CLUMPY: A public code for γ-ray and ν signals from dark matter structures.

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for the CLUMPY developers:

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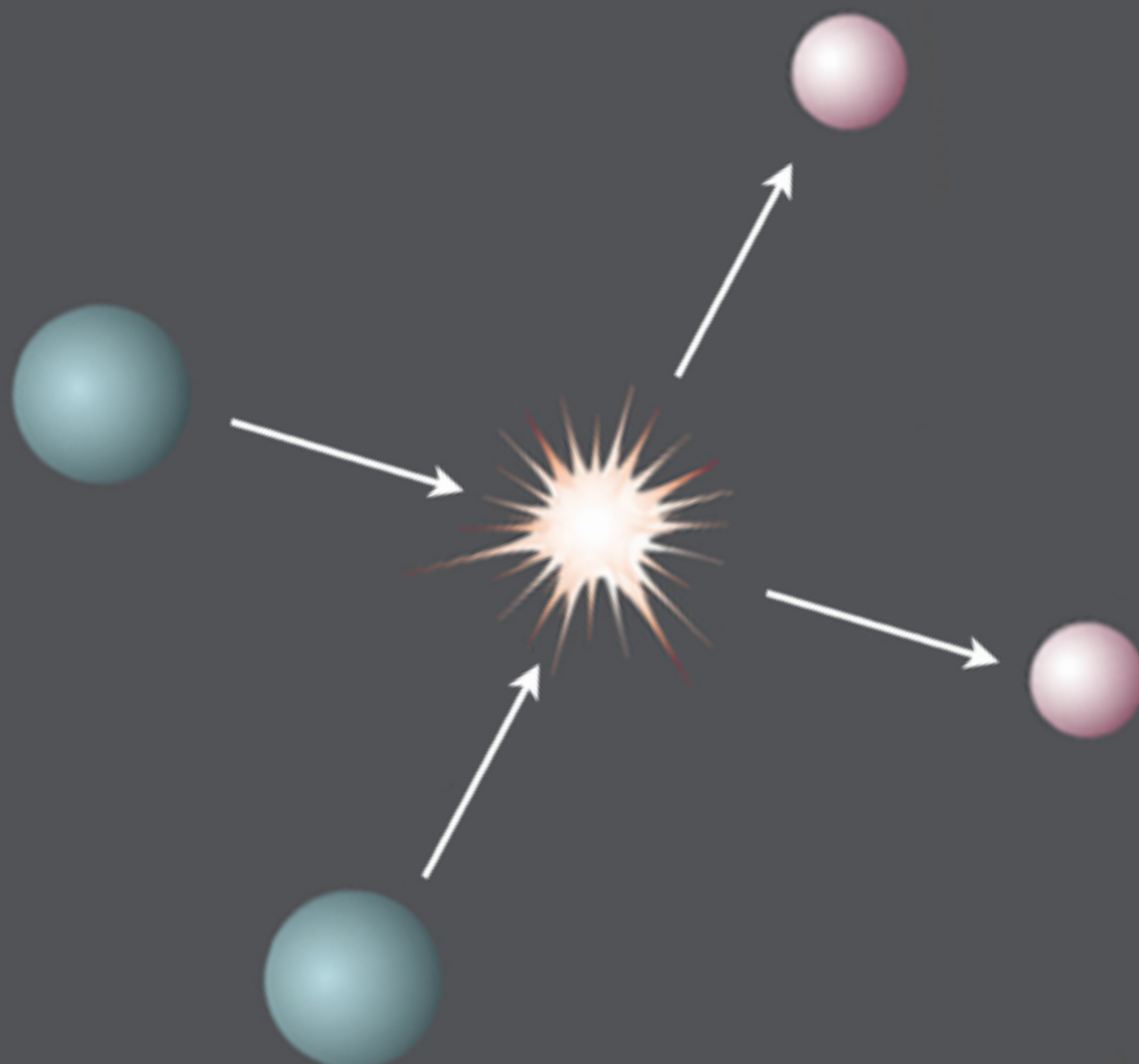
TOOLS 2017, Mon-Repos (Corfu)

https://lpsc.in2p3.fr/clumpy/

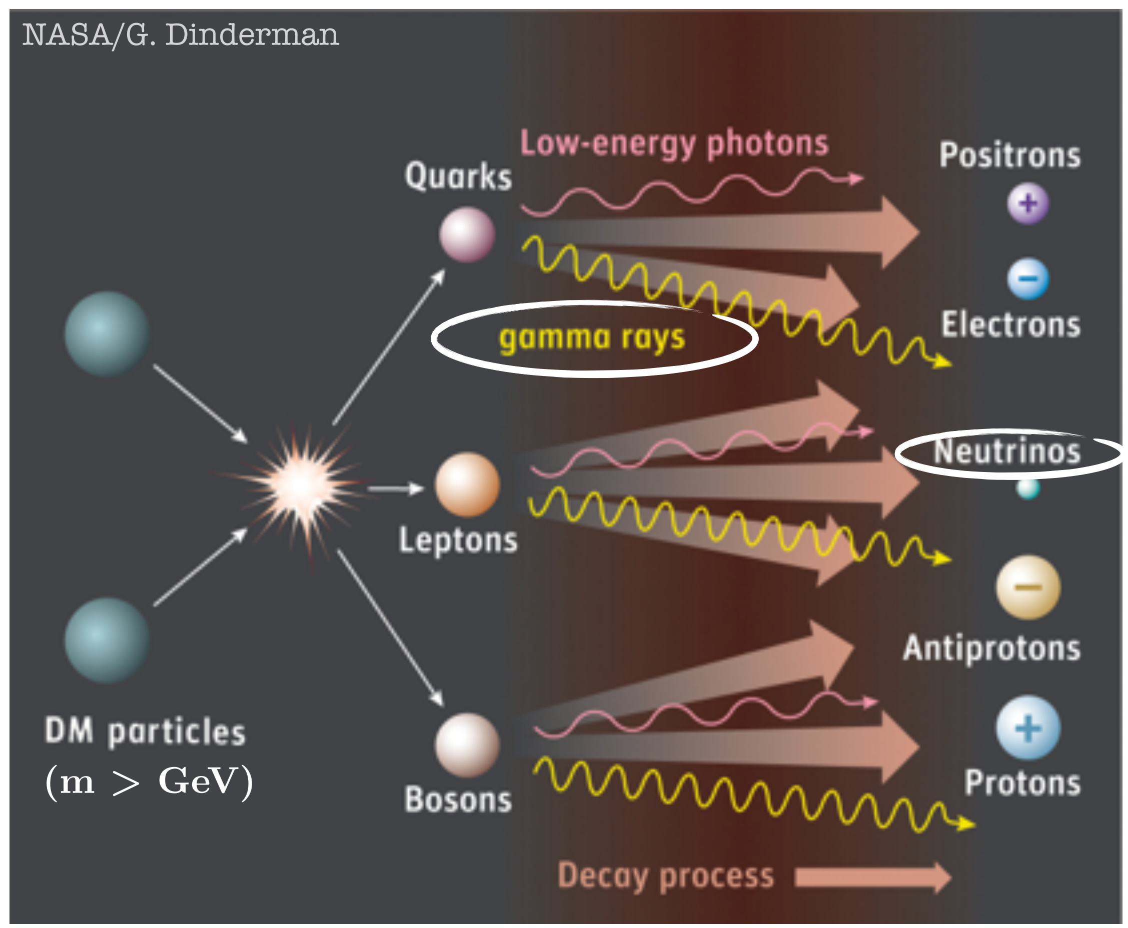
Bonnivard et al. (CPC, 2016), arXiv:**1506.07628**
Charbonnier et al. (CPC, 2012), arXiv:**1201.4728**

