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Pure Yang–Mills solutions on dS_4

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- Description of de Sitter space
- Reduction of Yang–Mills to matrix equations
- Further reduction to quintuple-well dynamics
- Yang–Mills configurations on de Sitter space
- Explicit examples
- Instantons on de Sitter space
- What about anti-de Sitter?
- Summary and outlook

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Description of de Sitter space

Four-dimensional de Sitter space dS_4 is a one-sheeted hyperboloid in $\mathbb{R}^{4,1}$ via

$$\delta_{ij} y^i y^j - (y^5)^2 \; = \; R^2 \qquad \text{where} \quad i,j = 1,\dots,4$$

Topologically, $\mathrm{dS}_4 \simeq \mathbb{R} \times S^3$. Closed-slicing global coordinates (τ,χ,θ,ϕ):

$$y^i = R\,\omega^i \cosh\tau \quad , \qquad y^5 = R\sinh\tau \qquad \text{with} \quad \tau \in \mathbb{R} \qquad \text{and} \qquad \delta_{ij}\,\omega^i\omega^j = 1$$

$\omega^i = \omega^i(\chi,\theta,\phi)$ embeds unit S^3 with metric $\mathrm{d}\Omega_3^2$ into $\mathbb{R}^4$:

$$\mathrm{d}s^2 \; = \; R^2 \left(-\mathrm{d}\tau^2 + \cosh^2\!\tau\, \mathrm{d}\Omega_3^2 \right)$$

Introduce orthonormal basis $\{e^a\}$, $a = 1,2,3$, of SU(2) left-invariant one-forms via

$$e^a \; = \; -\eta^a_{ij}\,\omega^i \mathrm{d}\omega^j \qquad \Rightarrow \qquad \mathrm{d}e^a + \varepsilon^a_{bc}\, e^b \wedge e^c \; = \; 0$$

with self-dual 't Hooft symbols η^a_{ij}

$$\mathrm{d}\Omega_3^2 \; = \; (e^1)^2 + (e^2)^2 + (e^3)^2$$

dS_4 is conf. equivalent to a finite Lorentzian cylinder $\mathcal{I} \times S^3$ via conformal coordinates

$$t \;=\; \arctan(\sinh\tau) \;=\; 2\arctan(\tanh\tfrac{\tau}{2}) \qquad \Leftrightarrow \qquad \frac{d\tau}{dt} \;=\; \cosh\tau \;=\; \frac{1}{\cos t}$$

Range: $\tau \in \mathbb{R} \quad \Leftrightarrow \quad t \in \mathcal{I} = (-\tfrac{\pi}{2}, +\tfrac{\pi}{2})$ open interval

metric: $$ds^2 \;=\; \frac{R^2}{\cos^2 t}\Big(-dt^2 + \delta_{ab}e^a e^b\Big) \;=\; \frac{R^2}{\cos^2 t}\, ds^2_{\text{cyl}}$$

Static coordinates $(\sigma, \rho, \theta, \phi)$ cover half of de Sitter space:

$$y^a = R\,\rho\,\lambda^a\,,\; y^4 = R\sqrt{1-\rho^2}\,\cosh\sigma\,,\; y^5 = R\sqrt{1-\rho^2}\,\sinh\sigma \quad \text{with} \quad \sigma\in\mathbb{R},\; \rho\in[0,1)$$

$$\lambda^1 = \sin\theta\,\sin\phi \quad , \qquad \lambda^2 = \sin\theta\,\cos\phi \quad , \qquad \lambda^3 = \cos\theta \qquad \Rightarrow \qquad \delta_{ab}\lambda^a\lambda^b = 1$$

Induced metric on dS_4:

$$ds^2 \;=\; R^2\Big(-(1-\rho^2)d\sigma^2 + \tfrac{d\rho^2}{1-\rho^2} + \rho^2 d\Omega_2^2\Big) \qquad \text{with} \qquad d\Omega_2^2 \;=\; d\theta^2 + \sin^2\theta\, d\phi^2$$

useful: $\rho = \sin\alpha \quad \Rightarrow \quad \sqrt{1-\rho^2} = \cos\alpha$

Reduction of Yang–Mills to matrix equations

Consider rank-N hermitian vector bundles over the cylinder $\mathcal{I}\times S^3$ conf. equiv. to dS$_4$

de Sitter space has a boundary $\Rightarrow$ frame the gauge bundle over the boundary:

gauge-group elements g subject to $g(\partial\mathrm{dS}_4) = \mathrm{Id}$ on $\partial\mathrm{dS}_4 = S^3_{t=+\frac{\pi}{2}} \cup S^3_{t=-\frac{\pi}{2}}$

Gauge potential $\mathcal{A}$ and gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ in $su(N)$ in temporal gauge $\mathcal{A}_0 = 0$

