

Carrollian fluids and flat holography

Marios Petropoulos

CPHT – Ecole Polytechnique – CNRS

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*with L. Ciambelli, C. Marteau, A. Petkou, K. Siampos
in memory of Ioannis Bakas*

Highlights

Foreword

Relativistic fluid/AdS gravity

Ricci-flat gravity/Carrollian fluid

Summary

Iannis liked gravitational instantons and dualities

Our papers

- ▶ *Geometric flows in Horava–Lifshitz gravity*, I. Bakas, F. Bourliot, D. Lüst and P.M. Petropoulos, JHEP **1004** (2010) 131
- ▶ *Mixmaster universe in Horava–Lifshitz gravity*, I. Bakas, F. Bourliot, D. Lüst and P.M. Petropoulos, Class. Quantum Grav. **27** (2010) 045013
- ▶ *Ricci flows and expansion in axion-dilaton cosmology*, I. Bakas, D. Orlando and P.M. Petropoulos, JHEP **0701** (2007) 040

An inspiring paper

I. Bakas, E.G. Floratos and A. Kehagias, *Octonionic gravitational instantons*, Phys. Lett. **445B** (1998) 69

AdS/CFT

Holographic correspondence: microscopic duality with $g_{\text{string}} \leftrightarrow 1/g_{\text{YM}}$

type IIB string theory on $AdS_5 \times S^5$



$N = 4$ Yang–Mills on $D = 4$ conformal boundary of AdS_5

Anti-de Sitter space: homogeneous spacetime with $\Lambda \neq 0$

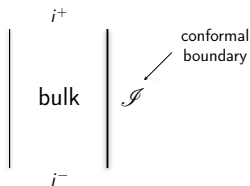


Figure: AdS₄ Penrose–Carter diagram

Thoughts on flat or de Sitter extensions

Minkowski spacetime:

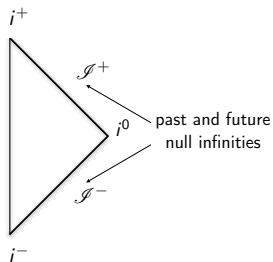


Figure: Minkowski Penrose–Carter diagram

Asymptotically flat holographic correspondence:

- ▶ *microscopically: perhaps in 2 + 1 dimensions or higher spins?*
- ▶ *macroscopically: better prospects*

Macroscopic approach: fluid/gravity correspondence

Branch of AdS/CFT: Einstein's and **relativistic** Euler's equations

[Bhattacharyya, Haack, Hubeny, Loganayagam, Minwalla, Rangamani, Yarom, ... '07]

Einstein locally AdS spacetime \mathcal{E} with $\Lambda \neq 0$



relativistic fluid on $\mathcal{I} = \partial\mathcal{E} \equiv$ conformal boundary

Historically: **non-relativistic incompressible-fluid** equations
emerge from perturbations of the black-hole horizon [Damour '79]

Scrutinizing the flat-spacetime holography:

1. Which surface \mathcal{S} would replace the AdS conf. bry. \mathcal{I} and what is its geometry?
2. Which are the degrees of freedom hosted by \mathcal{S} , what is their dynamics, how are the observables packaged?

Wide dispersion – mixing sometimes AdS [Strominger et al '10; Caldarelli, Taylor, Skenderis

et al '13; Klemm et al '14]

- ▶ Minkowski, surface at finite $r \dots$, relativistic, Rindler \dots
- ▶ incompressible classical fluids, relativistic fluids, Brown–York tensor, conserved energy–momentum tensor \dots

Our aim: unravel a clear pattern (3 + 1 dim)

Method: setting $k \rightarrow 0$ inside the expansion that reconstructs the bulk from the boundary ($\Lambda = -3k^2$)

Ricci-flat spacetime \leftrightarrow conformal Carrollian fluid on $\mathcal{S} \times \mathbb{R}$ at \mathcal{I}^+

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The bulk reconstruction

Given the “initial data”

- ▶ boundary metric ds^2 (neither flat nor conformally flat)
- ▶ conserved energy–momentum tensor T (generally viscous)

Two options exist to get perturbatively the asymptotically AdS bulk:

1. Fefferman–Graham expansion: mathematically robust
2. Derivative expansion: designed for fluid/gravity correspondence
requires an extra piece of bry. data – time-like hydrodynamic congruence u

Fefferman–Graham: expansion of the 3 + 1-dim Einstein metric for large r in a specific gauge (no lapse/shift): [Fefferman, Graham '85]

