Exact integrable RG flows and the c-theorem

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Based on:

- K.S., Nucl. Phys. **B880** (2014) 225
- and with: G. Georgiou, G. Itsios, K. Siampos & E. Sagkrioti in various combinations.
- and in progress.

For Ioannis

- We first met in 1989 in Trieste
- Several common projects from 1995 to 2002. Most important:
  - T-duality and world sheet supersymmetry
    Ioannis Bakas (CERN), Konstadinos Sfetsos (Utrecht U.),
    Phys.Lett. B349 (1995) 448-457
  - States and curves of five-dimensional gauged supergravity
    Ioannis Bakas (Patras U.), Konstadinos Sfetsos (CERN),
    Nucl.Phys. B573 (2000) 768-810
  - PP waves and logarithmic conformal field theories
    Ioannis Bakas, Konstadinos Sfetsos (Patras U.),
    Nucl.Phys. B639 (2002) 223-240
- We were colleagues in the U. of Patras for several years.
- I remember him for his distinct approach to problem solving and humor.
- On the non-scientific life of Ioannis see talk by T. Vladikas in Xmas Theoretical Physics Workshop @Athens 2016:
  https://sites.google.com/site/xmasathens2016/home2

Motivation

- Exact beta-function and anomalous dimensions in QFTs.
- Traditionally computed perturbatively. For example

$$\text{QED}: \qquad \mu\frac{de}{d\mu} = +\frac{e^3}{12\pi^2} + \cdots \ ,$$

$$\text{QCD}: \qquad \mu\frac{dg}{d\mu} = -\frac{7g^3}{16\pi^2} + \cdots$$

  It is a rare occasion to be able to compute them exactly.
- Systematic construction of new (integrable) deformations of (interacting) CFT's having explicit Lagrangian descriptions.
- Smooth RG flows (UV to IR) between CFTs.
- Type-II supergravity embeddings
- Applications/developments in an AdS/CFT context.

Outline

- The theories of interest
- Construction of effective actions (self- and mutual-interacting)
- Beta-functions and anomalous dimensions.

  Perturbative info + symmetry + analyticity $\Longrightarrow$ exact info

- Interplay between CFT and gravitational methods instrumental
- Exact C-function obeying Zamolodchikov's c-theorem (the first example in literature).
- Concluding remarks

The theories of interest

Let any 2-dim CFT with action S_k and a group G structure having holomorphic & anti-holomorphic currents $J^a(z)$ & $\bar{J}^a(\bar{z})$, obeying

$$J^a(z)J^b(w) = \frac{\delta_{ab}}{(z-w)^2} + \frac{f_{abc}}{\sqrt{k}}\frac{J^c(w)}{z-w}\ , \quad \text{similarly } \bar{J}^a(\bar{z})\bar{J}^b(\bar{z})\ .$$

We would be interested in deformations:

- Driven away from the conformal point by a self-interaction current bilinears

$$S_{k,\lambda} = S_k - \frac{k}{\pi}\int d^2z\ \lambda_{ab}J^a\bar{J}^b\ .$$

- Two decoupled theories driven away from the conformal point by mutual-interactions of current bilinears

$$S_{k_1,k_2,\lambda_1,\lambda_2} = S_{k_1} + S_{k_2} - \frac{1}{\pi}\int d^2z\ \left(k_1\lambda_1^{ab}J_1^a\bar{J}_2^b + k_2\lambda_2^{ab}J_2^a\bar{J}_1^b\right)\ .$$

We are aiming at:

- To compute the RG flow eqs

$$\frac{1}{2}\mu\frac{d\lambda_{ab}}{d\mu} = \cdots \ .$$

- Anomalous dimensions of currents and primary operators.
- Search for new fixed points under the RG flow towards the IR.
- Identity the CFT in the IR.
- We would like to do that exactly in λ and in k, unlike traditional approaches.
- Construct effective, all loop in λ, actions.