SU(2)-equivariant ansatz: $\mathcal{A} = X_a(t)\, e^a$ with $X_a \in su(N)$

Resulting gauge field:

$$\mathcal{F} = \mathcal{F}_{0a}\, e^0\wedge e^a + \tfrac{1}{2}\mathcal{F}_{bc}\, e^b\wedge e^c = \dot{X}_a\, e^0\wedge e^a + \tfrac{1}{2}\Big(-2\varepsilon^a_{bc} X_a + [X_b, X_c]\Big) e^b\wedge e^c$$

with $e^0 := \mathrm{d}t$ and $\dot{X}_a := \mathrm{d}X_a/\mathrm{d}t$

Yang–Mills equations:

$$\ddot{X}_a = -4\, X_a + 3\,\varepsilon_{abc}\,[X_b, X_c] - \Big[X_b, [X_a, X_b]\Big] \qquad \text{and} \qquad [\dot{X}_a, X_a] = 0$$

three coupled ordinary differential equations for three $N\times N$ matrix functions

Further reduction to quintuple-well dynamics

Restrict X_a to some $su(2) \subset su(N)$ by embedding spin-j of $su(2)$ into $su(2j{+}1)$

The three SU(2)-generators I_a are normalized to $C(\frac{1}{2}) = \frac{1}{2}, \quad C(1) = 2$

$[I_b, I_c] \;=\; 2\,\varepsilon^{a}_{bc} I_a$ and $\text{tr}(I_a I_b) \;=\; -4C(j)\,\delta_{ab}$ for $C(j) = \frac{1}{3}\,j(j{+}1)(2j{+}1)$

Simplest choice for the matrices X_a:

$$X_1 = \Psi_1 I_1\,, \quad X_2 = \Psi_2 I_2\,, \quad X_3 = \Psi_3 I_3 \qquad \text{with} \quad \Psi_a = \Psi_a(t) \in \mathbb{R}$$

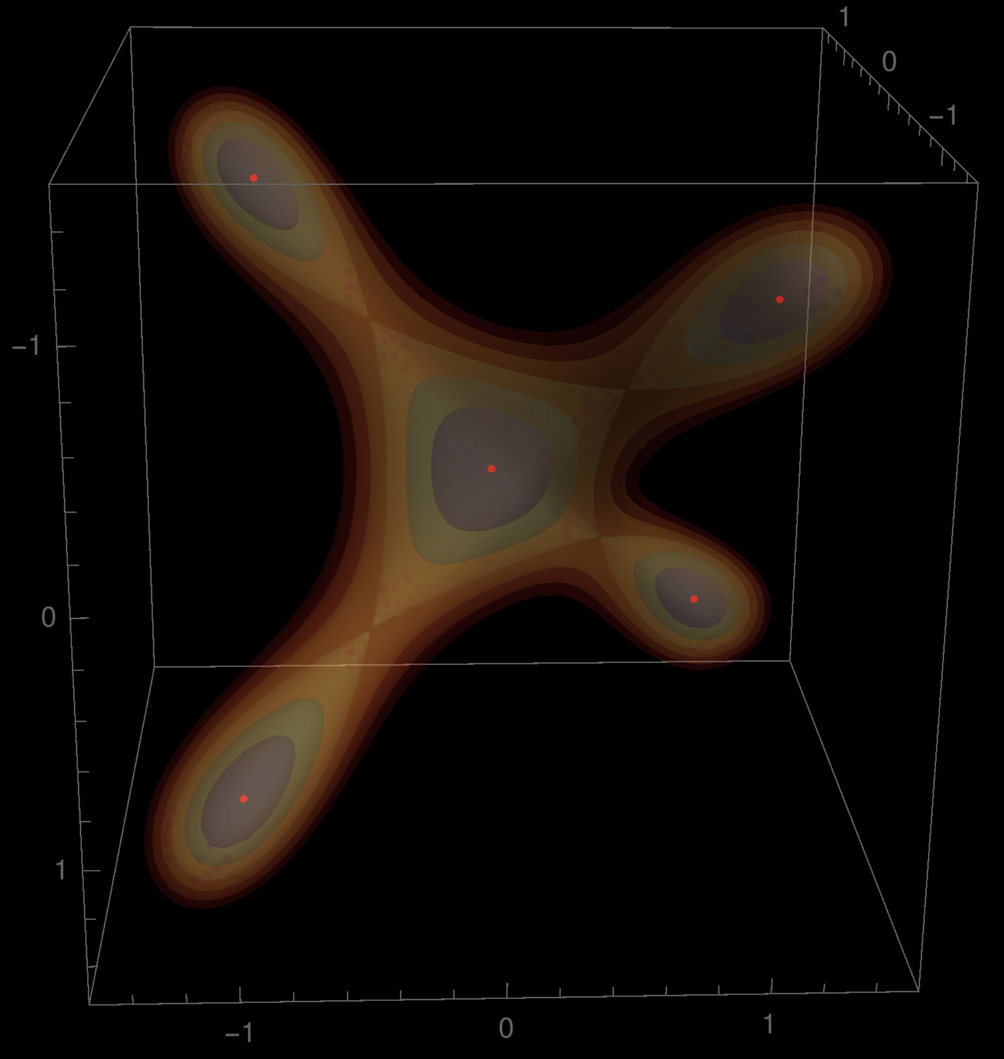
Resulting simplification of Yang–Mills Lagrangian density:

$$\mathcal{L} \;=\; \tfrac{1}{8}\,\text{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \;=\; -\tfrac{1}{4}\,\text{tr}\mathcal{F}_{0a}\mathcal{F}_{0a} \;+\; \tfrac{1}{8}\,\text{tr}\mathcal{F}_{ab}\mathcal{F}_{ab}$$

$$=\; 4\,C(j)\left\{\tfrac{1}{4}\dot{\Psi}_a\dot{\Psi}_a - (\Psi_1 - \Psi_2\Psi_3)^2 - (\Psi_2 - \Psi_3\Psi_1)^2 - (\Psi_3 - \Psi_1\Psi_2)^2\right\}$$

