were calculated in
   These modes correspond to spin-2 polarisation. We explicitly checked that for these modes [Ogawa, Kobayashi, Suyama'15].

$$\mathcal{G}_{\mu\nu} = \frac{\Lambda_{\text{eff}}}{\Lambda_{\text{bare}} + \Lambda_{\text{eff}}} \tilde{g}_{\mu\nu}$$

3. Stability window:

[EB,Charmousis,Esposito-Farese,Lehebel'17]

$$\text{either } \eta > 0,\ \beta < 0, \ \text{ and } \ \frac{1}{3}\Lambda_{\text{bare}} < -\frac{\eta}{\beta} < \Lambda_{\text{bare}},$$

$$\text{or } \eta < 0,\ \beta > 0, \ \text{ and } \ \Lambda_{\text{bare}} < -\frac{\eta}{\beta} < 3\Lambda_{\text{bare}}.$$

Conclusions

- There are solutions with exact equality of speeds of light end gravity, even in the vicinity of black hole.
- Hamiltonian vs stability.
- Stable of black holes for a range of parameters.