Effective actions

Self-interacting theories [KS 14]

The starting point is the action

$$S(g,\tilde{g}) = S_{\mathrm{WZW},k}(g) + S_{\mathrm{PCM},\kappa^2}(\tilde{g}) \ , \quad g,\tilde{g} \in G \ .$$

- $S_{\mathrm{WZW},k}(g)$: has a $G_{\mathrm{L,cur}} \times G_{\mathrm{R,cur}}$ current algebra symmetry.
- $S_{\mathrm{PCM}}(\tilde{g})$: it is integrable with global $G_L \times G_R$ symmetry.
- Introduce gauge fields and gauge the group acting as

$$g \to \Lambda^{-1} g \Lambda \ , \qquad \tilde{g} \to \Lambda^{-1} \tilde{g} \ , \qquad \Lambda \in G \ .$$

- After gauge fixing $\tilde{g} = \mathbb{1}$ integrating out the gauge fields

$$\boxed{S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi} \int J_+^a (\lambda^{-1} \mathbb{I} - D^T)^{-1}_{ab} J_-^b} \ ,$$

where $\lambda^{-1} = 1 + \kappa^2/k$ is the deformation parameter and

$$J_+^a = -i \mathrm{Tr}(t^a \partial_+ g g^{-1}) \ , \quad J_-^a = -i \mathrm{Tr}(t^a g^{-1} \partial_- g) \ , \quad D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1}) \ .$$

Some properties:

- Generalization to $\lambda\delta_{ab} \to \lambda_{ab}$ straightforward.
- For small λ_{ab}

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi}\int d^2\sigma\ \lambda_{ab} J_+^a J_-^b + \dots\ .$$

- This action has a duality-type symmetry [Itsios-KS-Siampos 14]

$$k \to -k\ , \qquad \lambda \to \lambda^{-1}\ , \qquad g \to g^{-1}\ .$$

  It should be reflected as a symmetry in physical quantities.
- An integrable theory [KS 13, Hollowood-Miramontes-Schmidtt 14]

$$\lambda_{ab} = \lambda\delta_{ab}\ .$$

Exact β-function and anomalous dims [Georgiou-Siampos-KS 15 & 16]

CFT and symmetry appoach

- We want to compute the 2-point functions

$$\langle J^a(x_1)J^b(x_2)\rangle_\lambda = \langle J^a(x_1)J^b(x_2)e^{-\frac{\lambda}{\pi}\int d^2z J^a(z)\bar{J}^a(\bar{z})}\rangle\ ,$$
$$\langle J^a(x_1)\bar{J}^b(x_2)\rangle_\lambda = \langle J^a(x_1)\bar{J}^b(x_2)e^{-\frac{\lambda}{\pi}\int d^2z J^a(z)\bar{J}^a(\bar{z})}\rangle\ ,$$

  perturbatively in λ by expanding the exponential.
- The basic correlators are

$$\langle J^a(x_1)J^b(x_2)\rangle = \frac{\delta_{ab}}{x_{12}^2}\ ,\quad \langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle = \frac{1}{\sqrt{k}}\frac{f_{abc}}{x_{12}x_{13}x_{23}}$$

  and similarly for the $\bar{J}^a$'s. Mixed $J\bar{J}$ correlators vanish.
- For higher correlators use Ward dentities

The perturbative β-function and anomalous dimensions

- The β-function is

$$\beta = \frac{1}{2}\mu\frac{d\lambda}{d\mu} = -\frac{c_G}{2k}\left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4)\right) ,$$

where c_G is the quadratic Casimir in the adjoint.

- The anomalous dimension of the currents is

$$\gamma^{(J)} = \mu\frac{d\ln Z^{1/2}}{d\mu} = \frac{c_G}{k}\left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4)\right) .$$

Task: Extend these exactly in λ?

Analyticity: λ-dependence of physical quantities

- Expand the action for $g = e^{ix^a t^a}$ around the identity

$$S_{k,\lambda} = \frac{k}{4\pi}\frac{1+\lambda}{1-\lambda}\int \partial_+ x^a \partial_- x^a + \cdots$$

- The β-function & anomalous dims may have poles at $\lambda = \pm 1$.
- The β-function & anomalous dims should be invariant under

$$k \to -k\ , \qquad \lambda \to \frac{1}{\lambda}\ ,$$

  for $k \gg 1$.
- Perturbative information to $\mathcal{O}(\lambda^2)$ and the above symmetry are enough to determine the β-function and the anomalous dimensions exactly in λ and to leading order in k.