Interpretation: $\{\Psi_a\} =$ particle coordinates in $\mathbb{R}^3$ $\Rightarrow$ Newtonian dynamics with

potential $\tfrac{1}{2}V(\Psi) \;=\; (\Psi_1{-}\Psi_2\Psi_3)^2 + (\Psi_2{-}\Psi_3\Psi_1)^2 + (\Psi_3{-}\Psi_1\Psi_2)^2$



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Euler-Langrange equations:

$$\begin{aligned} \tfrac{1}{4}\ddot{\Psi}_1 &= -\Psi_1 + 3\,\Psi_2\Psi_3 - \Psi_1(\Psi_2^2 + \Psi_3^2) \\ \tfrac{1}{4}\ddot{\Psi}_2 &= -\Psi_2 + 3\,\Psi_1\Psi_3 - \Psi_2(\Psi_1^2 + \Psi_3^2) \\ \tfrac{1}{4}\ddot{\Psi}_3 &= -\Psi_3 + 3\,\Psi_1\Psi_2 - \Psi_3(\Psi_2^2 + \Psi_3^2) \end{aligned}$$

Too hard to solve analytically in general, but invariant under S_4 (tetrahedral symmetry)

Try to find trajectories invariant under a maximal subgroup: A_4, D_8 or S_3

A_4 : $\Psi_1 = \Psi_2 = \Psi_3 = 0$ $\Rightarrow$ trivial

D_8 : $\Psi_1 = \Psi_2 = 0\,,\quad \Psi_3 =: \xi$ $\Rightarrow$ abelian

$\Rightarrow$ $V_\xi = 2\,\xi^2$ and $\ddot{\xi} = -4\,\xi$

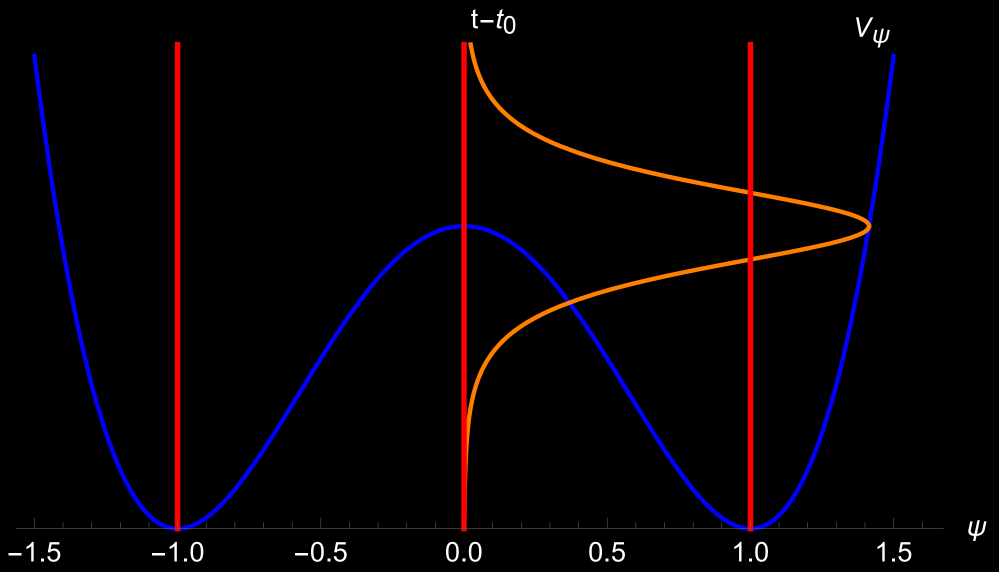
harmonic oscillator: $\xi(t) = -\frac{1}{2}\gamma\cos 2(t-t_0)$

S_3 : $\Psi_1 = \Psi_2 = \Psi_3 =: \frac{1}{2}(1+\psi)$ $\Rightarrow$ nonabelian

$\Rightarrow$ $V_\psi = \frac{1}{2}\,(1-\psi^2)^2$ and $\ddot{\psi} \;=\; 2\,\psi\,(1-\psi^2)$

double well: $\psi(t) = \pm 1$, $\psi(t) = 0$, $\psi(t) =$ bounce:

$$\psi(t) \;=\; \sqrt{2}\,\mathrm{sech}\big(\sqrt{2}(t-t_0)\big) \;=\; \frac{\sqrt{2}}{\cosh\big(\sqrt{2}(t-t_0)\big)}$$



‘energy conservation’:

$$\tfrac{1}{2}\,\dot{\psi}^2 = V_0 - V_\psi(\psi) = V_0 - \tfrac{1}{2}(1-\psi^2)^2$$

where V_0 is the potential at turning point

$V_0 \neq 0, \frac{1}{2}$: anharmonic oscillations

Yang–Mills configurations on de Sitter space

For the Lorentzian cylinder: substitute solution $\psi(t)$ into the gauge potential and field:

$$\mathcal{A} \;=\; \tfrac{1}{2}\,(1+\psi)\,e^a I_a \qquad \text{and} \qquad \mathcal{F} \;=\; \Big(\tfrac{1}{2}\,\dot{\psi}\,e^0\wedge e^a - \tfrac{1}{4}\,(1-\psi^2)\,\varepsilon^a_{\ bc}\,e^b\wedge e^c\Big) I_a$$

