Integrable deformations and non-commutativity

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based on 1608.08504 with Stijn van Tongeren
and ongoing work with Dieter Lüst

Motivation

- integrability of toy models, for example in AdS/CFT: type IIb GS superstring in $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM

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- integrability of toy models, for example in AdS/CFT: type IIb GS superstring in $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM
- 'deformations' of integrable (2d) σ-models
  - AdS/CFT: less symmetric examples?
  - symmetries behind integrable structures?
  - generating new supergravity solutions?
  - non-abelian T-dualities and non-geometric backgrounds

Overview

Yang-Baxter deformations - Definition and properties

- simplest setup: integrable deformations of principal chiral model on Lie group G ([Klimcik, 2008] & [Delduc et al., 2013])

$$S \propto \int \mathrm{d}^2\sigma \ \mathrm{Tr}\left((g^{-1}\partial_+ g)\frac{1}{\mathbb{1}-\eta R}(g^{-1}\partial_- g)\right)$$

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- Lax integrability (classical)
  - equations of motion equivalent to $\partial_+ L_- - \partial_- L_+ - [L_+, L_-] = 0$, with $L_\pm \equiv L_\pm(\lambda): \ \mathbb{C} \to \mathfrak{g}$ $\Rightarrow$ infinite tower of conserved charges
  - Deformed model is Lax integrable, if $R \in \mathrm{End}(\mathfrak{g})$ solution to (modified) classical Yang-Baxter equation: $[R(M), R(N)] - R\left([R(M), N] + [M, R(N)]\right) = C[M, N]$.

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- generalisation to symmetric space (super)coset models (e.g. superstring on $AdS_5 \times S^5$ possible)
- question: Is this deformation a supergravity solution?

Yang-Baxter deformations - Overview

- **abelian Yang-Baxter deformations**
  [Osten and van Tongeren, 2016]
  - based on R-operators with non-vanishing R^{ij} with $[t_i, t_j] = 0$
  - equivalent to θ-shifts from the $\mathrm{O}(d,d)$ T-duality group:
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- generalisation: **unimodular Yang-Baxter deformations**: [Borsato and Wulff, 2016]
  - based on unimodular R-operators: $R^{AB}[t_A, t_B] = 0$
  - R unimodular $\Leftrightarrow$ background is a supergravity solution.
  - conjectured to be related to non-abelian T-duality transformations in general (shown for unimodular R-operators of AdS_5 resp. $SO(2,4)$ in [Hoare and Tseytlin, 2016])

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- **η-deformations** [Delduc et al., 2013, Arutyunov et al., 2013]
  - generated by Drinfel'd-Jimbo R-operators solving mCYBE
  - quantum group deformations of isometry algebra
  - not a supergravity solution
  - *but:* Poisson-Lie T-dual to a supergravity solution (λ-deformation)

CFT-duals to Yang-Baxter deformed AdS-backgrounds

- Yang-Baxter deformation
  - closed string vs. open string picture [Seiberg and Witten, 1999]
  - $\leftrightarrow$ Drinfel'd twist of underlying Hopf algebra structure [van Tongeren, 2015]
- abelian twist of S^5 of $AdS_5 \times S^5$:
  - e.g. deformation based on $(\mathrm{U}(1))^3$ of $\mathrm{SO}(6)$ $\leftrightarrow$ deformation of $\mathcal{R}$-symmetry in $\mathcal{N} = 4$ SYM
  - represented by $\star$-product between the corr. scalars in Lagrangian, e.g.

$$\phi \star \psi = \exp\left(i\pi\gamma(q_1(\phi)q_2(\psi) - q_2(\phi)q_1(\psi)\right)\phi\psi,$$

    where q_1, q_2: charges of $\mathrm{U}(1)\times\mathrm{U}(1)$
- deformation of AdS_5 in $AdS_5 \times S^5$ $\leftrightarrow$ $\mathcal{N} = 4$ SYM on non-commutative spacetime (Groenewold-Moyal $\star$-product with $\Theta^{AB} \propto R^{AB}$)

Embedding into a 'doubled space'

