An Inverse Mass Expansion for Entanglement Entropy in Free Massive Scalar Field Theory

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Section 1

Introduction

The reduced density matrix

When a composite system lies in an entangled state, there is no answer to the question “what is the state of the subsystem *A*?”
The expected values of the measurables of a given subsystem cannot be reproduced by any state, however they are reproduced by the reduced density matrix

$$\rho_A = tr_{A^C} \rho = tr_{A^C} |\psi\rangle \langle\psi| \, ,$$

which in general describes a mixed state, even though the overall system lies in a pure one.

The more entangled the state the more dispersed the spectrum of the reduced density matrix.
Thus, a good measure of entanglement is Shannon's entropy applied to the spectrum of the reduced density matrix

$$S_{EE} = -tr \rho_A \ln \rho_A ,$$

known as entanglement entropy.
However, the whole information on entanglement is contained in the full spectrum of the reduced density matrix and not just S_{EE}.

Entanglement entropy in Field Theory

Modern approaches for the calculation of the entanglement entropy involve the replica trick and holographic calculations (Ryu-Takayanagi conjecture), mainly applied to conformal field theories.
There are some older more direct approaches for the calculation of entanglement entropy in quantum field theory:

- In 1986 Bombelli et.al.[1] sketched a method to calculate S_{EE} in free scalar field theory on a curved background and argued that it would be proportional to the area of the surface separating the two subsystems.
- In 1993, Srednicki[2] actually performed this calculation numerically, on a flat background and indeed found that S_{EE} is proportional to the area of the entangling surface.

[1] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, "A Quantum Source of Entropy for Black Holes", Phys. Rev. D **34**, 373 (1986)

[2] M. Srednicki, "Entropy and Area", Phys. Rev. Lett. **71**, 666 (1993) hep-th/9303048

Relation with Quantum Gravity?

This similarity of entanglement entropy with black hole entropy has initiated discussions on whether the latter can be attributed totally or at least partially to the former.
Such an approach would also imply that gravity itself could be an entropic force related not to thermal statistics but rather to quantum statistics due to quantum entanglement.
Holography and the Ryu-Takayanagi conjecture are also in line with such interpretation[3].
So we'd better understand entanglement in field theory in a more direct way

[3] N. Lashkari, M. B. McDermott and M. Van Raamsdonk, "Gravitational Dynamics from Entanglement Thermodynamics" JHEP **1404**, 195 (2014) arXiv:1308.3716 [hep-th]

We extend Srendicki's calculation:

- We consider a free real **massive** scalar field.
- We develop of a perturbative expansion for the **analytical** calculation of entanglement entropy **and** the reduced density matrix.

Section 2

Systems of Coupled Oscillators

Two Coupled Oscillators

Let's start out considerations with the simple example of two coupled harmonic oscillators.
The Hamiltonian of the system is

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2\right)$$

The ground state of the system is obviously

$$|0\rangle = |0\rangle_+ \otimes |0\rangle_-$$

One may mistakenly think that there is no entanglement in this state, since it can be written as a tensor product.
However, it is not a tensor product of states describing each of the "local" oscillators 1 and 2 but of the normal modes. This state is actually entangled.

Reduced Density Matrix

The reduced density matrix is

$$\rho(x, x') = \int_{-\infty}^{\infty} dy \psi(x, y)\psi^*(x', y) = \sqrt{\frac{\gamma - \beta}{\pi}} e^{-\frac{\gamma}{2}\left(x^2 + x'^2\right) + \beta x x'}$$

where

$$\beta - \frac{(\omega_+ - \omega_-)^2}{4\,(\omega_+ + \omega_-)}, \quad \gamma = \frac{\omega_+^2 + \omega_-^2 + 6\omega_+\omega_-}{4\,(\omega_+ + \omega_-)}, \quad \gamma - \beta = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}.$$

The eigenvalues and eigenfunctions of ρ are:

$$p_n = (1 - \xi)\xi^n, \qquad f_n = H_n(\sqrt{\alpha}x)e^{-\frac{\alpha}{2}x^2},$$

$$\xi = \frac{\beta}{\alpha + \gamma}, \qquad \alpha = \sqrt{\gamma^2 - \beta^2} = \sqrt{\omega_+\omega_-}$$

Entanglement Entropy

Finally, the Entanglement Entropy equals

$$S_{EE} = -\sum_{n=0}^{\infty} p_n \ln p_n = -\ln(1-\xi) - \frac{\xi}{1-\xi} \ln \xi$$

At the limit the coupling between the two oscillators k_1 goes to zero, entanglement entropy goes to zero.
The dynamics of the problem are not determining the entanglement. This is determined by the state. Any two-degrees of freedom system has the same entanglement when in the same state. Dynamics pick a state (as the ground state in this example) and determine the time evolution of entanglement.