- ▶ metric: leading term
- ▶ energy–momentum: subleading term

$$ds_{\text{bulk}}^2 = \frac{dr^2}{k^2 r^2} + r^2 ds_{\text{bry.}}^2 + \dots + \frac{16\pi G}{3(kr)} T_{\mu\nu} dx^\mu dx^\nu + \dots$$

Caveats

- ▶ not resumable
- ▶ does not discriminate locally vs. globally AdS bulks
- ▶ the limit $k \rightarrow 0$ is singular ($\Lambda = -3k^2$)

The derivative expansion – fluid/gravity [Bhattacharyya et al '07]

- ▶ Guideline: **Weyl covariance** – the bulk metric must be invariant under boundary conformal transformations
- ▶ Tool: Weyl connection $A = \frac{1}{k^2} (a - \frac{\Theta}{2} u)$ and Weyl covariant derivative $\mathcal{D} = \nabla + wA$ (a is the acceleration and $\Theta = \nabla \cdot u$)
- ▶ Output: $ds_{\text{bulk}}^2 =$ complicated expression based on the boundary data & their derivatives – order by order

Advantages [Leigh et al '10; Caldarelli et al '12; Mukhopadhyay et al '13; Gath et al '15]

- ▶ potentially resumable
- ▶ controls locally vs. globally AdS bulks
- ▶ the limit $k \rightarrow 0$ is regular: flat holography

Relativistic fluids

Obey $\nabla_\mu T^{\mu\nu} = 0$ with

$$T^{\mu\nu} = (\varepsilon + p) \frac{u^\mu u^\nu}{k^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{k^2} + \frac{u^\nu q^\mu}{k^2}$$

- ▶ $\|u\|^2 = -k^2$, $u^0 = \gamma k$, $u^i = \gamma v^i$
- ▶ ε , p : energy density and pressure
- ▶ $\tau^{\mu\nu}$, q^μ : viscous stress tensor and heat current – transverse

$$u^\mu q_\mu = 0 \quad u^\mu \tau_{\mu\nu} = 0$$

q^i and τ^{ij}

- ▶ carry all information on heat exchange and friction processes
- ▶ are usually expressed in terms of a u-derivative expansion

The resummation in 4 dimensions [Caldarelli et al '12; Mukhopadhyay et al '13; Gath et al '15]

Assuming *u shear-free* a resummation is performed:

$$ds_{\text{res. Einstein}}^2 = 2\frac{u}{k^2}(dr + rA) + r^2 ds^2 + \frac{S}{k^4} + \frac{u^2}{k^4 \rho^2} (8\pi G \varepsilon r + c\gamma)$$

- ▶ boundary metric $ds^2 = -k^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j$
- ▶ $u = -k^2 (\Omega dt - b_i dx^i) \Leftrightarrow u = \frac{1}{\Omega} \partial_t$ with $\|u\|^2 = -k^2$
- ▶ $\varepsilon = 2p$: conformal-fluid energy density
- ▶ $\rho^2 = r^2 + \frac{1}{2k^4} \omega_{\alpha\beta} \omega^{\alpha\beta} = r^2 + \gamma^2$
- ▶ $S = -2u \nabla_\nu \omega^\nu{}_\mu dx^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu} dx^\mu dx^\nu - \frac{1}{2} u^2 \mathcal{R}$
- ▶ $\mathcal{R} = R + 4 \nabla_\mu A^\mu - 2 A_\mu A^\mu$,
- ▶ c : Cotton (3rd-order derivative of the metric) $\nabla^\lambda C_{\lambda\mu} = 0$

$$C_{\mu\nu} = \frac{3c}{2} \frac{u_\mu u_\nu}{k} + \frac{ck}{2} g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}$$

Resummability conditions (on top of the absence of shear)

- ▶ transverse duality (with $\eta_{\mu\nu} = -\frac{u^\rho}{k} \eta_{\rho\mu\nu}$)

$$q_\mu = \frac{1}{8\pi G} \eta^\nu{}_\mu c_\nu \quad \tau_{\mu\nu} = -\frac{1}{8\pi G k^2} \eta^\rho{}_\mu c_{\rho\nu}$$

boundary “electric–magnetic gravitational duality” [Bakas '08]

- ▶ energy–momentum conservation $\nabla^\lambda T_{\lambda\mu} = 0$ for the boundary data ε , a_{ij} , Ω and b_i

*Output: algebraically special Einstein spacetimes – Goldberg–Sachs generalizations – asymptotically **locally** AdS*

- ▶ Kerr–Taub–NUT (perfect fluids)
- ▶ Robinson–Trautman
- ▶ Plebański–Demiański
- ▶ ...