SU(2) color electric and magnetic fields:

$$E_a \;=\; \mathcal{F}_{0a} \;=\; \tfrac{1}{2}\,\dot{\psi}\,I_a \qquad \text{and} \qquad B_a \;=\; \tfrac{1}{2}\varepsilon_{abc}\mathcal{F}_{bc} \;=\; -\tfrac{1}{2}\,(1-\psi^2)\,I_a$$

Their energy densities:

$$\rho_e \;=\; -\tfrac{1}{4}\mathrm{tr}\,E_aE_a \;=\; \tfrac{3}{4}\,C(j)\,\dot{\psi}^2 \qquad \text{and} \qquad \rho_m \;=\; -\tfrac{1}{4}\mathrm{tr}\,B_aB_a \;=\; \tfrac{3}{4}\,C(j)\,(1-\psi^2)^2$$

Total field energy:

$$\mathcal{E}_t \;=\; \int_{S^3} e^1\wedge e^2\wedge e^3\,(\rho_e+\rho_m) \;=\; \tfrac{3}{4}\,C(j)\,\mathrm{vol}(S^3)\,\Big(\dot{\psi}^2+(1-\psi^2)^2\Big) \;=\; 3\pi^2\,C(j)\,V_0$$

For de Sitter space: time variable is $\tilde{\tau} = R\,\tau$ thus

$$\mathcal{E}_{\tilde{\tau}} \;=\; \frac{\mathrm{d}t}{\mathrm{d}\tilde{\tau}}\,\mathcal{E}_t \;=\; \frac{1}{R}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathcal{E}_t \;=\; \frac{1}{R\,\cosh\tau}\,\mathcal{E}_t \;=\; \frac{3\pi^2\,C(j)\,V_0}{R\,\cosh\tau} \qquad \text{finite at all times}$$

Action functional on the Lorentzian cylinder:

$$S \;=\; \tfrac18 \int_{\mathcal{I}\times S^3} e^0\wedge e^1\wedge e^2\wedge e^3\ \mathrm{tr}\big(-2\mathcal{F}_{0a}\mathcal{F}_{0a}+\mathcal{F}_{ab}\mathcal{F}_{ab}\big) \;=\; \int_{\mathcal{I}} \mathrm{d}t\ \mathrm{vol}(S^3)\,(\rho_e-\rho_m)$$

$$=\; \tfrac32\pi^2\,C(j)\int_{-\pi/2}^{\pi/2}\mathrm{d}t\,\Big(\dot\psi^2-(1-\psi^2)^2\Big) \;=\; 3\pi^3\,C(j)\,V_0 \;-\; 6\pi^2\,C(j)\int_{-\pi/2}^{\pi/2}\mathrm{d}t\ V_\psi\big(\psi(t)\big)$$

Action functional on de Sitter space:

$$\mathcal{A} \;=\; \tilde{\mathcal{A}}_a\,\tilde e^a \qquad \text{and} \qquad \mathcal{F} \;=\; \tilde{\mathcal{F}}_{0a}\,\tilde e^0\wedge\tilde e^a+\tfrac12\tilde{\mathcal{F}}_{bc}\,\tilde e^b\wedge\tilde e^c$$

with ON system $\tilde e^0 := R\,\mathrm{d}\tau$ and $\tilde e^a := R\,\cosh\tau\,e^a$ hence

$$\mathcal{A}_a = R\,\cosh\tau\,\tilde{\mathcal{A}}_a\ , \quad \mathcal{F}_{bc} = R^2\cosh^2\!\tau\,\tilde{\mathcal{F}}_{bc}\ , \quad \mathcal{F}_{0a} = \partial_t\mathcal{A}_a = R^2\cosh^2\!\tau\,\partial_{\tilde\tau}\tilde{\mathcal{A}}_a$$

Result:

$$S \;=\; \tfrac18\int_{\mathrm{dS}_4}\tilde e^0\wedge\tilde e^1\wedge\tilde e^2\wedge\tilde e^3\ \mathrm{tr}\big(-2\tilde{\mathcal{F}}_{0a}\tilde{\mathcal{F}}_{0a}+\tilde{\mathcal{F}}_{ab}\tilde{\mathcal{F}}_{ab}\big) \;=\; \int_{\mathbb{R}}\mathrm{d}\tau\ \mathrm{vol}(S^3)\,\frac{\rho_e-\rho_m}{\cosh\tau}$$

agrees with the value on Lorentzian cylinder ✓ finite and bounded below

For a very explicit representation pick some coordinates on S^3:

$$\omega^1 = \sin\chi\, \sin\theta\, \sin\phi\,, \quad \omega_2 = \sin\chi\, \sin\theta\, \cos\phi\,, \quad \omega^3 = \sin\chi\, \cos\theta\,, \quad \omega^4 = \cos\chi$$