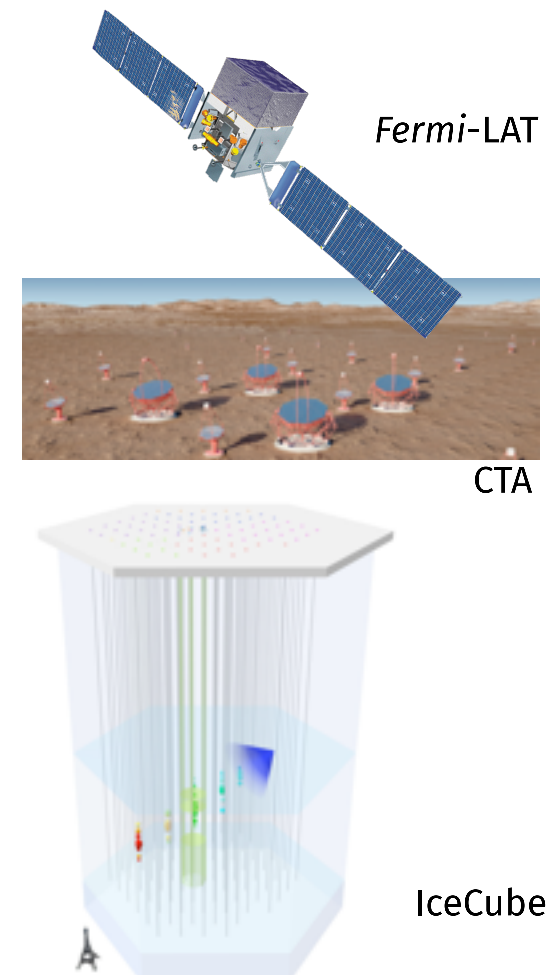
Introduction - the physics case



Indirect DM detection in γ-rays and ν

Dense & massive astrophysical DM budget → Annihilation or decay of the DM → Detectors for astrophysical γ-rays and neutrinos



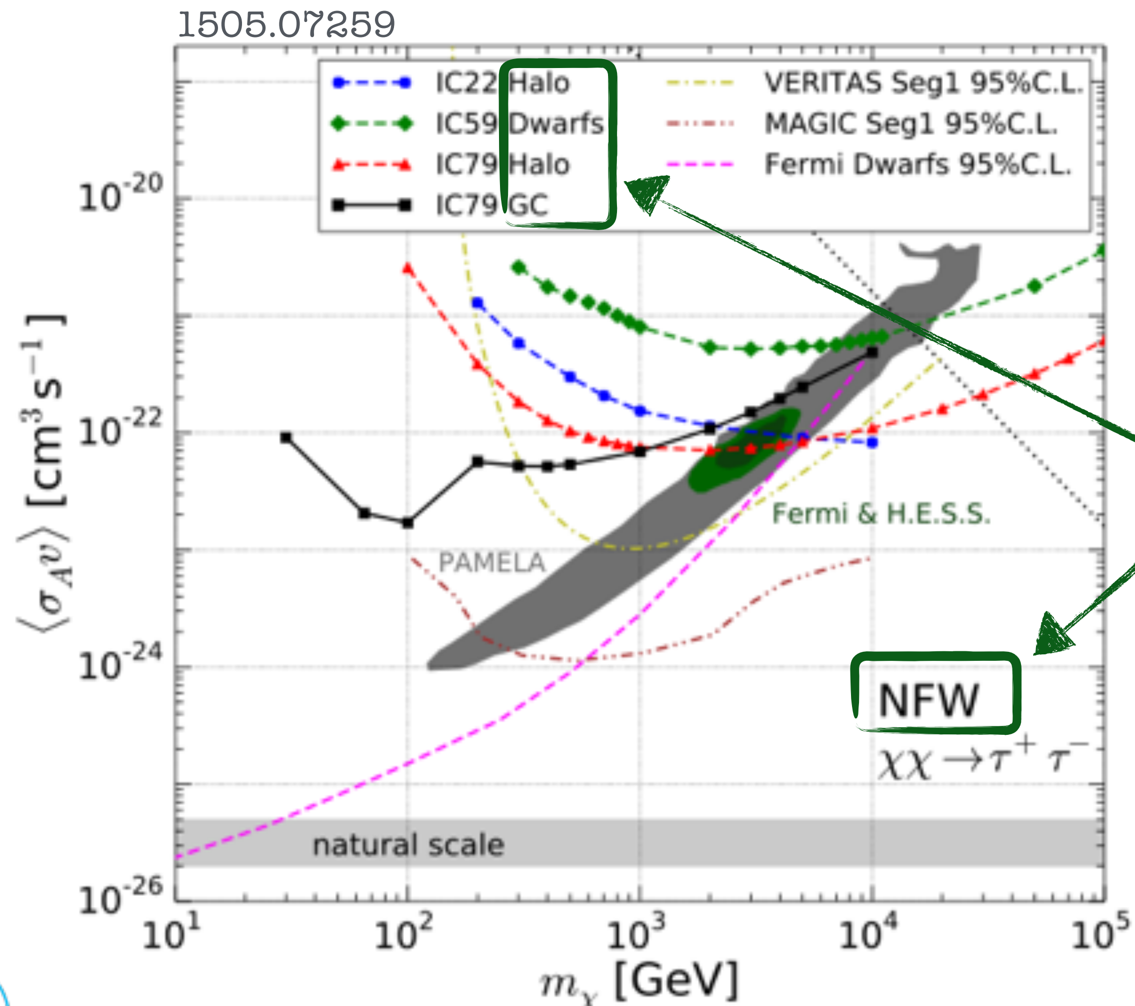


Indirect DM detection in γ-rays and neutrinos

Prompt γ-ray/ν flux in case of DM **annihilation** (local Universe):