Highlights

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Summary

Galilean vs. Carrollian contractions of Poincaré group

[Lévy-Leblond '65]

Both non-relativistic limits with decoupling of time

$$\text{Lorentz boosts: } \begin{cases} \Delta t' = \gamma \left(\Delta t - \beta_i \frac{\Delta x^i}{c} \right) \\ \Delta x^{i'} = \gamma \left(\Delta x^i - \beta^i c \Delta t \right) \end{cases}$$

with $\beta^i = v^i/c$ and $\gamma = 1/\sqrt{1-\beta^2}$

- ▶ Galilean limit $c \rightarrow \infty$:
$$\begin{cases} \Delta t' = \Delta t \\ \Delta x^{i'} = \Delta x^i - v^i \Delta t \end{cases}$$
- ▶ Carrollian limit $c \rightarrow 0$: everything is at rest $\Rightarrow \begin{cases} \beta_i \rightarrow 0 \\ \gamma \rightarrow 1 \end{cases}$

$$\text{but } \lim_{c \rightarrow 0} \beta_i/c = b_i \Rightarrow \begin{cases} \Delta t' = \Delta t - b_i \Delta x^i \\ \Delta x^{i'} = \Delta x^i \end{cases}$$

The Ricci-flat limit $\Lambda = -3k^2 \rightarrow 0$

In the boundary data: $k \equiv$ speed of light

$$ds^2 = -k^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j \quad \|u\|^2 = -k^2$$

$\xrightarrow[k \rightarrow 0]$ Carrollian limit: $\mathcal{I} \rightarrow \mathcal{I} \times \mathbb{R}$ reached at \mathcal{I}^+

The boundary conformal fluid $\xrightarrow[k \rightarrow 0]$ conformal Carrollian fluid

- ▶ data: $\varepsilon = 2p$, $q_i = Q_i + k^2 \pi_i + \mathcal{O}(k^4)$,
 $\tau_{ij} = \frac{1}{k^2} \Sigma_{ij} + \Xi_{ij} + \mathcal{O}(k^2)$
- ▶ dynamics: conformal Carrollian $\lim_{k \rightarrow 0} \nabla_\mu T^{\mu\nu} = 0$

$$\begin{cases} 0 = \frac{k}{\Omega} \nabla_\mu T^\mu{}_0 = \frac{1}{k^2} \mathcal{F} + \mathcal{E} + \mathcal{O}(k^2) \\ 0 = \nabla_\mu T^{\mu i} = \frac{1}{k^2} \mathcal{H}^i + \mathcal{G}^i + \mathcal{O}(k^2) \end{cases}$$

Carrollian hydrodynamics

Scalar equations

- ▶ $\mathcal{E} = -\frac{1}{\Omega}\partial_t\varepsilon - (\varepsilon + p)\theta - \hat{\nabla}_i Q^i - 2\varphi_i Q^i + \Xi^{ij}\zeta_{ij} + \frac{1}{2}\Xi^i{}_i\theta = 0$
- ▶ $\mathcal{F} = \Sigma^{ij}\zeta_{ij} + \frac{1}{2}\Sigma^i{}_i\theta = 0$

Vector equations

- ▶ $\mathcal{G}_j = \hat{\partial}_j p + (\varepsilon + p)\varphi_j + \frac{1}{\Omega}\partial_t\pi_j + \pi_j\theta + 2Q^i\omega_{ij} - \hat{\nabla}_i\Xi^i{}_j - \varphi_i\Xi^i{}_j = 0$
- ▶ $\mathcal{H}^i = \frac{a^{ij}}{\Omega}\partial_t Q_j + Q^i\theta - \hat{\nabla}_j\Sigma^{ji} - \varphi_j\Sigma^{ji} = 0$

Remark

φ_i , θ , ζ_{ij} , ω_{ij} : kinematic observables (relativistic origin: acceleration, expansion, shear, vorticity)

The Ricci-flat derivative expansion

The limit of the Einstein derivative expansion is well-defined and Weyl-covariant

$$\begin{aligned} \lim_{k \rightarrow 0} ds_{\text{res. Einstein}}^2 &= ds_{\text{res. flat}}^2 \\ &= -2 (\Omega dt - \mathbf{b}) \left(dr + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2} dt \right) + r^2 d\ell^2 \\ &\quad + \mathbf{s} + \frac{(\Omega dt - \mathbf{b})^2}{\rho^2} (8\pi G\varepsilon r + c * \omega) \end{aligned}$$

- ▶ $d\ell^2$, Ω , \mathbf{b} , $\boldsymbol{\alpha}$, \mathbf{s} , θ , $*\omega$: Carrollian-geometric data
- ▶ $\varepsilon = 2p$: conformal Carrollian-fluid energy density
- ▶ c : descendent of the Cotton