Corresponding left-invariant one-forms:

$$e^1 = \sin\theta \sin\phi\, \mathrm{d}\chi + \sin\chi \cos\chi\, (\tan\chi \cos\phi + \cos\theta \sin\phi)\, \mathrm{d}\theta + \sin^2\chi \sin\theta\, (\cot\chi \cos\phi - \cos\theta \sin\phi)\, \mathrm{d}\phi$$

$$e^2 = \sin\theta \cos\phi\, \mathrm{d}\chi - \sin\chi \cos\chi\, (\tan\chi \sin\phi - \cos\theta \cos\phi)\, \mathrm{d}\theta - \sin^2\chi \sin\theta\, (\cot\chi \sin\phi + \cos\theta \cos\phi)\, \mathrm{d}\phi$$

$$e^3 = \cos\theta\, \mathrm{d}\chi - \sin\chi \cos\chi \sin\theta\, \mathrm{d}\theta + \sin^2\chi \sin^2\theta\, \mathrm{d}\phi$$

Define three matrices I_* by decomposing

$$e^a I_a =: \mathrm{d}\chi\, I_\chi + \mathrm{d}\theta\, I_\theta + \mathrm{d}\phi\, I_\phi$$

In the fundamental (spin $j=\frac{1}{2}$) representation of $su(2)$:

$$I_\chi = -\mathrm{i}\begin{pmatrix} \cos\theta & -\mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & -\cos\theta \end{pmatrix}$$

$$I_\theta = -\mathrm{i}\,\sin\chi\cos\chi\begin{pmatrix} -\sin\theta & (\tan\chi-\mathrm{i}\cos\theta)\mathrm{e}^{\mathrm{i}\phi} \\ (\tan\chi+\mathrm{i}\cos\theta)\mathrm{e}^{-\mathrm{i}\phi} & \sin\theta \end{pmatrix}$$

$$I_\phi = -\mathrm{i}\,\sin^2\chi\sin\theta\begin{pmatrix} \sin\theta & (\cot\chi+\mathrm{i}\cos\theta)\mathrm{e}^{\mathrm{i}\phi} \\ (\cot\chi-\mathrm{i}\cos\theta)\mathrm{e}^{-\mathrm{i}\phi} & -\sin\theta \end{pmatrix}$$

Field-strength components:

$$E_\chi = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\chi \quad , \qquad E_\theta = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\theta \quad , \qquad E_\phi = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\phi$$

$$B_\chi = -\frac{1}{2}(1-\psi^2)\,I_\chi \quad , \qquad B_\theta = -\frac{1}{2}(1-\psi^2)\,I_\theta \quad , \qquad B_\phi = -\frac{1}{2}(1-\psi^2)\,I_\phi$$

Explicit examples

Minima $\psi = \pm 1 \quad \Longrightarrow \quad \mathcal{F} = 0$ vacua

Maximum $\psi = 0 \quad \Longrightarrow$

$$\mathcal{A} \;=\; \tfrac{1}{2}\, e^a\, I_a \;=\; \frac{\cos t}{2R}\,\tilde{e}^a\, I_a \;=\; \frac{1}{2\,R\,\cosh\tau}\,\tilde{e}^a\, I_a$$

$$\mathcal{F} \;=\; -\tfrac{1}{4}\,\varepsilon^a_{bc}\, e^b\wedge e^c\, I_a \;=\; -\frac{\cos^2 t}{4\,R^2}\,\varepsilon^a_{bc}\,\tilde{e}^b\wedge\tilde{e}^c\, I_a \;=\; -\frac{1}{4\,R^2\cosh^2\tau}\,\varepsilon^a_{bc}\,\tilde{e}^b\wedge\tilde{e}^c\, I_a$$

$\tilde{E}_a \;=\; \tilde{\mathcal{F}}_{0a} \;=\; 0$ and $\tilde{B}_a \;=\; \tfrac{1}{2}\varepsilon_{abc}\tilde{\mathcal{F}}_{bc} \;=\; -\dfrac{\cos^2 t}{2\,R^2}\, I_a \;=\; -\dfrac{1}{2\,R^2\cosh^2\tau}\, I_a$

Purely magnetic & spatially homogeneous action $S = -\frac{3}{2}\pi^3 C(j)$ minimal?

Bounce $\quad \psi = \sqrt{2}\,\mathrm{sech}\Big(\sqrt{2}(t-t_0)\Big) \quad \Longrightarrow$

$$\mathcal{A} = \frac{\cos t}{2\,R}\left\{1 + \frac{\sqrt{2}}{\cosh\Big(\sqrt{2}(t-t_0)\Big)}\right\}\tilde{e}^a\, I_a$$

$$\mathcal{F} = -\frac{\cos^2 t}{4\,R^2}\left\{4\frac{\sinh\Big(\sqrt{2}(t-t_0)\Big)}{\cosh^2\Big(\sqrt{2}(t-t_0)\Big)}\,\tilde{e}^0\wedge\tilde{e}^a + \left(1-\frac{2}{\cosh^2\Big(\sqrt{2}(t-t_0)\Big)}\right)\varepsilon^a_{\ bc}\,\tilde{e}^b\wedge\tilde{e}^c\right\} I_a$$

