Deformation quantization of non-geometric backgrounds in M-theory

Vladislav Kupriyanov

MPI Munich & UFABC

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Deformation quantization: BFFLS, '77

Let A be an algebra of functions on $\mathbb{R}^N$, e.g., $C^\infty(\mathbb{R}^N)$, $\mathrm{Poly}(\mathbb{R}^N)$. **Star product** is a formal deformation of the pointwise product on A in the direction of a given Poisson bivector field $P^{ij}(x)$.

1. A formal deformation,

$$f \cdot g \to f \star g = f \cdot g + \sum_{r=1}^{\infty} (i\hbar)^r C_r(f,g)\,.$$

2. The "Initial condition",

$$\lim_{\hbar\to 0} \frac{[f,g]_\star}{2i\hbar} = \{f,g\} = P^{ij}(x)\partial_i f\ \partial_j g\,.$$

3. The associativity condition, $(f \star g) \star h = f \star (g \star h)$.

The last condition

– requires Jacoby Identity for consistency:

$\{f,g,h\} := \{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$.

– allows to proceed to higher orders, $C_r(f,g)$, $r > 1$,

Existence: Formality theorem by M. Kontsevich, '97

- Magnetic charges through covariant momenta:

$$\{x^i, x^j\} = 0, \qquad \{x^i, \pi_j\} = \delta^i_j,$$
$$\{\pi_i, \pi_j\} = e\varepsilon_{ijk}B^k(x),$$
$$\{\pi_i, \pi_j, \pi_k\} = e\varepsilon_{ijk}\mathrm{div}\,\vec{B}\,.$$

For Dirac monopole, $\vec{B}(\vec{x}) = g\vec{x}/x^3$. For a constant uniform magnetic charge distribution one sets, $\vec{B}(\vec{x}) = \rho\vec{x}/3$, then $\mathrm{div}\,\vec{B} = \rho$.

- Constant R-flux [Blumenhagen, Plauschin & Lüst '10]:

$$\{x^i, x^j\} = \frac{\ell_s^3}{\hbar^2} R^{ijk} p_k\,, \qquad \{x^i, p_j\} = \delta^i_j \qquad \{p_i, p_j\} = 0\,,$$

Making, $p \to x$ and $x \to -p$, one obtains the algebra of a constant magnetic charge distribution.

Non-Poisson structures

- Magnetic charges through covariant momenta:

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Octonions, $1X = X1 = X$, and $|XY| = |X||Y|$.

$$X = k^0\,1 + k^A\,e_A,$$

where $k^0, k^A \in \mathbf{R}$, $A = 1, \ldots, 7$, while 1 is the identity element,

$$e_A\,e_B = -\delta_{AB}\,1 + \eta_{ABC}\,e_C\ ,$$

$\eta_{ABC} = +1$ for $ABC = 123,\ 435,\ 471,\ 516,\ 572,\ 624,\ 673$.

$$[e_A, e_B] := e_A\,e_B - e_B\,e_A = 2\,\eta_{ABC}\,e_C\ .$$

Introducing $f_i := e_{i+3}$ for $i = 1, 2, 3$, (e_i and 1 generate $\mathbf{H}$),

$$[e_i, e_j] = 2\,\varepsilon_{ijk}\,e_k \quad \text{and} \quad [e_7, e_i] = 2\,f_i\ ,$$

$$[f_i, f_j] = -2\,\varepsilon_{ijk}\,e_k \quad \text{and} \quad [e_7, f_i] = -2\,e_i\ ,$$

$$[e_i, f_j] = 2\,(\delta_{ij}\,e_7 - \varepsilon_{ijk}\,f_k)\ .$$

Octonions are non-associative, $[e_A, e_B, e_C] = -12\eta_{ABCD}\,e_D$, but alternative, i.e.,