- simple setup of a double field theory on Lie groups [Hull and Reid-Edwards, 2009, Hassler, 2017]
  - Drinfel'd double $\mathcal{D}$: $2d$-dim. Lie group with compatible $\mathrm{O}(d,d)$-metric η
  - polarisations $\hat{=}$ bialgebra decompositions, $\mathfrak{d} = \mathfrak{g}_i \oplus \mathfrak{g}_i^\star$, where $(\mathfrak{g}_i,\, \mathfrak{g}_i^\star)$: dual pair of *maximally isotropic subalgebras* w.r.t. to η
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- Yang-Baxter deformations, generated by solutions R of cYBe
  - $\begin{bmatrix} \text{Principal chiral model} \\ \text{polarisation: } (\mathfrak{g}, (\mathfrak{u}(1))^d) \end{bmatrix} \mapsto \begin{bmatrix} \text{Yang-Baxter def. model} \\ \text{polarisation: } (\mathfrak{g}, \mathfrak{g}^\star) \end{bmatrix}$
    with Lie algebra structure on $\mathfrak{g}^\star$: $\overline{f}_c{}^{ab} = f^{[a}{}_{cd} R^{b]d}$
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- conjectural note on η-deformations:
  - non-trivial β-shift, $[\beta, \beta]_S \propto \mathcal{C}^3_{\mathfrak{g}}$
  - $\rightarrow$ Q-/R-flux background?
  - a unified description of η-/λ-deformations in this way?
  - gerbe formulation needed?

Conclusion

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- overview over integrable deformations
- dual CFTs for deformed AdS
- possible direction for embedding into a 'doubled space'

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Open questions:

- non-abelian twists in AdS/CFT
- more elaborate double field theory on group manifolds
    - gerbe formulation
    - non-trivial fluxes
    - gauge algebra
- η-/λ-deformation as non-geometric flux backgrounds

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Thank you for your attention!

BACKUP 1: Classical Yang-Baxter Equation (CYBE)

- Common form:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \qquad \text{for } r \in \mathfrak{g} \otimes \mathfrak{g}$$

Transition from a skew-symmetric r-matrix to a R-operator:

$$r = a \wedge b := \frac{1}{2}(a \otimes b - b \otimes a) \quad \rightarrow \quad R(M) := \mathrm{Tr}_2(r \cdot (1 \otimes M))$$

- There are deformations based on solutions R of
  - CYBE: $[R(M), R(N)] - R([R(M), N] + [M, R(N)]) = 0$
  - mCYBE: $[R(M), R(N)] - R([R(M), N] + [M, R(N)]) = \pm[M, N]$

BACKUP 2: λ-deformation

Let G be a Lie group, and $g:\ \Sigma \to G$ with generators $\{t_a\}$ and structure constants $(f^c)_{ab}$. The action [Sfetsos, 2014]

$$S(g) = S_{WZW,k}(g) + \frac{\lambda}{\pi}\int \mathrm{d}^2\sigma\ (g^{-1}\partial_+ g)^a \left(\frac{1}{\mathbb{1} - \lambda \mathrm{Ad}_g^{-1}}\right)_{ab} (g^{-1}\partial_+ g)^b$$

describes a deformation of the WZW-model. The deformation parameter λ can be written in terms of the WZW-level k and the coupling of a principal chiral model κ is $\lambda = \frac{k^2}{\kappa^2+k}$. The limits are

- $\lambda \to 0$: undeformed WZW model
- $k \to \infty$: non-abelian T-dual of principal chiral model on G

$$S_{NATD} = \frac{1}{\pi}\int \mathrm{d}\sigma^2\ \partial_+\chi_a\left(\mathbb{1} - \chi_c(f^c)\right)^{-1,ab}\partial_-\chi_b$$

- $k \ll \kappa^2$: perturbed WZW

$$S(g) = S_{WZW,k}(g) + \frac{k^2}{\pi\kappa^2}\int \mathrm{d}\sigma^2\ (g^{-1}\partial_+ g)^a (g^{-1}\partial_+ g)^a$$

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