System of Coupled Oscillators

The same can be repeated for an arbitrary number N of harmonically coupled oscillators,

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j ,$$

where the matrix K is symmetric and positive definite.
We define the matrix Ω as the square root of K with positive eigenvalues

$$\Omega = \sqrt{K} .$$

We define as subsystem A the $N-n$ oscillators with index $i > n$. We define the blocks of the matrix Ω,

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

Reduced Density Matrix

It is matter of algebra to show that the reduced density matrix is given by

$$\rho\left(x;x'\right)=\frac{\left[\det\left(\gamma-\beta\right)\right]^{\frac{1}{2}}}{\pi^{\frac{N-n}{2}}}e^{-\frac{x^T\gamma x+x'^T\gamma x'}{2}+x^T\beta x'},$$

where

$$\beta=\frac{1}{2}B^TA^{-1}B,\quad \gamma=C-\frac{1}{2}B^TA^{-1}B.$$

It has the eigenvalues:

$$p_{n_{n+1},\ldots,n_N}=\prod_{i=n+1}^{N}\left(1-\xi_i\right){\xi_i}^{n_i},\qquad \xi_i=\frac{\beta_{Di}}{1+\sqrt{1-{\beta_{Di}}^2}}$$

where β_{D_i} are the eigenvalues of the matrix $\gamma^{-1}\beta$

Entanglement Entropy

It follows that the entanglement entropy equals:

$$S = \sum_{j=n+1}^{N} \left(-\ln\left(1-\xi_j\right) - \frac{\xi_j}{1-\xi_j} \ln \xi_j \right).$$

- The ground state is highly entangled.
- The specification of S_{EE} requires a non-perturbative calculation (the calculation of Ω and the diagonalization of $\gamma^{-1}\beta$).

QFT - Setting up the Problem

- We are interested in defining the subsystem A as the degrees of freedom in some spatial region in space, i.e. outside a sphere of radius R.
- We consider the system at its ground state
- The fact that the field theory under study is free does not mean that there is no entanglement at the ground state. Free means that different momenta modes do not interact. Local degrees of freedom always interact through the kinetic term.
- This problem is an appropriate continuous limit of the finite degrees of freedom systems studied before.

3+1 dimensions QFT discretization

The Hamiltonian of free scalar massive field theory in $3+1$ dimensions reads:

$$H = \frac{1}{2} \int d^3x \left[\pi^2 \left(\vec{x} \right) + \left| \vec{\nabla} \varphi \left(\vec{x} \right) \right|^2 + m^2 \varphi^2 \left(\vec{x} \right) \right].$$

We define the modes

$$\varphi_{\ell m} \left(x \right) = x \int d\Omega \, Y_{\ell m} \left(\theta, \varphi \right) \varphi \left(\vec{x} \right)$$

$$\pi_{\ell m} \left(x \right) = x \int d\Omega \, Y_{\ell m} \left(\theta, \varphi \right) \pi \left(\vec{x} \right),$$

Discretization Scheme

Only the radial coordinate is continuous. We introduce a lattice of spherical shells with radii $x = ja$, $j = 1, \cdots, N$. The discretized Hamiltonian may be found following the rules:

$$x \to ja, \qquad \varphi_{\ell m}(ja) \to \varphi_{\ell m,j}, \qquad \pi_{\ell m}(ja) \to \frac{\pi_{\ell m,j}}{a}$$

$$\left.\frac{\partial \varphi_{\ell m}(x)}{\partial x}\right|_{x=ja} \to \frac{\varphi_{\ell m,j+1} - \varphi_{\ell m,j}}{a}, \qquad \int_0^{(N+1)a} dx \to a \sum_{j=1}^{N}.$$

Regularization scheme

The regularization scheme is rather peculiar

- We introduced a UV radial cutoff $1/a$.
- We introduced an IR radial cutoff $1/(Na)$.
- We introduced no angular cutoff. There is no obvious connection of a with an energy cutoff.