$$[X, Y, Z] = 6\left((X\,Y)\,Z - X\,(Y\,Z)\right).$$

Defining the coordinates and momenta in terms of the imaginary octonions as

$$x^i = \frac{\sqrt{\lambda\,\ell_s^3\,R}}{2\hbar}\,f_i\ ,\quad p_i = -\tfrac{\lambda}{2}\,e_i\ ,\quad x^4 = \frac{\sqrt{\lambda^3\,\ell_s^3\,R}}{2\hbar}\,e_7\ ,$$

we obtain

$$\begin{aligned}
\{x^i, x^j\}_\lambda &= \tfrac{\ell_s^3}{\hbar^2}\,R^{4,ijk4}\,p_k \quad \text{and} \quad \{x^4, x^i\}_\lambda = \tfrac{\lambda\,\ell_s^3}{\hbar^2}\,R^{4,1234}\,p^i,\\
\{x^i, p_j\}_\lambda &= \delta^i_j\,x^4 + \lambda\,\varepsilon^i{}_{jk}\,x^k \quad \text{and} \quad \{x^4, p_i\}_\lambda = \lambda^2\,x_i\ ,\\
\{p_i, p_j\}_\lambda &= -\lambda\,\varepsilon_{ijk}\,p^k\ .
\end{aligned}$$

with λ being the M-theory radius.
Sending $\lambda \to 0$ one recover the R-flux algebra.

Non-associative star products

Main problem: for a non-Poisson P^{jk} what can be used instead of the associativity condition to restrict the higher order terms in star products? Why not,

$$f \star g = f \cdot g + \frac{i\hbar}{2}\{f, g\}\,?$$

The nonassociative star products should be:

(a$\star$) Hermitean:

$$(f \star g)^* = g^* \star f^* \ .$$

(b$\star$) Unital:

$$1 \star f = f = f \star 1 \ .$$

(c$\star$) Closed:

$$\int f \star g = \int f \cdot g \ .$$

(d$\star$) 3-cyclic:

$$\int (f \star g) \star h = \int f \star (g \star h) \ .$$

Alternativity and Malcev-Poisson identity

def. $\star$ is *alternative* if $A_\star(f,g,h)$ is completely antisymmetric in its arguments, or 'alternating'. For such products we have

$$f\star(g\star h)-(f\star g)\star h=\frac{1}{6}[f,g,h]_\star\ ,$$
$$[f,g,h]_\star:=[f,[g,h]_\star]_\star+[h,[f,g]_\star]_\star+[g,[h,f]_\star]_\star\ .$$

Each alternative algebra defines the Malcev algebra when product is substituted by the commutator, $f\star g\to[f,g]_\star$. It satisfys:

$$[f,g,[f,h]_\star]_\star=[[f,g,h]_\star,f]_\star\ .$$

In the semi-classical limit it implies the Malcev-Poisson identity:

$$\{f,g,\{f,h\}\}=\{\{f,g,h\},f\}\ .$$

For the constant R-flux taking $f=x^1$, $g=x^3\,p_1$ and $h=x^2$, one finds on the r.h.s.: $\frac{3\ell_s^3}{\hbar^2}\,R$, while the l.h.s. vanishes. The same is true for the M-theory R-flux.

Weyl star products

def. Weyl star products satisfy

$$(x^{i_1}\dots x^{i_n})\star f=\sum_{P_n}\frac{1}{n!}P_n(x^{i_1}\star(\dots\star(x^{i_n}\star f)\dots),$$

e.g.,

$$(x^i x^j)\star f=\frac{1}{2}\left(x^i\star(x^j\star f)+x^j\star(x^i\star f)\right).$$

Theorem [KVG & Vassilevich, '15]: *For any bivector field $P^{ij}(x)$ there is unique Hermitian, unital, strictly triangular, Weyl star product.*

Remarks:

- Weyl $\star$ is alternative on monomials (Schwartz functions?).
- It is neither closed, $\int f\star g\neq\int f\cdot g$, nor 3-cyclyc.
- Constructive order by order procedure.