$$\frac{d\Phi_\gamma^{\text{ann.}}}{dE_\gamma} = \frac{1}{4\pi}\frac{\langle\sigma v\rangle}{2m_\chi^2} \times \frac{dN_\gamma}{dE_\gamma} \times \int_{\Delta\Omega}\int_{l.o.s.} \rho_{\text{DM}}^2 \, dl \, d\Omega$$

Flux = Particle physics × J : Astrophysical factor $\overset{\propto}{\sim} \frac{1}{d^2}\frac{M^2}{V}$



Detection or non-detection:

J-factor and uncertainty must be well-known to put constraints on DM candidate

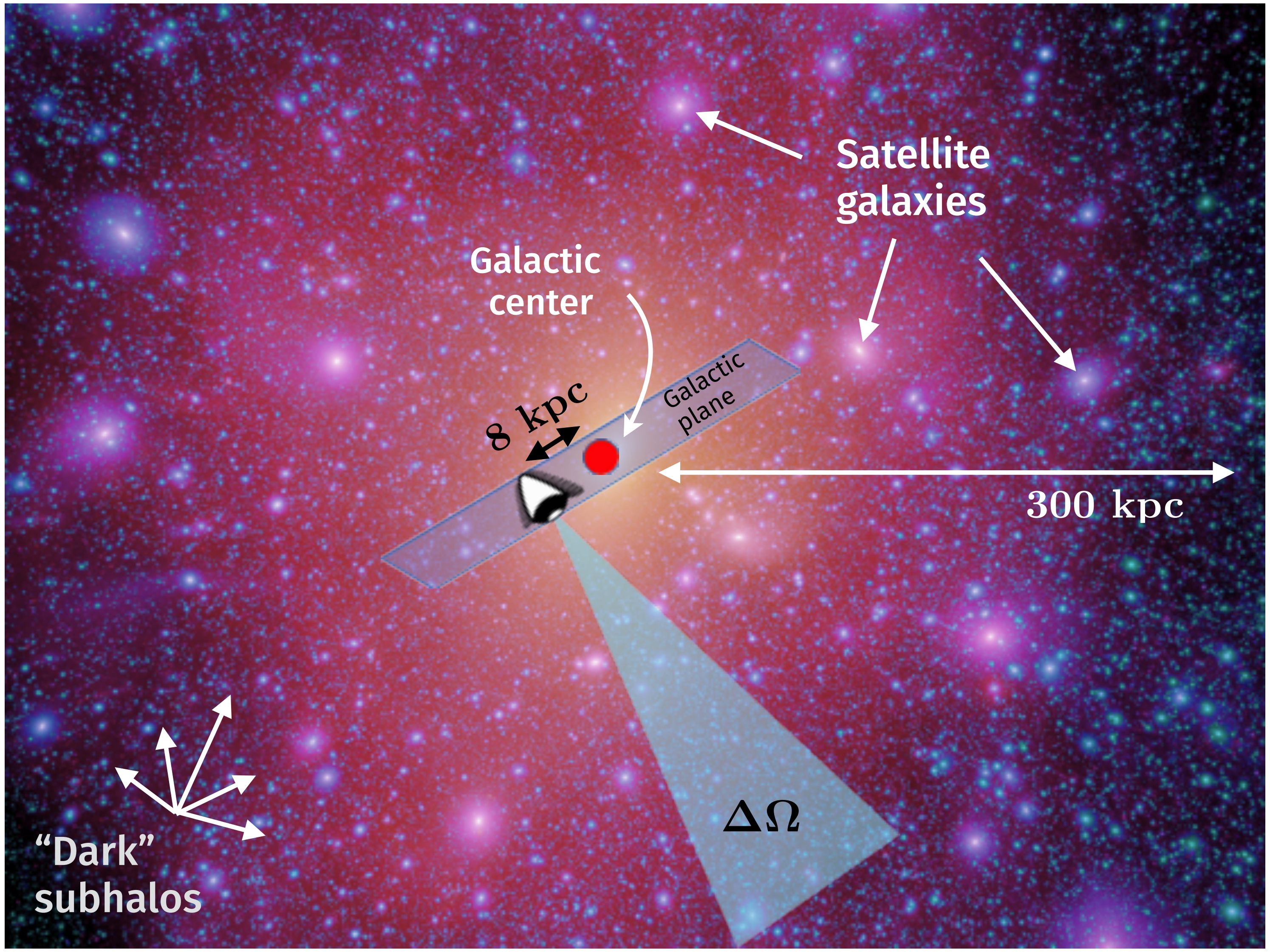
Annihilation: Signal depends crucially on DM target and distribution (smooth + substructures)

(CLUMPY can also do all calculations for DM **decay**, but skipped for this talk)

Indirect DM detection in γ-rays and neutrinos

Where to look?

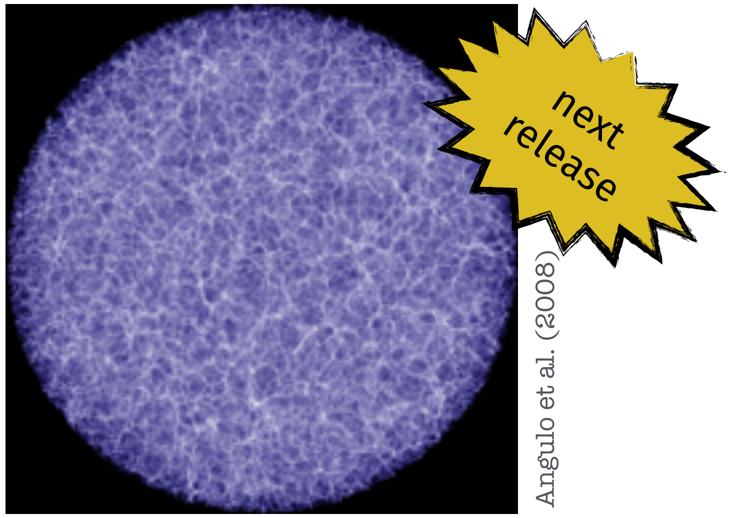
Massive & dense (M^2/V) vs. close ($1/d^2$) vs. little astrophysical background