3+1 dimensions QFT discretization

The discretized Hamiltonian assumes the form:

$$H = \frac{1}{2a} \sum_{\ell,m} \sum_{j=1}^{N} \left[\pi_{\ell m,j}^2 + \left(j + \frac{1}{2} \right)^2 \left(\frac{\varphi_{\ell m,j+1}}{j+1} - \frac{\varphi_{\ell m,j}}{j} \right)^2 + \left(\frac{\ell(\ell+1)}{j^2} + m^2 a^2 \right) \varphi_{\ell m,j}^2 \right] \equiv \frac{1}{2a} \sum_{\ell,m} H_\ell$$

As a result the entanglement entropy can be found as:

$$S_{\rm EE}(N,n) = \sum_{\ell=0}^{\infty} (2\ell+1) S_\ell(N,n),$$

where S_ℓ is the entanglement entropy at the ground state of the Hamiltonian H_ℓ.

The couplings matrix

The discretized Hamiltonian H_ℓ corresponds to a problem of finite harmonically coupled oscillators like those treated above. The corresponding couplings matrix reads

$$K_{ij}^{(3+1)} = \left\{ 2 + \frac{\ell(\ell+1)+1/2}{i^2} + m^2 a^2 \right\} \delta_{i,j} - \frac{(i+1/2)^2}{i\,(i+1)} \delta_{i+1,j} - \frac{(j+1/2)^2}{j\,(j+1)} \delta_{i,j+1}$$

Section 3

Perturbation Theory

A perturbative expansion

We consider as subsystem A the degrees of freedom with $j > n_R$. These are the degrees of freedom outside a sphere of radius $R = n_R a$.

- We are left with the problem of specifying the square root of the couplings matrix specified above. Is there a limit where we may calculate it perturbatively?
- The answer is yes. When the diagonal elements of the couplings matrix are much larger than the non-diagonal ones, the latter can be treated as a perturbation.
- This clearly corresponds to very massive fields. At this limit, the local oscillators can be considered decoupled and the ground state of the system is disentangled. Thus, the zero-th order entanglement entropy in this approach vanishes.
- This expansion converges when the relative matrices are diagonally dominant, i.e. the sum of the absolute values of all non-diagonal elements does not exceed the diagonal one in all rows and columns.

The perturbative expansion at leading order

The matrix K is of the form

$$K_{ij} = K_i \delta_{ij} + L_i \left(\delta_{i+1,j} + \delta_{i,j+1} \right),$$

where K_i is of order m^2 and L_i is of order 1. We define

$$K_i := \frac{k_i^2}{\varepsilon^2} \qquad L_i := l_i \left(k_i + k_{i+1} \right).$$

Thus, ε is our expansive parameter. Expansion in ε is a semiclassical expansion, since recovering the fundamental constants yields

$$\varepsilon \sim \hbar / \left(mac \right).$$

The above is consistent with the vanishing zero-th order result.

The perturbative expansion at leading order

Now it is a matter of simple algebra to show that at first non-vanishing order the matrix Ω equals

$$\Omega_{ij} = k_i \delta_{ij} \varepsilon^{-1} + l_i \left(\delta_{i+1,j} + \delta_{i,j+1} \right) \varepsilon + \mathcal{O}\left(\varepsilon^3 \right) .$$

$$\beta_{ij} = \frac{l_n^2}{2k_n} \delta_{i,1} \delta_{j,1} \varepsilon^3 + \mathcal{O}\left(\varepsilon^5 \right) ,$$

$$\left(\gamma^{-1} \beta \right)_{ij} = \frac{l_n^2}{2k_n k_{n+1}} \delta_{i,1} \delta_{j,1} \varepsilon^4 + \mathcal{O}\left(\varepsilon^6 \right) .$$

The sole non-vanishing element of $\gamma^{-1}\beta$ is obviously its sole non-vanishing eigenvalue. Thus, the entanglement entropy at first non-vanishing order equals

$$S_{\mathrm{EE}\ell} = \frac{l_n^2}{4k_n k_{n+1}} \left(1 - \ln \frac{l_n^2 \varepsilon^4}{4k_n k_{n+1}} \right) \varepsilon^4 + \mathcal{O}\left(\varepsilon^8 \right) .$$

Locality

Locality got imprinted in the above result to the fact that the matrix $\gamma^{-1}\beta$ contains only one non-vanishing element corresponding to the coupling between the degrees of freedom just inside and just outside the entangling surface. This fact leads to the area law.