$$
\begin{aligned}
(f \star g)(x) &= f \cdot g + \frac{i\hbar}{2} P^{ij} \partial_i f \partial_j g \\
&-\frac{\hbar^2}{8} P^{ij} P^{kl} \partial_i \partial_k f \partial_j \partial_l g - \frac{\hbar^2}{12} P^{ij} \partial_j P^{kl} \left(\partial_i \partial_k f \partial_l g - \partial_k f \partial_i \partial_l g \right) \\
&-\frac{i\hbar^3}{8} \left[\frac{1}{3} P^{nl} \partial_l P^{mk} \partial_n \partial_m P^{ij} \left(\partial_i f \partial_j \partial_k g - \partial_i g \partial_j \partial_k f \right) \right. \\
&+\frac{1}{6} P^{nk} \partial_n P^{jm} \partial_m P^{il} \left(\partial_i \partial_j f \partial_k \partial_l g - \partial_i \partial_j g \partial_k \partial_l f \right) \\
&+\frac{1}{3} P^{ln} \partial_l P^{jm} P^{ik} \left(\partial_i \partial_j f \partial_k \partial_n \partial_m g - \partial_i \partial_j g \partial_k \partial_n \partial_m f \right) \\
&+\frac{1}{6} P^{jl} P^{im} P^{kn} \partial_i \partial_j \partial_k f \partial_l \partial_n \partial_m g \\
&\left. +\frac{1}{6} P^{nk} P^{ml} \partial_n \partial_m P^{ij} \left(\partial_i f \partial_j \partial_k \partial_l g - \partial_i g \partial_j \partial_k \partial_l f \right) \right] + \mathcal{O}\left(\hbar^4\right) .
\end{aligned}
$$

and so on.

Exemple: quantization of a constant R-flux

Consider the algebra of non-Poisson brackets,

$$\{x^I, x^J\} = \Theta^{IJ}(x) = \begin{pmatrix} \frac{\ell_s^3}{\hbar^2} R^{ijk} p_k & -\delta^i_j \\ \delta^i_j & 0 \end{pmatrix} \qquad \text{with} \qquad x = (x^I) = (\mathbf{x}, \mathbf{p}).$$

The quantization is given by [Mylonas, Schupp, Szabo '12]:

$$f \star_R g = \int \frac{d^6 k}{(2\pi)^6} \frac{d^6 k'}{(2\pi)^6} \tilde{f}(k) \tilde{g}(k')\, e^{i\mathcal{B}(k,k')\cdot x} = f(x)\, e^{\frac{i\hbar}{2} \overleftarrow{\partial}_I \Theta^{IJ}(x) \overrightarrow{\partial}_J} g(x),$$

where

$$\mathcal{B}(k,k')\cdot x := (\mathbf{k}+\mathbf{k}')\cdot \mathbf{x} + (\mathbf{l}+\mathbf{l}')\cdot \mathbf{p} - \frac{\ell_s^3}{2\hbar} R\, \mathbf{p}\cdot(\mathbf{k} \times_\varepsilon \mathbf{k}') + \frac{\hbar}{2}\left(\mathbf{l}\cdot\mathbf{k}' - \mathbf{k}\cdot\mathbf{l}'\right),$$

It is alternative on Schwartz functions and monomials.
Also, it is Weyl; Hermitean, unital, closed and 3-cyclic.

Deformed vector sums

For each $\vec{p}, \vec{p}\,'$ from the unit ball $|\vec{p}| \leq 1$ in $\mathbf{R}^7$, define the map

$$\vec{p} \circledast_\eta \vec{p}\,' = \epsilon_{(\vec{p},\vec{p}')} \left(\sqrt{1 - |\vec{p}\,'|^2}\; \vec{p} + \sqrt{1 - |\vec{p}\,|^2}\; \vec{p}\,' - \vec{p} \times_\eta \vec{p}\,' \right) .$$