The exact β-function and anomalous dimensions

The exact β-function and anomalous dimensions are of the form

$$\beta_\lambda = -\frac{c_G}{2k}\frac{f(\lambda)}{(1+\lambda)^2}\ , \qquad \gamma^{(J)} = \frac{c_G}{k}\frac{g(\lambda)}{(1-\lambda)(1+\lambda)^3}\ ,$$

where $f(\lambda)$ and $g(\lambda)$ are analytic in λ.

- They have a well defined non-Abelian and pseudodual limits as $\lambda \to \pm 1$ and $k \to \infty$.
- Due to the symmetry $(k,\lambda) \mapsto (-k,\lambda^{-1})$ we have that

$$\lambda^4 f(1/\lambda) = f(\lambda)\ , \qquad \lambda^4 g(1/\lambda) = g(\lambda)\ .$$

- $f(\lambda)$ and $g(\lambda)$ are polynomials of degree four, fixed by the above symmetry and the two-loop perturbative result.

- The final result for the beta-function is

$$\beta_\lambda = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} \leqslant 0$$

  In agreement with [Kutasov 89] and [Gerganov-LeClair-Moriconi 01].
- For the anomalous dimension

$$\gamma^{(J)} = \frac{c_G}{k}\frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \geqslant 0 \ .$$

  Agreement with perturbation theory to $\mathcal{O}(\lambda^3)$ and $\mathcal{O}(\lambda^4)$.
- Similarly for current 3-point functions and correlators of primary fields [Georgiou-KS-Siampos 16].

Gravitational approach

The one-loop β-functions are [Ecker-Honerkamp 71, Friedan 80, Braaten-Curtright-Zachos 85, Fridling-van de Ven 86]

$$\frac{\mathrm{d}G_{\mu\nu}}{\mathrm{d}t} + \frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}t} = R^-_{\mu\nu} + \nabla^-_\nu \xi_\mu \ , \quad t = \ln \mu^2 \ .$$

- Covariant derivatives & tensors with torsion. The ξ^μ's are diffs.
- Then the RG-flow eqs are [KS-Siampos 14, KS-Siampos-Sagkrioti 18]

$$\frac{d\lambda_{ab}}{dt} = \frac{1}{2k}\mathrm{Tr}\left(\mathcal{N}_a(\lambda,\lambda_0^{-1})\mathcal{N}_b(\lambda^T,\lambda_0)\right) \ ,$$

where the matrices $\mathcal{N}_a$ have elements

$$(\mathcal{N}_a(\lambda,\lambda_0^{-1}))_b{}^c = \left(\lambda_{ae}\lambda_{bd}f_{edf} - \lambda_0^{-1}f_{abe}\lambda_{ef}\right)g^{fc} \ ,$$

$$g_{ab} = (\mathbb{1}-\lambda^T\lambda)_{ab} \ , \quad k = \sqrt{k_1 k_2} \ , \quad \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1 \ .$$

Mutually-interacting theories [Georgiou-KS 16,17]

Modifying the gauging procedure produces an action with two coupling matrices. $\lambda_{1,ab}$ and $\lambda_{2,ab}$.

- If $\lambda_i = \lambda_i \delta_{ab}$ then it is integrable.
- Beta-function for $\lambda_{ab} = \lambda \delta_{ab}$

$$\frac{d\lambda}{dt} = -\frac{c_G}{2k} \frac{\lambda^2(\lambda - \lambda_0)(\lambda - \lambda_0^{-1})}{(1-\lambda^2)^2} \ .$$

  where λ is either λ_1 or λ_2.
- A new fixed point in the IR at

$$\lambda = \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1 \ .$$

- Anomalous dimensions also computed.

Smooth RG flows

The parameter λ runs with the energy scale.

- There must be a CFT at the IR at the fixed point at λ_0.
- Indeed one may show that the flow is [Georgiou-KS 17]

$$\underbrace{G_{k_1} \times G_{k_2}}_{\text{At the UV, } \lambda=0} \quad \stackrel{\text{RG flow}}{\Longrightarrow} \quad \underbrace{\frac{G_{k_1} \times G_{k_2-k_1}}{G_{k_2}} \times G_{k_2-k_1}}_{\text{At the IR, } \lambda=\lambda_0} ,$$

- $c_{\rm IR} < c_{\rm UV}$, i.e. in accordance with Zamolodchikov's c-theorem.

Exact Zamolodchikov's C-function, i.e. $C = C(\lambda, \lambda_0)$?