Electric-magnetic spatially homogeneous configuration with $\mathcal{E}_{\tilde{\tau}} = \frac{3\pi^2 C(j)}{2\,R\,\cosh\tau}$

$$\frac{S}{C(j)} = -\tfrac{3}{2}\pi^3 + 12\,\pi^2\int_{-\pi/2}^{\pi/2}\mathrm{d}t\,\frac{\sinh^2\Big(\sqrt{2}(t-t_0)\Big)}{\cosh^4\Big(\sqrt{2}(t-t_0)\Big)}$$

$$= -\tfrac{3}{2}\pi^3 + \sqrt{8}\pi^2\Big(\tanh^3\big(\tfrac{\pi}{\sqrt{2}}+\delta\big) + \tanh^3\big(\tfrac{\pi}{\sqrt{2}}-\delta\big)\Big)$$

Bounce modulus $\delta = \sqrt{2}\,t_0$ is nontrivial because $t \in \mathcal{I} = (-\frac{\pi}{2},\frac{\pi}{2}) \neq \mathbb{R}$

Abelian $\quad \xi = -\frac{1}{2}\gamma \cos 2(t-t_0) \quad \Longrightarrow$

$$\mathcal{A} \;=\; -\tfrac{1}{2}\gamma \cos 2(t-t_0)\, e^3\, I_3 \;=\; -\tfrac{\gamma}{2\,R} \cos t\, \cos 2(t-t_0)\, \tilde{e}^3\, I_3$$

$$\mathcal{F} \;=\; \mathrm{d}\mathcal{A} \;=\; \tfrac{\gamma}{R^2} \cos^2 t \left\{ \sin 2(t-t_0)\, \tilde{e}^0 \wedge \tilde{e}^3 \;+\; \cos 2(t-t_0)\, \tilde{e}^1 \wedge \tilde{e}^2 \right\} I_3$$

$$\widetilde{E}_3 \;=\; \tfrac{\gamma}{R^2} \cos^2 t\, \sin 2(t-t_0)\, I_3 \qquad \text{and} \qquad \widetilde{B}_3 \;=\; \tfrac{\gamma}{R^2} \cos^2 t\, \cos 2(t-t_0)\, I_3$$

$$\rho_e \;=\; \gamma^2 C(j)\, \sin^2 2(t-t_0) \qquad \text{and} \qquad \rho_m \;=\; \gamma^2 C(j)\, \cos^2 2(t-t_0)$$

$$\mathcal{E}_{\tilde{\tau}} \;=\; \frac{\mathrm{d}t}{\mathrm{d}\tilde{\tau}} \int_{S^3} e^1 \wedge e^2 \wedge e^3\, (\rho_e + \rho_m) \;=\; \frac{2\pi^2 \gamma^2\, C(j)}{R\, \cosh \tau}$$

$$S \;=\; \int_{\mathcal{I}} \mathrm{d}t\; \mathrm{vol}(S^3)\, (\rho_e - \rho_m) \;=\; 2\pi^2\, \gamma^2\, C(j) \int_{\mathcal{I}} \mathrm{d}t \left(\sin^2 2(t-t_0) - \cos^2 2(t-t_0) \right) \;=\; 0$$

Instantons on de Sitter space

$$\mathrm{dS}_4 \quad \overset{\text{Wick rotation}}{\longrightarrow} \quad S^4 \quad \overset{\text{conf. equiv.}}{\longrightarrow} \quad \mathbb{R}\times S^3$$

$$(\tau,\chi,\theta,\phi) \quad \longrightarrow \quad (\varphi,\chi,\theta,\phi) \quad \longrightarrow \quad (\tfrac{r}{T},\chi,\theta,\phi)$$

$$\tau = \mathrm{i}\,(\varphi - \tfrac{\pi}{2}) \qquad \varphi = 2\arctan\tfrac{r}{R} \qquad \tfrac{r}{R} = \mathrm{e}^T \quad \Rightarrow \quad \sin\varphi = \frac{1}{\cosh T}$$

$$\mathrm{d}s^2 \;=\; R^2(\mathrm{d}\varphi^2 + \sin^2\!\varphi\,\mathrm{d}\Omega_3^2) \;=\; \frac{4\,R^4}{(r^2+R^2)^2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2) \;=\; \frac{R^2}{\cosh^2\!T}(\mathrm{d}T^2 + \mathrm{d}\Omega_3^2)$$

Euclidean dS_4 is conformally equivalent to Euclidean cylinder over S^3

Stereographic coordinates $\quad x^i = r\,\omega^i(\chi,\theta,\psi)$: $\qquad \mathrm{d}s^2 \;=\; \dfrac{4\,R^4}{(r^2+R^2)^2}\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$

Instanton equation: $\mathcal{F}_{ij} = \frac{1}{2}\sqrt{\det g}\,\varepsilon_{ijkl}\,\mathcal{F}^{kl}$

Go to the Euclidean cylinder with $\mathrm{d}s^2_{\text{cyl}} = \mathrm{d}T^2 + \mathrm{d}\Omega^2_3$ and temporal gauge $\mathcal{A}_0 = 0$

SU(2)-equivariant ansatz: $\mathcal{A} = X_a(T)\,e^a$ with $X_a \in su(N)$

$\Rightarrow$ $\mathcal{F}_{4a} = \frac{\mathrm{d}X_a}{\mathrm{d}T}$ and $\mathcal{F}_{ab} = -2\,\varepsilon_{abc}X_c + [X_a, X_b]$