Aquarius simulation – Springel et al. (Nature, 2008)

- single galaxy clusters ($d >$ Mpc)

- ensemble average of extra-galactic DM ($d >$ Gpc)

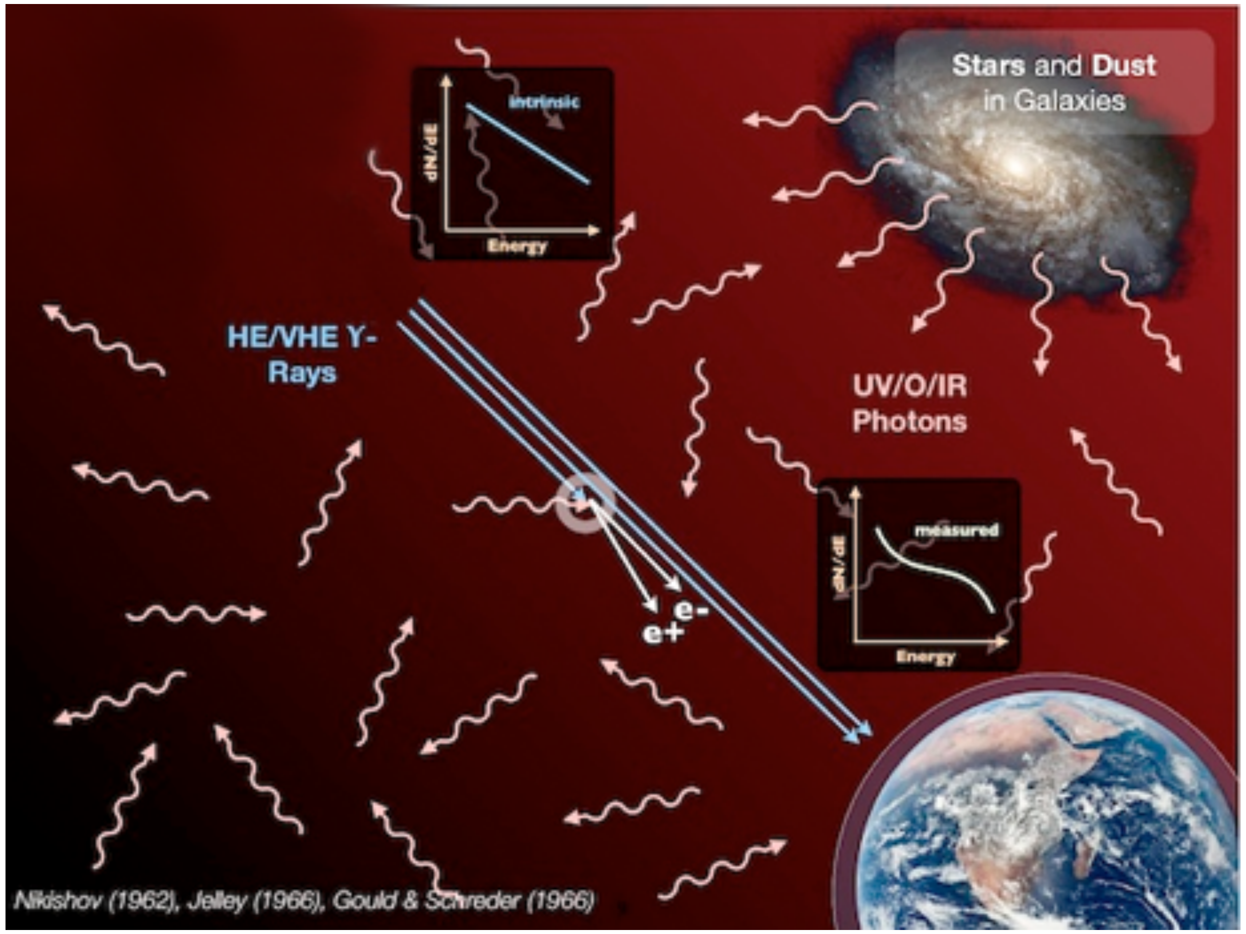


CLUMPY calculates J-factors/fluxes for all the various targets

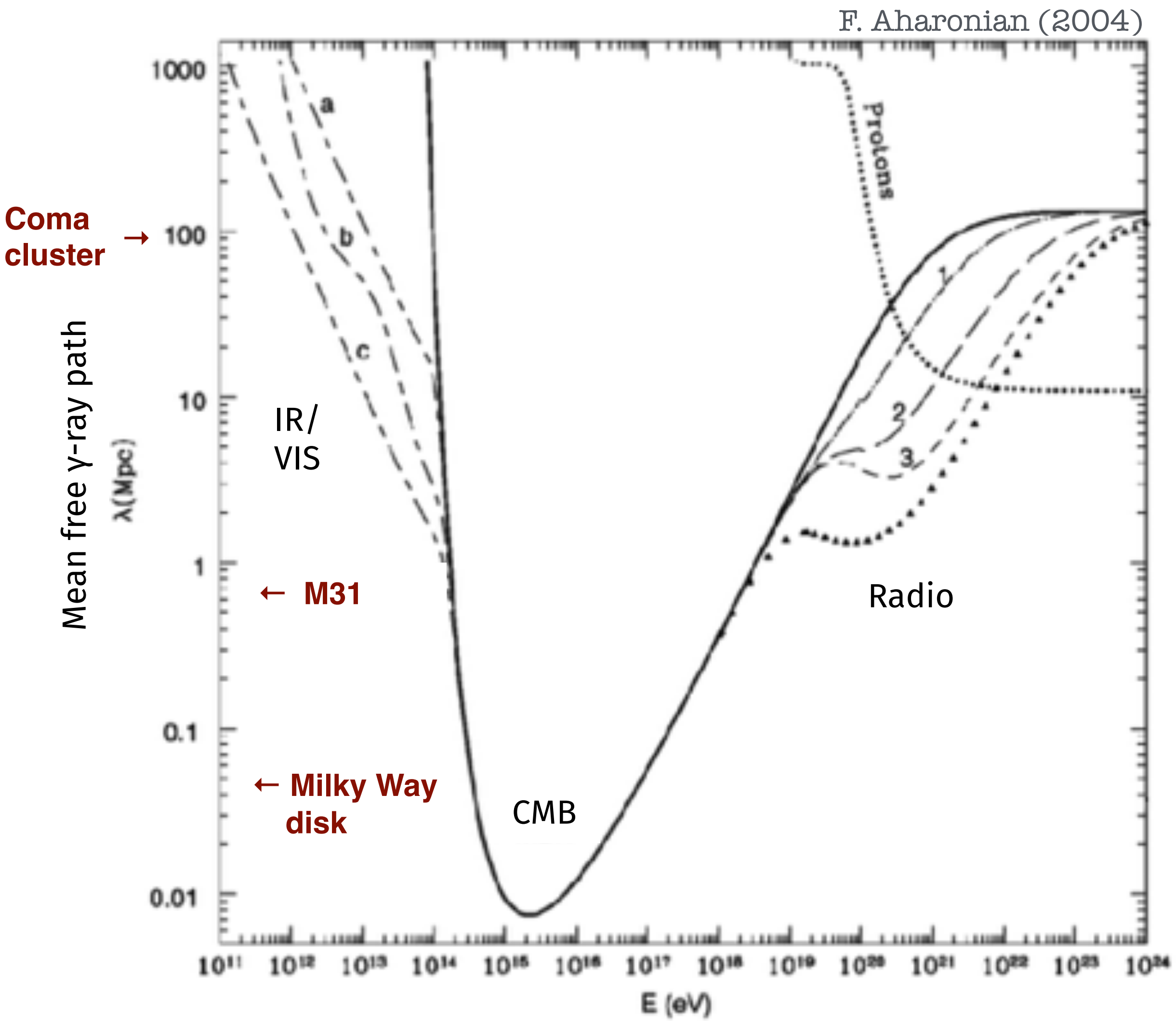
Indirect DM detection in extragalactic γ-rays

Consider for γ-rays from outside the local Universe

- Redshifting of the γ-rays/ neutrino energy loss
- γ-rays absorption by pair-production with photons of the extragalactic background light (EBL)



LEXI, University of Hamburg



Indirect DM detection in extragalactic γ-rays

- More intricate form of flux equation (single extragalactic object):

EBL absorption

$$\frac{d\Phi_\gamma^{\mathrm{ann.}}}{dE_\gamma^{\mathrm{obs}}} = \frac{1}{4\pi}\frac{\langle\sigma v\rangle}{2m_\chi^2} \times \frac{dN_\gamma}{dE_\gamma^{\mathrm{source}}}\left([1+z]\,E_\gamma^{\mathrm{obs}}\right) \times e^{-\tau(z,E_\gamma)}$$