(V1) Vector $\vec{p} \circledast_\eta \vec{p}\,'$ belongs to the unit ball in V,

$$1 - |\vec{p} \circledast_\eta \vec{p}\,'|^2 = \left(\sqrt{1 - |\vec{p}\,|^2}\,\sqrt{1 - |\vec{p}\,'|^2} - \vec{p} \cdot \vec{p}\,' \right)^2 \geq 0;$$

(V2) Commutator reproduces the cross product,

$$\vec{p} \circledast_\eta \vec{p}\,' - \vec{p}\,' \circledast_\eta \vec{p} = \frac{1}{2}\, \vec{p}\,' \times_\eta \vec{p}; \qquad (\vec{p} \times_\eta \vec{p}\,')_A = \eta_{ABC}\, p_B\, p'_C\,.$$

(V3) It is alternative,

$$\vec{A}_\eta(\vec{p}, \vec{p}\,', \vec{p}\,'') := (\vec{p} \circledast_\eta \vec{p}\,') \circledast_\eta \vec{p}\,'' - \vec{p} \circledast_\eta (\vec{p}\,' \circledast_\eta \vec{p}\,'') = \tfrac{2}{3}\, \vec{J}_\eta(\vec{p}, \vec{p}\,', \vec{p}\,'') .$$

For $\mathbf{q} \in \mathbf{R}^3$, vector star product $\mathbf{q} \circledast \mathbf{q}\,'$ is associative.

Deformed vector sums

To extend $\vec{p} \circledast_\eta \vec{p}\,'$ over the entire space V we introduce the maps

$$\vec{p} = \frac{\sin(\hbar\,|\vec{k}|)}{|\vec{k}|}\;\vec{k} \qquad \text{and} \quad \vec{k} = \frac{\sin^{-1}|\vec{p}\,|}{\hbar\,|\vec{p}\,|}\;\vec{p}\;.$$

The deformed vector sum is defined as

$$\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,) = \frac{\sin^{-1}|\vec{p}\circledast_\eta \vec{p}\,'\,|}{\hbar\,|\vec{p}\circledast_\eta \vec{p}\,'\,|}\;\vec{p}\circledast_\eta \vec{p}\,'\Big|_{\vec{p}=\vec{k}\,\sin(\hbar\,|\vec{k}|)/|\vec{k}|}\;.$$

(B1) $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,) = -\vec{\mathcal{B}}_\eta(-\vec{k}',-\vec{k}\,)$;

(B2) $\vec{\mathcal{B}}_\eta(\vec{k},\vec{0}\,) = \vec{\mathcal{B}}_\eta(\vec{0},\vec{k}\,) = \vec{k}$;

(B3) $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,) = \vec{k} + \vec{k}' - 2\,\hbar\,\vec{k}\,\times_\eta\,\vec{k}' + O(\hbar^2)$;

(B4) The associator

$$\vec{\mathcal{A}}_\eta(\vec{k},\vec{k}',\vec{k}''\,) := \vec{\mathcal{B}}_\eta(\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,),\vec{k}''\,) - \vec{\mathcal{B}}_\eta(\vec{k},\vec{\mathcal{B}}_\eta(\vec{k}',\vec{k}''\,))$$

is antisymmetric in all arguments.

Quantization of imaginary octonions, $[e_A, e_B] = 2\,\eta_{ABC}\, e_C$.

$$(f \star_\eta g)(\vec{\xi}\,) = \int \frac{d^7\vec{k}}{(2\pi)^7}\,\frac{d^7\vec{k}'}{(2\pi)^7}\,\tilde{f}(\vec{k}\,)\,\tilde{g}(\vec{k}'\,)\,e^{i\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)\cdot\vec{\xi}} .$$

This star product satisfies,

(S1) $(f \star_\eta g)^* = g^* \star_\eta f^*$;

(S2) $f \star_\eta 1 = 1 \star_\eta f = f$;

(S3) provides quantization of imaginary octonions,

$$f \star_\eta g = f\cdot g + \frac{i\hbar}{2}\{f,g\}_\eta + O(\hbar^2), \qquad \{\xi_A,\xi_B\}_\eta = 2\,\eta_{ABC}\,\xi_C;$$

(S4) It is alternative on monomials and Schwartz functions.