Third non-vanishing order

At third order a second non-vanishing eigenvalue emerges. Indicatively, the part of the eigenvalues that is relevant for the Area Law (the dominant part for large entangling sphere radii) is

$$\lambda_1 = \frac{1}{8K_r^2} + \frac{5}{16K_r^4} + \frac{1875}{2048K_r^6}$$

and

$$\lambda_2 = \frac{1}{2048K_r^6}$$

This is a persisting pattern. A new eigenvalue emerges every second order, while the previous ones accept corrections at any new order. In general the effect of the corrections is more important than the effect of the new eigenvalues.

Section 4

Entanglement and Area

Expanding for large entangling sphere radii

Our first goal is understanding the scaling properties of entanglement entropy with the size of the entangling sphere.
We consider that the entangling sphere lies exactly in the middle between the sites n and $n+1$ such that $R = ra$, $r = n + 1/2$. In order to specify the entanglement entropy, we have to sum all the ℓ sectors. Of course the discrete sum cannot be analytically performed, however, it may be approximated by an integral,

$$\begin{aligned} S_{\mathrm{EE}} &= \sum_{\ell=0}^{\infty} (2\ell+1)\, S_{\mathrm{EE}\ell}\left(n, \ell(\ell+1)\right) \\ &\simeq \int_0^{\infty} d\ell\, (2\ell+1)\, S_{\mathrm{EE}\ell}\left(n, \ell(\ell+1)\right). \end{aligned}$$

The “Area law”

At third order in ε we find that the leading contribution for large r is:

$$S^{\text{Area}} = \left(\frac{3 + 2\ln\left(4(2+m^2a^2)\right)}{16(2+m^2a^2)} + \frac{167 + 492\ln\left(4(2+m^2a^2)\right)}{4608(2+m^2a^2)^3} + \frac{-11 + 2940\ln\left(4(2+m^2a^2)\right)}{15360(2+m^2a^2)^5} + \mathcal{O}\left(m^{-14}\right) \right) r^2 + \mathcal{O}\left(r^0\right)$$

This is the so called area law behaviour of entanglement entropy, analytically calculated for the first time directly in quantum field theory.

The Euler MacLaurin formula

The approximation of the series with an integral can be performed more formally with the use of Euler MacLaurin summation formula

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \frac{f(b)+f(a)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right),$$

where B_k are the Bernoulli numbers defined so that $B_1 = 1/2$.

First subleading term

For example at r^0 order, apart from the $S_0/2$ term, we have only one more contribution at r^0 order, namely the $k = 1$ term, and specifficaly the part of this term where the derivative acts on the factor $2\ell + 1$ and not on S_ℓ. Bearing in mind that $B_2 = 1/6$, the contribution to the constant term by the terms of Euler-Maclaurin formula apart from the integral one are $S_0/3$,

$$S = \int_0^\infty (2\ell + 1) S_\ell d\ell + \frac{S_0}{3}.$$

One can easily see that only the integral term can contribute to the “area law” term ($\sim r^2$).