redshift

$$\times\,(1+z)^3 \int_{\Delta\Omega}\int_{l_c} \rho_{\mathrm{DM}}^2\,dl_c\,d\Omega$$

annihilation boost in smaller proper volume

description in comoving coordinates

- Separation in particle physics/astrophysics term **breaks down** if considering a signal originating from multiple redshift shells

(see Marco Cirelli's talk for the full complex equation)

J_{tot} integration, substructures, and boost factor

- Simple or „smooth“ DM density profile:

  No analytic J-factor expression for usual NFW, Einasto,.. DM density profiles

$$\frac{\mathrm{d}J_{\mathrm{sm}}}{\mathrm{d}\Omega}(\theta) = \int_{l_{\mathrm{min}}}^{l_{\mathrm{max}}} \rho_{\mathrm{sm}}^2 \,\mathrm{d}l$$

  - numeric line-of-sight integration needed

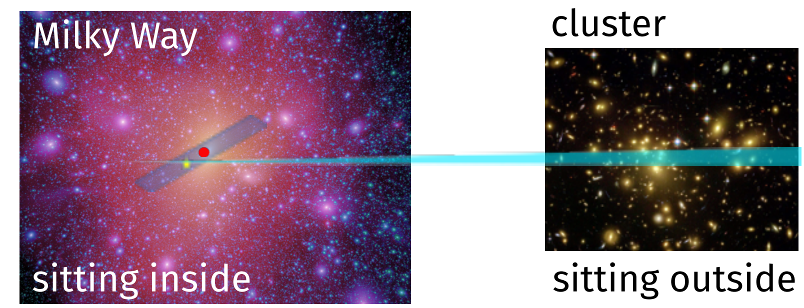
$$\frac{\mathrm{d}J}{\mathrm{d}\Omega} = \int_{l_{\mathrm{min}}}^{l_{\mathrm{max}}} \rho^2 \,\mathrm{d}l$$

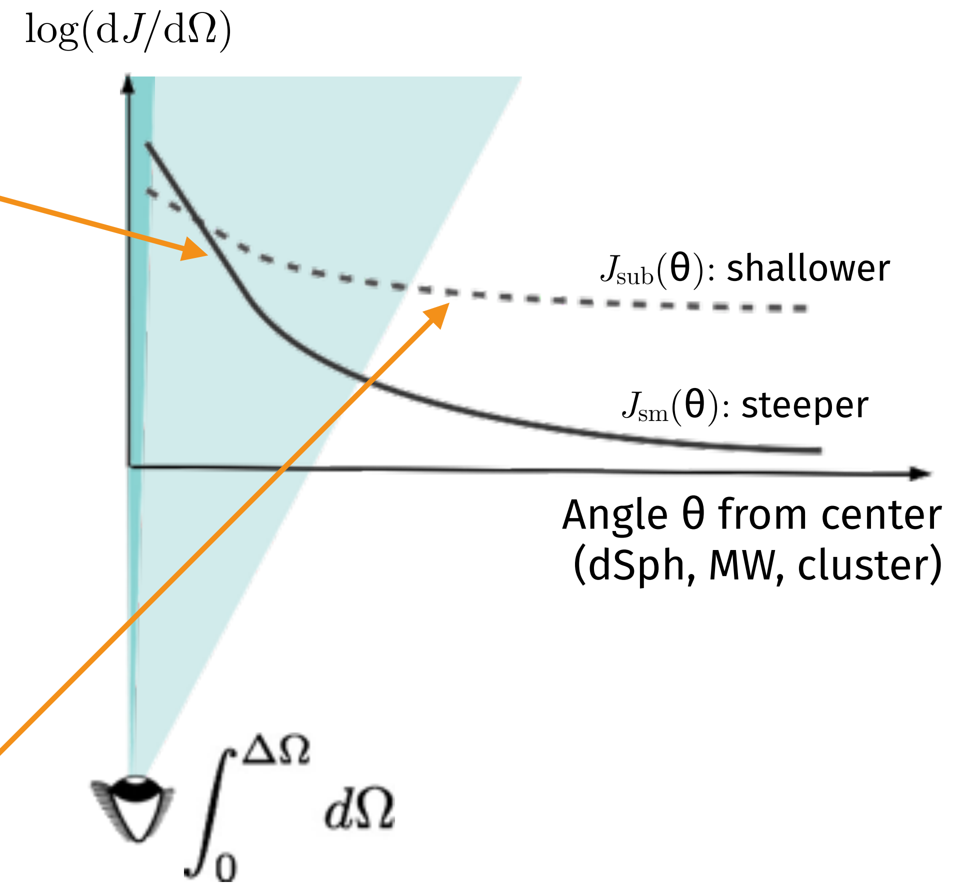
- DM substructure:

  For DM annihilation, we have a **boost** from unresolved substructures in the halo, $J = J_{\mathrm{sm}} + \langle J_{\mathrm{sub}} \rangle = \mathrm{boost} \times J_{\mathrm{sm}}$

$$\left\langle \frac{\mathrm{d}J_{\mathrm{sub}}}{\mathrm{d}\Omega}(\theta) \right\rangle = N_{\mathrm{tot}} \int_{l_{\mathrm{min}}}^{l_{\mathrm{max}}} \frac{\mathrm{d}P_V}{\mathrm{d}V}(l)\,\mathrm{d}l \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} \mathcal{L}(M) \frac{\mathrm{d}P_M}{\mathrm{d}M}\,\mathrm{d}M$$

  - depending on $\mathrm{d}P/\mathrm{d}V$, $J_{\mathrm{sub}}(\theta)$ not proportional to $J_{\mathrm{sm}}(\theta)$!





dSph: $\mathrm{boost}(\theta) \sim 1 - 2$

Galaxy cluster: $\mathrm{boost}(\theta) \sim 10 - 100$

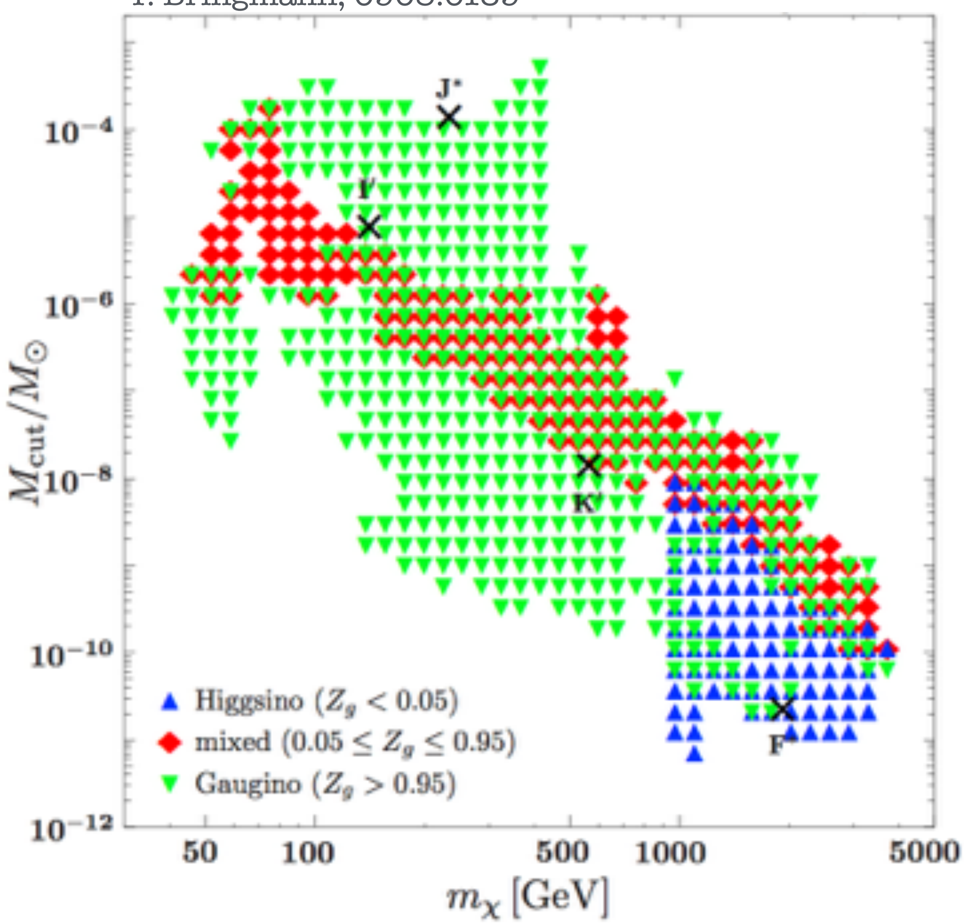
$J_{\rm tot}$ integration, substructures, and boost factor

Boost factor is main uncertainty - depends on particle physics and small scale physics