It will be the first such example in literature.

The exact C-function [In progress]

Zamolodchikov's c-theorem [A.B. Zamolodchikov 86]

In a 2-dim QFT with couplings λ^i:

- There exist a function $C(\lambda) \geqslant 0$ such that

$$\mu \frac{dC}{d\mu} = 2\beta^i \frac{\partial C}{\partial \lambda^i} \geqslant 0 \ .$$

- The equality is reached at the fixed points λ^i_* of the RG flow. Then $C(\lambda_*)$ equals the central charge of a CFT.
- The proof is technically simple: Based on renormalizability, rotational and translational symmetry and positivity.

Physical content:

- Theories at high energies have more degrees of freedom than at low energies.
- UV information is lost irreversibly towards the IR.

The exact C-function – one coupling case

On general grounds near a fixed point

$$G_{\lambda\lambda}\beta_\lambda \equiv \frac{\mu}{2}\partial_\mu\lambda = \frac{1}{24}\partial_\lambda C + \cdots \ ,$$

where $G_{\lambda\lambda}$ is the metric in the coupling space. In our case $G_{\lambda\lambda} = \frac{\dim G}{(1-\lambda^2)^2}$.

- The beta-function near $\lambda = 0$ is

$$\beta_\lambda(\lambda, \lambda_0) = -\frac{c_G}{2k}\lambda^2 + \cdots \ .$$

- At the UV CFT $G_{k_1} \times G_{k_2}$ for $k \gg 1$ is

$$c_{\rm UV} = 2\dim G - \frac{c_G \dim G}{2k}(\lambda_0 + \lambda_0^{-1}) + \cdots \ .$$

- Therefore by a simple integration

$$C(\lambda, \lambda_0) = c_{\rm UV} - 4\frac{c_G \dim G}{k}\lambda^3 + \mathcal{O}(\lambda^4)$$

Proceeding as before, i.e.

- Symmetry considerations, i.e. $k_{1,2} \mapsto -k_{2,1}$, $\lambda \to 1/\lambda$
- Demanding finite limiting behavior for $\lambda = \pm 1$ and $k \to \infty$
- Matching with the perturbative results

The exact in λ (and leading in $1/k$) C-function is

$$\boxed{C(\lambda, \lambda_0) = 2 \dim G - \frac{c_G \dim G}{k} \frac{(\lambda_0 + \lambda_0^{-1})(1 - 3\lambda^2 - 3\lambda^4 + \lambda^6) + 8\lambda^3}{2(1-\lambda^2)^3}}\ .$$

Properties and checks

- It is positive, i.e. $C(\lambda, \lambda_0) \geqslant 0$.
- Near the IR fixed point at $\lambda = \lambda_0$

$$C = \underbrace{2\dim G - \frac{c_G \dim G}{k} \frac{1+\lambda_0^4}{2\lambda_0(1-\lambda_0^2)}}_{\text{CFT central charge at IR}} + \underbrace{\frac{6c_G \dim G}{k} \frac{\lambda_0}{(1-\lambda_0^2)^3}(\lambda-\lambda_0)^2}_{\text{obeys } \partial_\lambda C \sim \beta_\lambda} + \dots$$

- In addition

$$\mu \frac{dC}{d\mu} = 2\beta_\lambda \partial_\lambda C = 12 \frac{c_G^2 \dim G}{k^2} \frac{\lambda^4(\lambda-\lambda_0)^2(\lambda-\lambda_0^{-1})^2}{(1-\lambda^2)^6} \ ,$$

Hence, C is monotonically increasing from the IR to the UV.

Concluding remarks

- Computed exactly the beta-function and anomalous dimensions of operators in interacting current algebra theories.
- Based on leading order perturbative results and symmetries.
- New integrable σ-model theories, as all loop effective actions for current-current interactions of one, two or more exact CFT CFTs.
- Non-trivial smooth flows between exact CFTs.
- Computation of Zamolodchikov's C-function as a complete function of the coupling (1st example in literature).
- Future directions:
  - For $k_1 \neq k_2$ embed to type-II supergravity.
    As in [KS-Thompson 14, Demulder-KS-Thompson 15, Borsato-Tseytlin 15]. Prototype example $AdS_3 \times S^3 \times S^3 \times S^1$.
  - Check C-function to higher orders in perturbation.