The boundary Cotton tensor $\xrightarrow[k \rightarrow 0]{} \text{conformal Carrollian 3rd-derivative}$

$c, \chi_i, \psi_i, X_{ij}, \Psi_{ij}$ obeying equations similar to the those for $\varepsilon, Q_i, \pi_i, \Sigma_{ij}, \Xi_{ij}$

The Carrollian-boundary resummability conditions: $\zeta_{ij} = 0$

$$Q_i = \frac{1}{8\pi G} \eta^j{}_i \chi_j \quad \Sigma_{ij} = \frac{1}{8\pi G} \eta^l{}_i X_{lj} \quad \Xi_{ij} = \frac{1}{8\pi G} \eta^l{}_i \Psi_{lj}$$

Output : algebraically special Ricci-flat spacetimes – Goldberg–Sachs
In general: asymptotically **locally** flat

Example I: stationary

Boundary data: $d\ell^2 = \frac{2}{\rho^2} d\zeta d\bar{\zeta}$, $\Omega = 1$, $\mathbf{b} = b_\zeta d\zeta + b_{\bar{\zeta}} d\bar{\zeta}$

$\xi_{ij} = 0$, $\varphi_i = 0$, $\theta = 0$, $\boldsymbol{\omega} = \frac{1}{2} d\mathbf{b}$

Perfect fluid: $\varepsilon, \boldsymbol{\pi} = 0$, $\mathbf{Q} = \frac{1}{8\pi G} * \boldsymbol{\chi} = 0$, $\boldsymbol{\Sigma} = \frac{1}{8\pi G} * \mathbf{X} = 0$,
 $\boldsymbol{\Xi} = \frac{1}{8\pi G} * \boldsymbol{\Psi} = 0$

fluid equations: $\varepsilon = M/4\pi G$, $P(\zeta, \bar{\zeta})$, $b_\zeta(\zeta, \bar{\zeta})$, $b_{\bar{\zeta}}(\zeta, \bar{\zeta})$

Resummation: Ricci-flat Kerr–Taub–NUT family

Example II: time-dependent

Boundary data: $d\ell^2 = \frac{2}{P^2}d\zeta d\bar{\zeta}$, $\Omega = 1$, $b_i = 0$

- ▶ $\xi_{ij} = 0$, $\omega_{ij} = 0$, $\varphi_i = 0$, $\theta = -2\partial_t \ln P$
- ▶ “Cotton”: $c = 0$, $\boldsymbol{\psi} = 0$, $\boldsymbol{\chi} = \frac{i}{2} \left(\partial_\zeta K d\zeta - \partial_{\bar{\zeta}} K d\bar{\zeta} \right)$, $\boldsymbol{\Psi} = 0$,
 $\boldsymbol{X} = \frac{i}{P^2} \left(\partial_\zeta (P^2 \partial_t \partial_\zeta \ln P) d\zeta^2 - \partial_{\bar{\zeta}} (P^2 \partial_t \partial_{\bar{\zeta}} \ln P) d\bar{\zeta}^2 \right)$

$(K = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta \ln P)$

Fluid data: ε , $\boldsymbol{\pi} = 0$, $\boldsymbol{Q} = \frac{1}{8\pi G} * \boldsymbol{\chi}$, $\boldsymbol{\Sigma} = \frac{1}{8\pi G} * \boldsymbol{X}$, $\boldsymbol{\Xi} = 0$

- ▶ momentum equation: $\partial_i \varepsilon = 0 \Rightarrow \varepsilon(t) = M(t)/4\pi G$
- ▶ energy equation: $\Delta \Delta \ln P + 12M \partial_t \ln P - 4\partial_t M = 0$

Resummation: Ricci-flat Robinson–Trautman

Highlights

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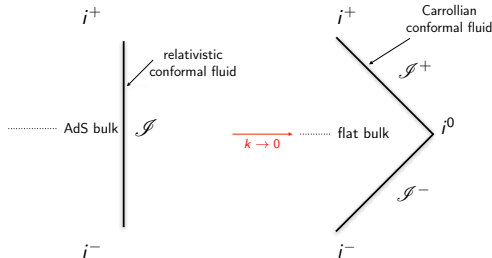
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Summary

About flat holography and Carrollian fluids

1. Which surface \mathcal{S} would replace the AdS conf. bry. \mathcal{I} and what is its geometry? **Spatial surface at \mathcal{I}^+ equipped with Carrollian geometry**
2. Which are the degrees of freedom hosted by \mathcal{S} , what is their dynamics, how are the observables packaged? **Carrollian conformal fluid with $\varepsilon, Q_i, \pi_i, \Sigma_{ij}, \Xi_{ij}$ obeying Carrollian fluid equations**