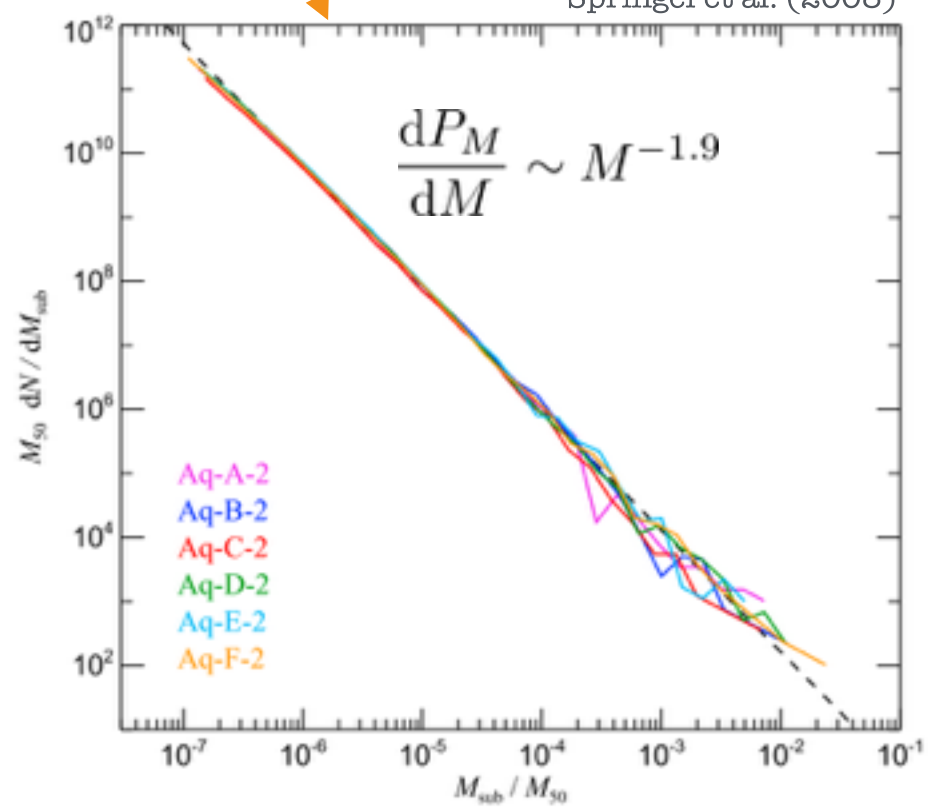
dSph: $\mathrm{boost}(\theta) \sim 1 - 2$
Galaxy cluster: $\mathrm{boost}(\theta) \sim 10 - 100$?

$$\left\langle \frac{\mathrm{d}J_{\rm sub}}{\mathrm{d}\Omega}(\theta) \right\rangle = N_{\rm tot} \int_{l_{\rm min}}^{l_{\rm max}} \frac{\mathrm{d}P_V}{\mathrm{d}V}(l)\,\mathrm{d}l \int_{M_{\rm min}}^{M_{\rm max}} \mathcal{L}(M) \frac{\mathrm{d}P_M}{\mathrm{d}M}\,\mathrm{d}M$$

T. Bringmann, 0903.0189



Springel et al. (2008)



Mass converges for $M_{\rm min} \to 0$, but not annihilation boost!

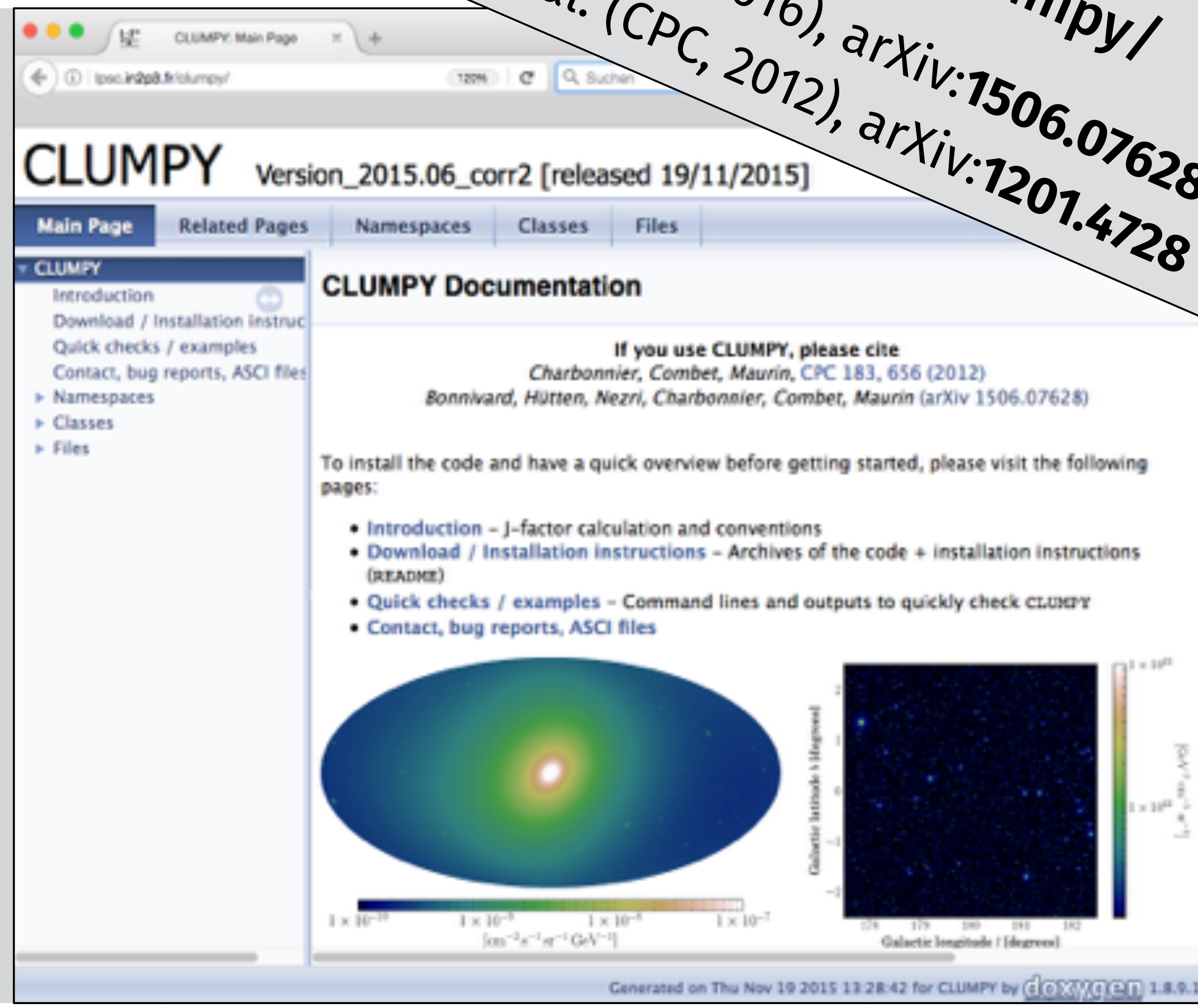
```
j_1D.assign(n_1D, 1.e-40);

// now calculate 1D-halo profile between phi_min and phi_max:
if (is_interpolate) {
   // do it like in ClumpyV1:

   if (phi_min > fivedeg_rad) is_simple_interp = true;

   if (is_simple_interp) printf("        ... Fill interpolation function (%d lin-steps) ...\n", n_base);
   else printf("        ... Fill interpolation function (%d log-steps) ...\n", n_base);

   j_1D_base.assign(n_base, 1.e-40);
   phi_base.assign(n_base, 1.e-40);
   iphi_inbase.assign(n_1D, -1);
   double delta_phi_base;
   if (is_simple_interp) delta_phi_base = (phi_max - phi_min) / double(n_base - 1);
   else delta_phi_base = pow(phi_max / alpha_quad_start, 1. / double(n_base - 1));


   for (int i = 0; i < n_base; ++i) {
      if (is_simple_interp) phi_base[i]  = phi_min + i * delta_phi_base;
      else phi_base[i] = alpha_quad_start * pow(delta_phi_base, i);
      double jopt = 1.e-40;
      if (switch_j == 0) {
         if (f_dm > 1.e-3)
            jopt = jsmooth_mix(mtot, par_tot, phi_base[i], theta_1D, lmin, lmax, eps, f_dm, par_dpdv);
         else
            jopt = jsmooth(par_tot, phi_base[i], theta_1D, lmin, lmax, eps);
      } else if (switch_j == 1) {
         // N.B.: we have to take into account all mass decades
         for (int k = 0; k < n_mass; ++k) {
            if (l_crit[k] < lmax)
               jopt += jsub_continuum(ntot_subs, par_dpdv, phi_base[i], theta_1D,
                                      l_crit[k], lmax,  par_subs, m1[k], m2[k]);
         }
      } else if (switch_j == 2) {
         // N.B.: we have to take into account all mass decades
         for (int k = 0; k < n_mass; ++k) {
            if (l_crit[k] < lmax)
               jopt += frac_nsubs_in_m1m2(&par_subs[8], m1[k], m2[k], gSIM_EPS)
                       * jcrossprod_continuum(mtot, par_tot, phi_base[i], theta_1D,
                                              l_crit[k], lmax, eps, f_dm, par_dpdv);
         }
      }
      if (jopt == 0.) jopt = 1.e-40;
      j_1D_base[i] = jopt;
   }
   // Set indices for phi_base[iphi_inbase] for phi_tab[i]
   // Search (only once) for interpolation indices for angles
   for (int i = 0; i < n_1D; ++i)
      iphi_inbase[i] = TMath::BinarySearch(n_base, &phi_base[0], phi_tab[i]);

if (is_interpolate) printf("        ... and interpolate for %d l.o.s. directions ...\n", n_1D);
```