3+1 dimensions

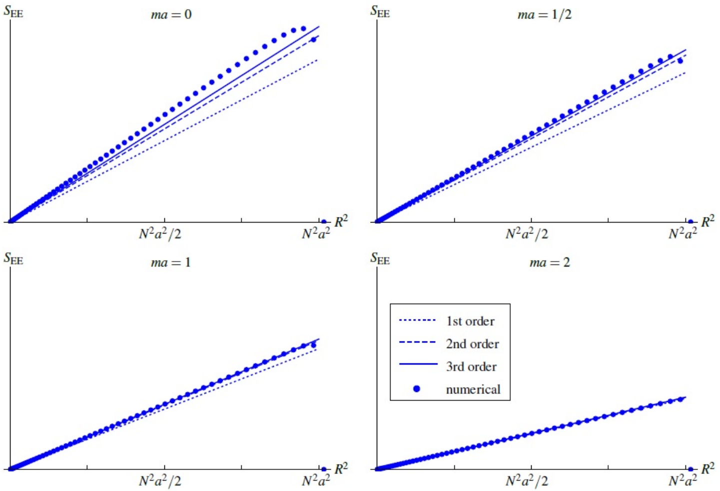
Area law term

$$S^{\text{Area}} = \left(\frac{3 + 2\ln\left(4(2+m^2a^2)\right)}{16(2+m^2a^2)} + \frac{167 + 492\ln\left(4(2+m^2a^2)\right)}{4608(2+m^2a^2)^3} + \frac{-11 + 2940\ln\left(4(2+m^2a^2)\right)}{15360(2+m^2a^2)^5} + \mathcal{O}\left(m^{-14}\right) \right) r^2$$

Constant Term

$$S^{\text{const}} = -\frac{1}{48\left(2+m^2a^2\right)} - \frac{1 + 2\log\left(4\left(2+m^2a^2\right)\right)}{96\left(2+m^2a^2\right)^2} + \left(\frac{-127 + 90\log\left(4\left(2+m^2a^2\right)\right)}{9600\left(2+m^2a^2\right)^3} - \frac{1 + 164\log\left(4\left(2+m^2a^2\right)\right)}{3072\left(2+m^2a^2\right)^4} \mathcal{O}\left(m^{-10}\right) \right)$$

3+1 dimensions comparison with numerical



3+1 dimensions massless limit

Interestingly enough, the perturbative expansion appears to converge even in the massless limit to the numerical results.
This happens because the parameter of expansion is not exactly the inverse of the square of the mass, but rather it is equal to

$$\varepsilon \simeq \frac{1}{\sqrt{m^2 a^2 + 2}}.$$

The series converges even at $m = 0$,

$$S \simeq \left(\frac{3 + 2\ln 8}{32} + \frac{167 + 492\ln 8}{36864} + \frac{-11 + 2940\ln 8}{491520} \right) r^2$$

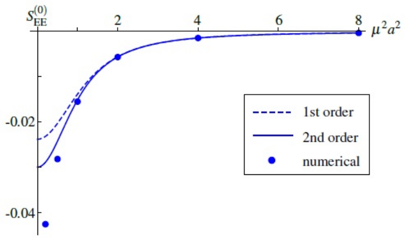
$$S \simeq (0.224 + 0.032 + 0.012)\, r^2 = 0.268 r^2$$

allowing us to compare to the famous result by Srendicki, who numerically found

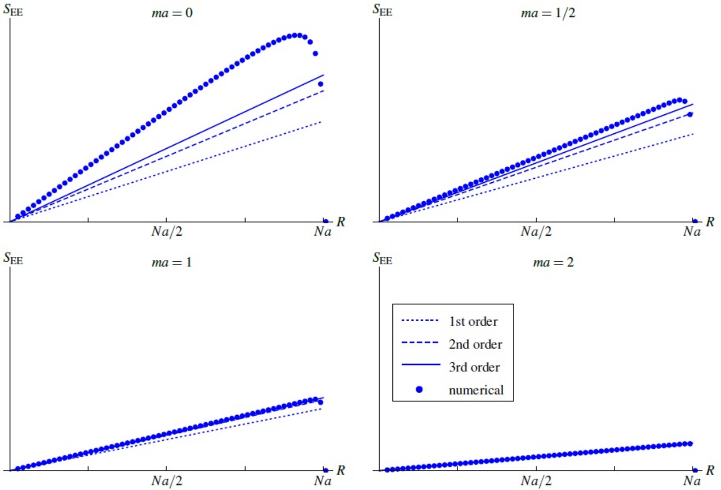
$$S \simeq 0.30 r^2.$$

3+1 dimensions subleading term

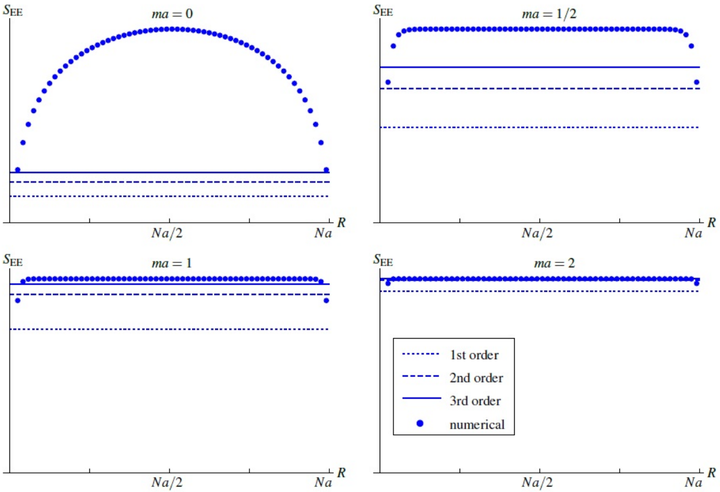
The perturbative result for the subleading term is also in good agreement with the numerical results.