Confirms scattered observations

- ▶ *Ricci-flat limit is related to some ultra-relativistic contraction of Poincaré algebra* [Barnich et al. '10; Bagchi et al. '10–12; Duval et al. '14; Jensen, Karch '15]
- ▶ *Null infinity plays a privileged role for hosting the degrees of freedom* [He, Kapec, Mitra, Pasterski, Raclariu, Shao, Strominger '16–'17]

Raises fundamental questions

- ▶ *Carrollian fluids: understand the microscopic behaviour at $c \rightarrow 0$: Boltzmann equation, QFT*
- ▶ *Flat holography: is there a genuine microscopic perspective?*

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Galilean covariance in d spatial dimensions

Geometry: $d\ell^2 = a_{ij}(t, \mathbf{x})dx^i dx^j$, $\Omega = \Omega(t)$, $\mathbf{w} = w^i(t, \mathbf{x})\partial_i$

- ▶ Galilean diffs.: $t' = t'(t)$, $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$
- ▶ Jacobian: $J(t) = \frac{\partial t'}{\partial t}$, $j^i(t, \mathbf{x}) = \frac{\partial x'^i}{\partial t}$, $J_j^i(t, \mathbf{x}) = \frac{\partial x'^i}{\partial x^j}$
- ▶ transfs.: $a'_{ij} = a_{kl} J^{-1k}_i J^{-1l}_j$, $\Omega' = \frac{\Omega}{J}$, $w'^k = \frac{1}{J} (J_i^k w^i + j^k)$
- ▶ absolute Newtonian time (invariant): $\Omega(t)dt$

[Cartan, Bekaert, Bergshoeff, Duval, Gibbons, Gomis, Hartong, Horvathy, Longhi, Morand, Obers]

Do not confuse with the Galilean group of invariance present when $a_{ij} = \delta_{ij}$, $\Omega = 1$, $w^i = \text{constant}$

$$\begin{cases} t' = t + t_0, \\ x'^k = R_i^k x^i + V^k t + x_0^k \end{cases}$$

Simple realization and relativistic uplift

Particle: $x^i = x^i(t)$, $v^i = dx^i/dt$, $\mathbf{v} = v^i \partial_i$

- ▶ transfs.: $v'^k = \frac{1}{J} (J_i^k v^i + j^k)$, $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ d -dim vector
- ▶ free dynamics: $\mathcal{L}(\mathbf{v}, \mathbf{x}, t) = \frac{1}{2\Omega^2} a_{ij} (v^i - w^i) (v^j - w^j)$

Relativistic uplift: $d + 1$ -dim Zermelo form

- ▶ $ds^2 = -\Omega^2 c^2 dt^2 + a_{ij} (dx^i - w^i dt) (dx^j - w^j dt)$
- ▶ form-invariant under $J_V^\mu(x) = \frac{\partial x'^\mu}{\partial x^v} = \begin{pmatrix} J(t) & 0 \\ \frac{j^i(t, \mathbf{x})}{c} & J_j^i(t, \mathbf{x}) \end{pmatrix}$
- ▶ relevant limit: $c \rightarrow \infty$

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Non-relativistic Galilean fluid

Relativistic fluid on Zermelo at $c \rightarrow \infty$: Galilean fluid

$$u^0 = \frac{c}{\Omega} + \mathcal{O}(1/c) \quad u^i = \frac{v^i}{\Omega} + \mathcal{O}(1/c^2)$$

- ▶ v^i
- ▶ e, p, ϱ
- ▶ $q_i \rightarrow Q_i$ and $\tau_{ij} \rightarrow -\Sigma_{ij}$

Galilean-covariant equations on $a_{ij}(t, \mathbf{x}), \Omega(t), w^i(t, \mathbf{x})$

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Geometry on \mathcal{S} : $d\ell^2 = a_{ij}(t, \mathbf{x})dx^i dx^j$ $\Omega(t, \mathbf{x})$ $\mathbf{b} = b_i(t, \mathbf{x})dx^i$

▶ Carrollian diffs.: $t' = t'(t, \mathbf{x})$ $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$

▶ Jacobian: $J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}$ $j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}$ $J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}$

▶ transfs.: $a'_{ij} = a_{kl} J^{-1k}_i J^{-1l}_j$ $\Omega' = \frac{\Omega}{J}$ $b'_k = (b_i + \frac{\Omega}{J} j_i) J^{-1i}_k$

Do not confuse with the Carrollian group of invariance present when $a_{ij} = \delta_{ij}$, $\Omega = 1$, $b_i = \text{constant}$ (here realized in tangent space)

$$\begin{cases} t' = t + B_i x^i + t_0, \\ x'^k = R_i^k x^i + x_0^k \end{cases}$$