Instanton equation reduces to generalized Nahm equation

$$\frac{\mathrm{d}X_a}{\mathrm{d}T} = 2\,X_a - \frac{1}{2}\,\varepsilon_{abc}\,[X_b, X_c]$$

Same ansatz as before:

$$X_1 = \Psi_1 I_1\,, \quad X_2 = \Psi_2 I_2\,, \quad X_3 = \Psi_3 I_3 \qquad \text{with} \quad \Psi_a = \Psi_a(T) \in \mathbb{R}$$

yields Wick-rotated Newtonian dynamics in $\mathbb{R}^3$, or $V(\Psi) \to -V(\Psi)$, and thus

$$\ddot{\Psi}_a = +\frac{\partial V}{\partial \Psi_a} \quad \Leftarrow \quad \dot{\Psi}_a = \frac{\partial U}{\partial \Psi_a} \quad \text{with} \quad V = \tfrac{1}{2}\frac{\partial U}{\partial \Psi_a}\frac{\partial U}{\partial \Psi_a}$$

for a superpotential

$$U(\Psi) = \Psi_1^2 + \Psi_2^2 + \Psi_3^2 - 2\,\Psi_1\Psi_2\Psi_3$$

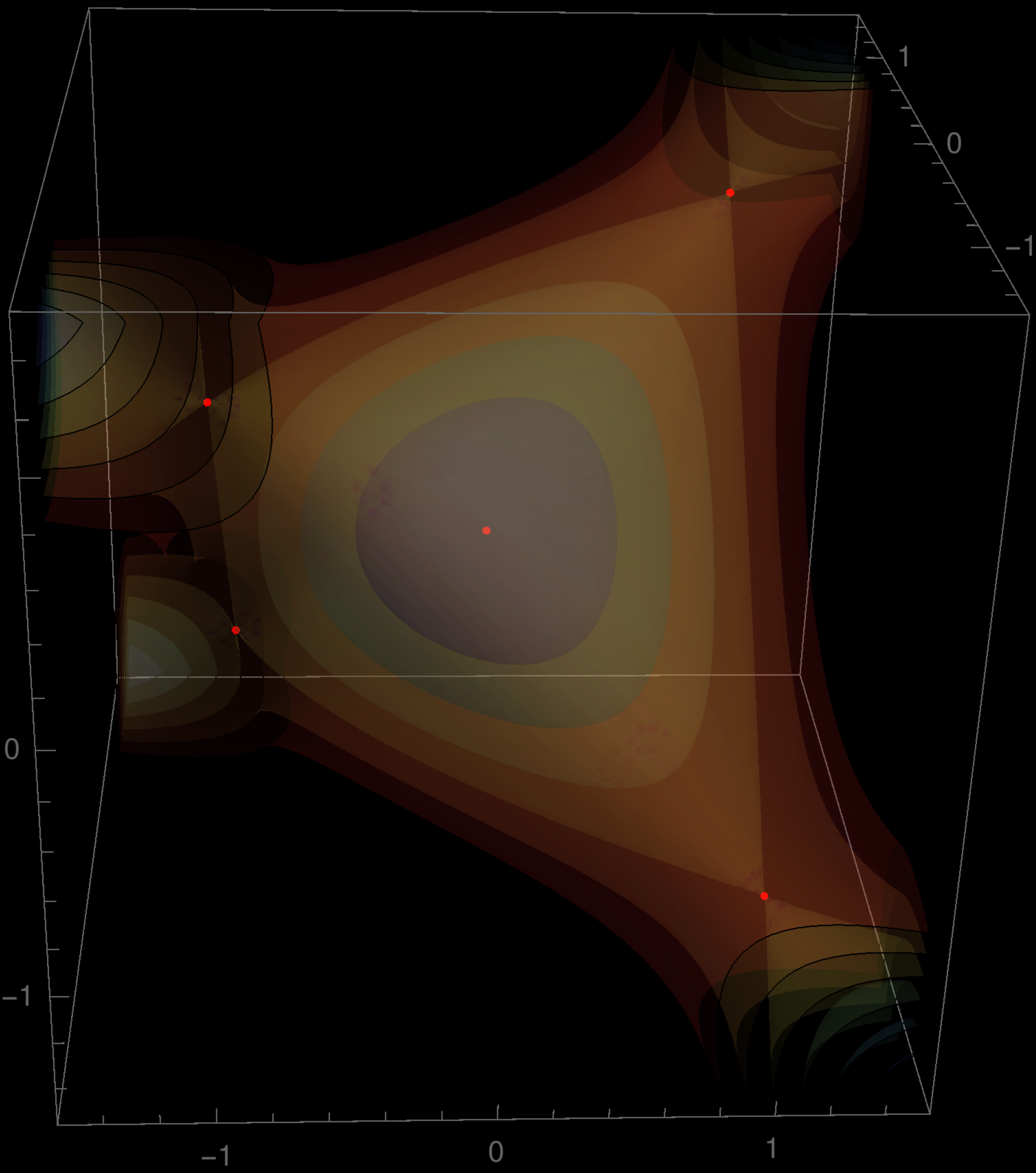
Take again

$$\Psi_1 = \Psi_2 = \Psi_3 = \tfrac{1}{2}(1+\psi) \quad \Rightarrow \quad U_\psi(\psi) = \psi - \tfrac{1}{3}\psi^3 \quad \& \quad \dot{\psi} = 1-\psi^2$$