(S5) For functions f, g on three-dimensional subspace endowed with $su(2)$ Lie algebra $[e_i, e_j] = 2\,\varepsilon_{ijk}\,e_k$, the $\star_\eta$ defines the associative star product $(f \star_\varepsilon g)(\xi, \mathbf{0}, 0)$.

Neither $\star_\eta$, nor $\star_\varepsilon$ is closed, $\int f \star g \neq \int f\, g$.

Closure and cyclicity

def. Star products $\bullet$ and $\star$ are equivalent if

$$f \bullet g = \mathcal{D}^{-1}\big(\mathcal{D}f \star \mathcal{D}g\big) \qquad \text{with} \quad \mathcal{D} = 1 + O(\hbar) \ .$$

Physically, equivalent star products represent different quantizations for the same classical system preserving the main properties.

$$6\big(f \star (g \star h) - (f \star g) \star h\big) = [f,g,h]_\star \iff$$
$$6\big(f \bullet (g \bullet h) - (f \bullet g) \bullet h\big) = [f,g,h]_\bullet .$$

The closed, $\int f \bullet g = \int f\, g$, alternative star product is 3-cyclic:

$$\int (f \bullet g) \bullet h = \int f \bullet (g \bullet h) \ .$$

Octonionic closed star product is given by:

$$f \bullet_\eta g = \mathcal{D}^{-1}\big(\mathcal{D}f \star_\eta \mathcal{D}g\big) \qquad \text{with} \quad \mathcal{D} = \Big(\big(\hbar \triangle_{\vec{\xi}}^{1/2}\big)^{-1} \sinh\big(\hbar \triangle_{\vec{\xi}}^{1/2}\big)\Big)^6,$$

[KVG '16].

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[KVG '16].

The GLM algebra is obtained from, $\{\xi_A, \xi_B\} = 2\eta_{ABC}\xi_C$, by linear transformation,

$$\vec{x} = (x^A) = (\mathbf{x}, x^4, \mathbf{p}) := \Lambda\,\vec{\xi} = \tfrac{1}{2\hbar}\left(\sqrt{\lambda\,\ell_s^3\,R}\,\sigma\,,\ \sqrt{\lambda^3\,\ell_s^3\,R}\,\sigma^4\,,\ -\lambda\,\hbar\,\xi\right).$$

Define a star product of functions on the seven-dimensional M-theory phase space by the prescription

$$(f\star_\lambda g)(\vec{x}) = (f_\Lambda \bullet_\eta g_\Lambda)(\vec{\xi}\,)\big|_{\vec{\xi}=\Lambda^{-1}\,\vec{x}}$$

where $f_\Lambda(\vec{\xi}\,) := f(\Lambda\,\vec{\xi}\,)$.
It satisfies all required properties. Moreover,

$$\lim_{\lambda\to 0}\,(f\star_\lambda g)(\vec{x}) = (f\star_R g)(x)\ ,$$

in the limit $\lambda \to 0$ the non-geometric M-theory star product reduces exactly to constant R-flux star product, [KVG & Szabo '17].

Discussion

1. Why we cannot just set

$$f \star g = f \cdot g + \frac{i\hbar}{2}\{f, g\}\,?$$

   The condition of the *alternativity* of the star product, at least on some class of functions, seems to be reasonable condition to restrict non-associative star products.

2. G_2-symmetric star product $\star_\eta$ is used to quantize non-geometric M-theory background, construct $\star_\lambda$.
   - The restriction of $\star_\lambda$ to a proper subspace defines an associative $su(2)$ star product $\star_\varepsilon$.
   - The limit $\lambda \to 0$ of $\star_\lambda$ reproduces the constant R-flux star product $\star_R$. The limit $R \to 0$ of $\star_R$ gives Moyal.
   - Explicit all orders formulas for everything.

3. Not the end of the story! What kind of physics will we have here?

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