Simple realization and relativistic uplift

Extended object: $t = t(\mathbf{x})$, $\beta_i = \Omega \partial_i t - b_i$, $\boldsymbol{\beta} = \beta_i dx^i$

- ▶ transfs.: $\beta'_k = \beta_i J^{-1k}_i$ (d -dim form)
- ▶ free dynamics: $\mathcal{L}(\boldsymbol{\partial}t, t, \mathbf{x}) = \frac{1}{2} a^{ij} (\Omega \partial_i t - b_i) (\Omega \partial_j t - b_j)$

Relativistic uplift: $d + 1$ -dim Randers–Papapetrou form

- ▶ relevant limit: $c \rightarrow 0$
- ▶ $ds^2 = -c^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j$
- ▶ form-invariant under Carrollian diffeomorphisms ($x^0 = ct$)

$$J^\mu_\nu(t, \mathbf{x}) = \frac{\partial x^\mu}{\partial x^\nu} = \begin{pmatrix} J(t, \mathbf{x}) & c j_j(t, \mathbf{x}) \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix}$$

More on Carrollian geometries

- ▶ *isometries*
- ▶ *time and space connections, covariant derivatives, curvatures*
- ▶ *time and space Weyl connection, Weyl curvature*

Example: Carrollian space derivative $\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t$

- ▶ *transfs.: $\hat{\partial}'_i = J^{-1j}_i \hat{\partial}_j$*
- ▶ *connection: $\hat{\gamma}^i_{jk} = \frac{a^{il}}{2} \left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk} \right)$*
- ▶ *covariant metric-compatible derivative: $\hat{\nabla} = \hat{\partial} + \hat{\gamma}$*

Similarly: Weyl-covariant metric-compatible derivatives $\hat{\mathcal{D}}_i, \hat{\mathcal{D}}_t$

built on $\varphi_i = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega)$ and $\theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a}$

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Carrollian limit: Carrollian fluid

Relativistic fluid on Randers–Papapetrou at $c \rightarrow 0$: kinematics

v^i must vanish faster than c :

$$v^i = c^2 \Omega \beta^i + \mathcal{O}(c^4)$$

avoids blow-ups without trivializing

- ▶ kinematic variable $\beta^i = -\frac{\Omega u_i}{c u_0} - b_i = v_i / c^2 \Omega \left(1 - \frac{v^j b_j}{\Omega}\right)$
- ▶ $u^0 = \gamma c = c / \Omega + \mathcal{O}(c^3)$ $u^i = \gamma v^i = c^2 \beta^i + \mathcal{O}(c^4)$
- ▶ $u_0 = -c \Omega + \mathcal{O}(c^3)$ $u_i = c^2 (b_i + \beta_i) + \mathcal{O}(c^4)$

[worth comparing with de Boer, Hartong, Obers, Sybesma, Vandoren '17]

Limit inside the fluid data (microscopic justification yet to come)

- ▶ ε, p
- ▶ $q^i \rightarrow Q^i + c^2 \pi^i$ and $\tau^{ij} \rightarrow -\frac{1}{c^2} \Sigma^{ij} - \Xi^{ij}$

Inside the perfect-fluid energy–momentum tensor

$$\gamma = \frac{1 + c^2 \boldsymbol{\beta} \cdot \mathbf{b}}{\Omega \sqrt{1 - c^2 \boldsymbol{\beta}^2}} = \frac{1}{\Omega} \left(1 + \frac{c^2}{2} \boldsymbol{\beta} \cdot (\boldsymbol{\beta} + 2\mathbf{b}) + \mathcal{O}(c^4) \right)$$

$$\begin{cases} T_{\text{perf } 0}^0 = -\varepsilon - c^2(\varepsilon + p)\beta^k (b_k + \beta_k) + \mathcal{O}(c^4) \\ c\Omega T_{\text{perf } i}^0 = c^2(\varepsilon + p)(b_i + \beta_i) + \mathcal{O}(c^4) \\ \frac{c}{\Omega} T_{\text{perf } 0}^j = -c^2(\varepsilon + p)\beta^j + \mathcal{O}(c^4) \\ T_{\text{perf } i}^j = p\delta_i^j + c^2(\varepsilon + p)\beta^j (b_i + \beta_i) + \mathcal{O}(c^4) \end{cases}$$

Inside the relativistic-fluid equations

$$\begin{cases} 0 = \frac{c}{\Omega} \nabla_{\mu} T^{\mu}_0 = \frac{1}{c^2} \mathcal{F} + \mathcal{E} + \mathcal{O}(c^2) \\ 0 = \nabla_{\mu} T^{\mu i} = \frac{1}{c^2} \mathcal{H}^i + \mathcal{G}^i + \mathcal{O}(c^2) \end{cases}$$

→ Carrollian equations

- ▶ scalar equations: $\mathcal{E} = 0 \quad \mathcal{F} = 0$
- ▶ vector equations: $\mathcal{G}^j = 0 \quad \mathcal{H}^i = 0$

→ Covariant under Carrollian diffs.