2+1 dimensions



1+1 dimensions



Universal terms

In $3+1$ dimensions, the first subleading term is a constant even in the massless case.
The usual treatment of entanglement entropy in $3+1$ dimensions in either conformal field theory or in theories with holographic duals through the Ryu-Takayanagi conjecture predicts an expansion for entanglement entropy of the form

$$S_{\mathrm{EE}} = c_2 \frac{R^2}{a^2} + c_0 + c \ln \frac{a}{R} + \mathcal{O}\left(a^{-2}\right).$$

So, how is the absence of the logarithmic term in our expansion explained?
The reason is quite complicated and related to the failure to capture the leading entanglement entropy contribution in $1+1$ dimensions.
In a similar manner our perturbation theory is unable to capture the constant term in massless $2+1$ field theory.
These terms that we cannot capture are universal, they depend on the global characteristics of the entangling surface and play the role of order parameter in geometric phase transitions.

Universal terms

The formulae used in our perturbation theory for the square root of matrix K, as well as the formulae for the inverse of matrices A and C, present some "edge effects".

Such "edge effects" can be treated analytically in our expansion as long as the order of the expansion is kept lower than the dimension of the matrices. If this is not the case, these "edge effects" will get reflected at the ends of the matrices and spread all over the matrix elements.

Universal terms

- The reflections of these "edge effects" lead to matrix elements that depend on all the elements of the matrix K thus, contributions to the entanglement entropy that depend on the global characteristics of the entangling surface.
- Such "universal" terms cannot be captured at any finite order in our perturbation series. They are rather non-perturbative effects in this expansion. Of course they are visible in the numerical calculations.

Universal terms

- The terms we capture depend on the local characteristics of the entangling surface. This is depicted to the fact that the perturbative expressions for the elements of the matrices Ω, A^{-1} and C^{-1} depend on a finite number of the elements of matrix K.
- This is the reason our method is appropriate to capture the “area law”, as well as subleading terms that scale with smaller powers of the entangling sphere radius. Our method is appropriate to study the dependence of such terms on local geometric characteristics of the entangling surface, such as curvature.
- The introduction of a mass exponentially dumps the propagation of these “edge effects”. As a result, our expansive calculations accurately converge to the numerical calculations.

The regularization scheme

In our analysis, we have applied a peculiar, inhomogeneous regularization. We have imposed a cutoff in the radial direction, but not in the angular directions.

Thus, the measurables that we have calculated, are those measured by a peculiar observer who has access to radial excitations of the theory up to an energy scale $1/a$ and to arbitrary high energy azimuthal excitations.

An angular cutoff

We could have applied a different more homogeneous regularization imposing an azimuthal cutoff by constraining the summation series in ℓ to a maximum value equal to ℓ_{max}. Such a prescription would make our approach more similar to a traditional square lattice regularization.
Locality enforces the area law term to depend on the characteristics of the underlying theory in the region of the entangling surface.
Therefore, a natural selection for an azimuthal cutoff ℓ_{max}, when considering a d-dimensional entangling surface should have the following property: the total number of harmonics with $\ell \leq \ell_{\text{max}}$ should equal the area of the entangling surface divided by a^d. Such a cutoff is of the form $\ell_{\text{max}} = cR/a$, where c is a constant.

The expansion with an angular cutoff

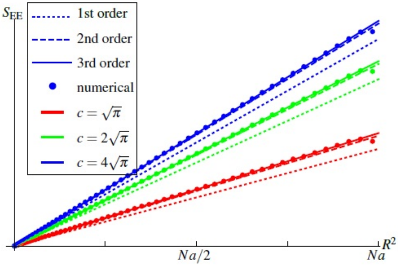
It is not difficult to repeat our analysis including this azimuthal cutoff. The only extra necessary steps are the introduction of a finite upper bound in the definite integral and similarly the inclusion of the terms calculated at $x = \ell_{\max}$ in the Euler-Maclaurin formula.