Simplest solution is the kink:

$$\psi(T) = \tanh 2(T-T_0) \quad \Rightarrow$$

$$X_a(T) = \big[1+\exp(-2(T-T_0))\big]^{-1} I_a = \frac{r^2}{r^2+\Lambda^2}\, I_a \qquad \text{with} \quad \Lambda^2 = \mathrm{e}^{2T_0}R^2$$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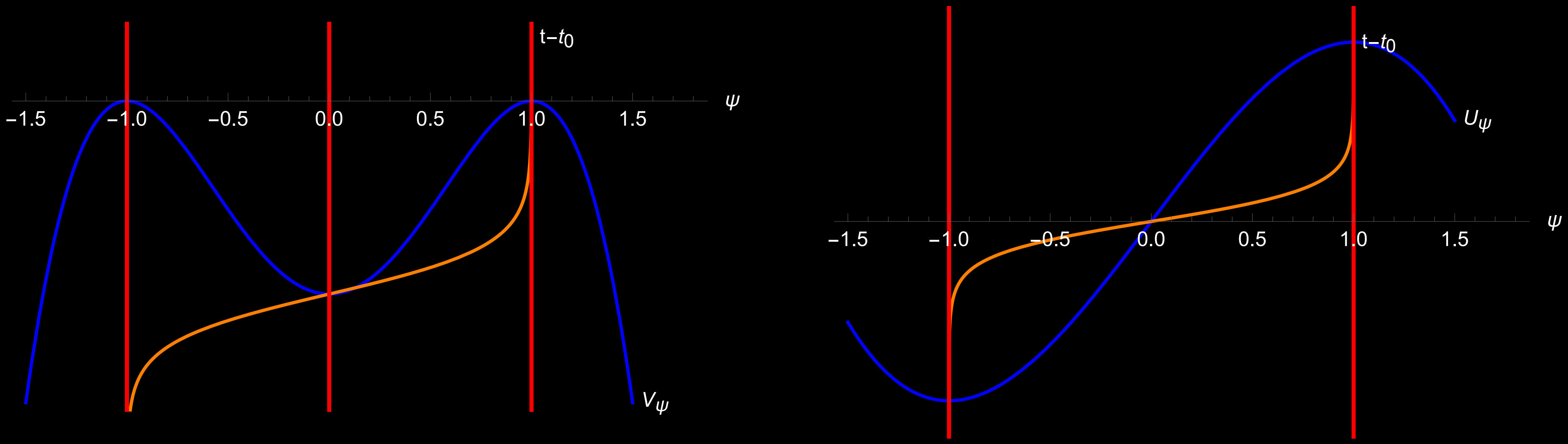

Gauge potential and field strength:

$$\mathcal{A} \;=\; X_a\, e^a \;=\; -\frac{1}{r^2+\Lambda^2}\,\eta^a_{ij}\, I_a\, x^i \mathrm{d}x^j$$

$$\mathcal{F} \;=\; -\frac{\Lambda^2}{(r+\Lambda^2)^2}\,\eta^a_{ij}\, I_a\, \mathrm{d}x^i \wedge \mathrm{d}x^j$$

This is the familiar BPST instanton extended from $\mathbb{R}^4$ to S^4 ✓

What about *anti*-de Sitter?

Four-dimensional anti-de Sitter space AdS$_4$ is embedded in $\mathbb{R}^{3,2}$ via

$$(y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2 - (y^5)^2 \;=\; -R^2$$

AdS$_4$ is conformally equivalent to $\mathcal{I} \times$ AdS$_3$ $\simeq$ $\mathcal{I} \times$ PSL$(2,\mathbb{R})$
and also conformally equivalent to $S^1 \times S^3_+$ (upper hemisphere)

Construction for gauge group SU(N) is similar to the one on dS$_4$ but . . .

- conformal factor now depends on a spatial coordinate χ
- AdS$_3$ slicing: energy and action $\sim$ vol(PSL$(2,\mathbb{R})$) are infinite ↯
- S^3_+ slicing: can import time-periodic dS$_4$ solutions and restrict $S^3 \to S^3_+$
- these solutions have again finite energy and action (again $\geq -\frac{3}{2}\pi^3 C(j)$) ✓
- all field components vanish exponentially towards the AdS boundary
- finite-action instantons exist but are unstable; they become stable on $\widetilde{\text{AdS}}_4$

Summary and outlook

- Finite-action Yang–Mills solutions (without Higgs) exist in 4D de Sitter space
- Rotationally symmetric solutions are spatially homogeneous and everywhere smooth
- Ansatz reduces Yang–Mills to matrix equations & then to special Newtonian dynamics
- Nonabelian fields from 3D particle trajectories in a tetrahedric quintuple-well potential
- fields $\sim (R \cosh\tau)^{-2}$, energy $\sim (R \cosh\tau)^{-1}$, action finite and $\geq -\frac{3}{2}\pi^3 C(j)$
- Instantons extended to S^4 are reconstructed as a byproduct
- Analog configurations on AdS_4 with H^3 slicing also enjoy finite energy and action
- These classical fields may be important for Yang–Mills vacuum structure on (A)dS_4

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THANK YOU !

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