- ▶ $\mathcal{E}' = \mathcal{E} \quad \mathcal{F}' = \mathcal{F}$
- ▶ $\mathcal{G}'^i = J_j^i \mathcal{G}^j \quad \mathcal{H}'^i = J_j^i \mathcal{H}^j$

Carrollian hydrodynamics with $\beta = 0$

Scalar equations

- ▶ $\mathcal{E} = -\frac{1}{\Omega}\partial_t\varepsilon - (\varepsilon + p)\theta - \hat{\nabla}_i Q^i - 2\varphi_i Q^i + \Xi^{ij}\zeta_{ij} + \frac{1}{2}\Xi^i_i\theta = 0$
- ▶ $\mathcal{F} = \Sigma^{ij}\zeta_{ij} + \frac{1}{2}\Sigma^i_i\theta = 0$

Vector equations

- ▶ $\mathcal{G}_j = \hat{\Delta}_j p + (\varepsilon + p)\varphi_j + \frac{1}{\Omega}\partial_t\pi_j + \pi_j\theta + 2Q^i\omega_{ij} - \hat{\nabla}_i \Xi^i_j - \varphi_i \Xi^i_j = 0$
- ▶ $\mathcal{H}^i = \frac{a^{ij}}{\Omega}\partial_t Q_j + Q^i\theta - \hat{\nabla}_j \Sigma^{ji} - \varphi_j \Sigma^{ji} = 0$

Remarks

- ▶ more involved for $\beta^i \neq 0$
- ▶ more elegant for conformal fluids

$\varphi_i, \theta, \zeta_{ij}, \omega_{ij}$: kinematic observables

Relativistic origin: *acceleration, expansion, shear, vorticity*

- ▶ $\frac{a_i}{c^2} = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega) + \mathcal{O}(c^2) = \varphi_i + \mathcal{O}(c^2)$
- ▶ $\Theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a} + \mathcal{O}(c^2) = \theta + \mathcal{O}(c^2)$
- ▶ $\sigma_{ij} = \frac{1}{\Omega} \left(\frac{1}{2} \partial_t a_{ij} - \frac{1}{d} a_{ij} \partial_t \ln \sqrt{a} \right) + \mathcal{O}(c^2) = \zeta_{ij} + \mathcal{O}(c^2)$
- ▶ $\frac{\omega_{ij}}{c^2} = \partial_{[i} b_{j]} + \frac{1}{\Omega} b_{[i} \partial_{j]} \Omega + \frac{1}{\Omega} b_{[i} \partial_t b_{j]} + \mathcal{O}(c^2) = \omega_{ij} + \mathcal{O}(c^2)$

Remarks

- ▶ all Carrollian-covariant
- ▶ purely geometric
- ▶ more terms if $\beta^i \neq 0$

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Conformal Carrollian geometry

Weyl transformation on Carrollian geometry

$$a_{ij} \rightarrow \frac{a_{ij}}{\mathcal{B}^2} \quad b_i \rightarrow \frac{b_i}{\mathcal{B}} \quad \Omega \rightarrow \frac{\Omega}{\mathcal{B}}$$

Spatial Weyl derivative for a weight- w vector V^I

$$\hat{\mathcal{D}}_j V^I = \hat{\nabla}_j V^I + (w - 1)\varphi_j V^I + \varphi^I V_j - \delta_j^I V^i \varphi_i$$

Temporal Weyl derivative for a weight- w vector V^I

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V^I = \frac{1}{\Omega} \partial_t V^I + \frac{w}{2} \theta V^I + \zeta^I{}_i V^i$$

Conformal Carrollian fluids $\beta = 0$

From relativistic to Carrollian conformal properties

- ▶ $\varepsilon = dp$ and $\tau^\mu{}_\mu = 0 \xrightarrow{c \rightarrow 0} \Xi^i{}_i = \Sigma^i{}_i = 0$
- ▶ $\varepsilon \rightarrow \mathcal{B}^{d+1}\varepsilon$, $\pi_i \rightarrow \mathcal{B}^d\pi_i$, $Q_i \rightarrow \mathcal{B}^d Q_i$,
 $\Xi_{ij} \rightarrow \mathcal{B}^{d-1}\Xi_{ij}$, $\Sigma_{ij} \rightarrow \mathcal{B}^{d-1}\Sigma_{ij}$