As an indicative example, in $3 + 1$ dimensions, the area law term calculated at second order in the inverse mass expansion assumes the form

$$S_{\mathrm{EE}} = \left(\frac{3 + 2\ln\left[4\left(\mu^2 a^2 + 2\right)\right]}{\left(\mu^2 a^2 + 2\right)} - \frac{3 + 2\ln\left[4\left(\mu^2 a^2 + 2 + c^2\right)\right]}{\left(\mu^2 a^2 + 2 + c^2\right)} + \frac{167 + 492\ln\left[4\left(\mu^2 a^2 + 2\right)\right]}{4608\left(\mu^2 a^2 + 2\right)^3} - \frac{167 + 492\ln\left[4\left(\mu^2 a^2 + 2 + c^2\right)\right]}{4608\left(\mu^2 a^2 + 2 + c^2\right)^3} + \mathcal{O}\left(\mu^{-10}\right) \right) \frac{R^2}{a^2}.$$

Comparison with numerical results

An azimuthal cutoff of the form $\ell_{\max} = cR/a$ preserves the dominance of the area law term in entanglement entropy. The inverse mass expansion is still a good approximation when such a regularization scheme is chosen.



- The area law term, as well as the subleading terms, are strongly affected by the regularization scheme. This is the expected behaviour. The only terms that do not depend on the regularization scheme are the universal terms, which cannot be captured by our perturbation theory.
- The introduction of an azimuthal cutoff would also set the perturbative calculation of the entanglement entropy finite at higher number of dimensions, where the respective integral term diverges as $\ell_{\text{max}} \to \infty$.
- Srednicki's calculation, which is equivalent to the specific choice $c \to \infty$, is an upper bound for the area law coefficient. The fact that the integral terms in more than $3+1$ dimensions diverge, implies that such an upper bound exists only in $2+1$ and $3+1$ dimensions.

Section 5

Discussion

Summary

- We managed to find a perturbative method to calculate S_{EE} analytically, using as expansive parameter the inverse mass of the field.
- The calculation indicates that the major contribution to entanglement entropy is a term proportional to the area of the entangling surface, i.e. the “area law” term. The perturbative calculation of the coefficient of this term agrees with the numerical calculation of entanglement entropy. Subleading terms can also be perturbatively calculated.
- The inverse mass expansion and the entangling sphere radius expansions can be performed simultaneously, but they are not parallel in any sense. The leading term in the entangling sphere radius expansion, i.e. the area law term, as well as the subleading terms, receive contributions at all orders in the inverse mass expansion.

Summary

- The area law term, as well as the subleading ones are dependent on the regularization scheme. Universal terms that depend on the global characteristics of the entangling surface are non-perturbative contributions in this expansive approach.
- The coefficient of the area law term in $2+1$ and $3+1$ dimensions has an upper bound, for any regularization scheme. The latter does not exist in higher dimensions.
- The perturbation series converges even in the massless field case. In the case of free massless scalar field in $3+1$ dimensions the inverse mass series for the coefficient of the area law term approaches the value 0.295 found in the literature.

Summary

- The perturbative method is not limited to the calculation of S_{EE}, but it provides the full spectrum of the reduced density matrix. The latter, unlike S_{EE}, contains the full information of the entanglement between the considered subsystems.
  This is clearly an advantage in comparison to holographic or replica trick calculations.
- Locality is encoded into the couplings matrix K as the absence of non-diagonal elements apart from the elements of the superdiagonal and subdiagonal. This results in an hierarchy for the eigenvalues of ρ_A. This hierarchy in the spectrum of ρ_A depicts the fact that locality enforces entanglement between the interior and the exterior of the sphere to be dominated by the entanglement between pairs of neighbouring degrees of freedom that are separated by the entangling surface.
  The latter are clearly proportional to the area and not the volume of the entangling sphere.

Future Directions

- Field theory at finite temperature. Cosmological implications.
- Field theory at excited states.
- More general entangling surfaces. For example entangling surfaces that correspond to the elliptic minimal surfaces found in AdS_4.
- Field theory on a curved background, e.g. dS or AdS.

Thank you!