Scalar equations

- ▶ $\mathcal{E} = -\frac{1}{\Omega}\hat{\mathcal{D}}_t\varepsilon - \hat{\mathcal{D}}_i Q^i + \Xi^{ij}\zeta_{ij} = 0$
- ▶ $\mathcal{F} = \Sigma^{ij}\zeta_{ij} = 0$

Vector equations

- ▶ $\mathcal{G}_j = \frac{1}{d}\hat{\mathcal{D}}_j\varepsilon + \frac{1}{\Omega}\hat{\mathcal{D}}_t\pi_j + \pi_i\zeta^i{}_j + 2Q^i\omega_{ij} - \hat{\mathcal{D}}_i\Xi^i{}_j = 0$
- ▶ $\mathcal{H}_j = \frac{1}{\Omega}\hat{\mathcal{D}}_t Q_j + Q_i\zeta^i{}_j - \hat{\mathcal{D}}_i\Sigma^i{}_j = 0$

Highlights

Galilean covariance

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Curvature and Cotton

The Conformal Carrollian curvature tensors in 2 dim

The Ricci tensor (space)

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k{}_{ikj} = \hat{r}_{ij} + a_{ij} \hat{\nabla}_k \varphi^k = \hat{s}_{ij} + \mathcal{K} a_{ij} + \hat{\mathcal{A}} \eta_{ij}$$

$$\mathcal{K} = \frac{1}{2} a^{ij} \hat{\mathcal{R}}_{ij} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \frac{1}{2} \eta^{ij} \hat{\mathcal{R}}_{ij}$$

\hat{r}_{ij} is the Ricci of $\hat{\nabla}_k$ and $2\hat{K} = \hat{r} = a^{ij} \hat{r}_{ij}$

The vector (time)

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_t \varphi_i - \frac{1}{2} (\hat{\partial}_i + \varphi_i) \theta$$

Reminder: the Cotton tensor

In 3 dim the Weyl tensor vanishes – conformal properties are captured by the Cotton tensor

$$C_{\mu\nu} = \eta_{\mu}^{\rho\sigma} \nabla_{\rho} \left(R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right)$$

$$(\eta_{\mu\nu\sigma} = \sqrt{-g} \epsilon_{\mu\nu\sigma})$$

- ▶ symmetric and traceless
- ▶ conformally covariant of weight 1
- ▶ identically conserved: $\nabla_{\mu} C^{\mu\nu} = 0$

Decomposition wrt u

$$C_{\mu\nu} = \frac{3c}{2} \frac{u_\mu u_\nu}{k} + \frac{ck}{2} g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}$$

At large k with $u = \frac{1}{\Omega} \partial_t$

- ▶ $c_i = \chi_i + k^2 \psi_i$
- ▶ $c_{ij} = X_{ij} + k^2 \Psi_{ij}$
- ▶ weight 2 and 1 respectively

The "Cotton" in two-dimensional Carrollian geometry

$$\left\{ \begin{array}{l} c = (\hat{\mathcal{D}}_l \hat{\mathcal{D}}^l + 2\mathcal{K}) * \omega \\ \chi_j = \frac{1}{2} \eta^l_j \hat{\mathcal{D}}_l \mathcal{K} + \frac{1}{2} \hat{\mathcal{D}}_j \mathcal{A} - 2 * \omega \hat{\mathcal{R}}_j \\ \psi_j = 3\eta^l_j \hat{\mathcal{D}}_l * \omega^2 \\ X_{ij} = \frac{1}{2} \eta^l_j \hat{\mathcal{D}}_l \hat{\mathcal{R}}_i + \frac{1}{2} \eta^l_i \hat{\mathcal{D}}_j \hat{\mathcal{R}}_l \\ \Psi_{ij} = \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \omega - \frac{1}{2} a_{ij} \hat{\mathcal{D}}_l \hat{\mathcal{D}}^l * \omega - \eta_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t * \omega^2 \end{array} \right.$$

Conservation identities

$$\left\{ \begin{array}{l} \frac{1}{\Omega} \hat{\mathcal{D}}_t c + \hat{\mathcal{D}}_i \chi^i = 0 \\ \frac{1}{2} \hat{\mathcal{D}}_j c + 2\chi^i \omega_{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi_j - \hat{\mathcal{D}}_i \Psi^i_j = 0 \\ \frac{1}{\Omega} \hat{\mathcal{D}}_t \chi_j - \hat{\mathcal{D}}_i X^i_j = 0 \end{array} \right.$$