The Code

What is CLUMPY?

https://lpsc.in2p3.fr/clumpy/
Bonnivard et al. (CPC, 2016), arXiv:**1506.07628**
Charbonnier et al. (CPC, 2012), arXiv:**1201.4728**

- **Open-source**, written in C/C++
- Depends on:
  - CERN's ROOT
  - Heasarc's cfitsio
  - ~~HEALPix~~ (next release shipped with frozen HEALPix version)
  - gsl
  - GreAT (lpsc.in2p3.fr/great, optional)
- Runs on **Linux** and **MacOS X**

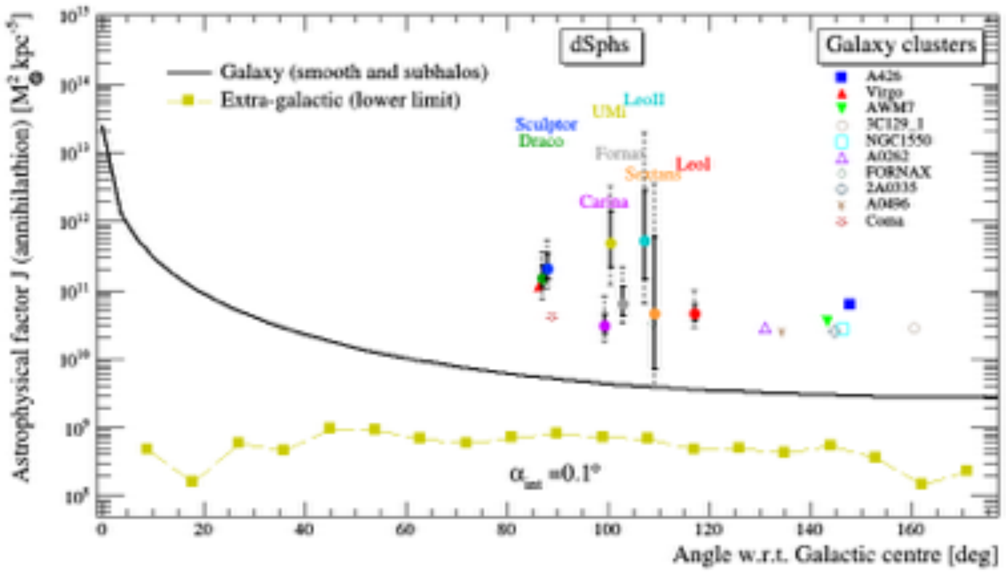


Open source code to provide the community with reproducible and comparable models for J-factors and prompt γ-ray/ν fluxes

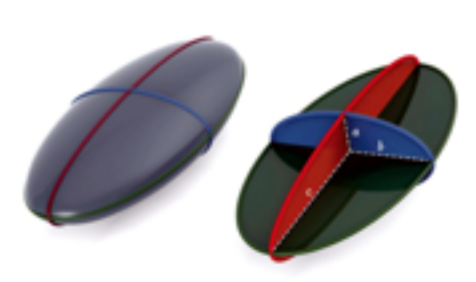
CLUMPY features (I): $\rho_{\rm sm} + \rho_{\rm subs} \rightarrow J$-factor/flux

J-factors/fluxes of individual objects (e.g. dSph's) from **pre-defined DM profiles**

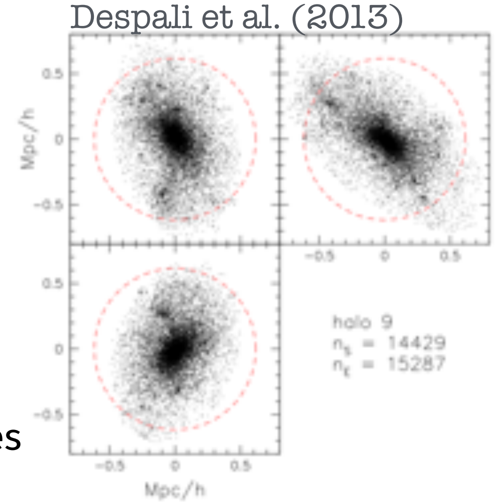
- Propagate **error bars from DM profiles to J-factors and limits on DM**
  (Bonnivard ApJ, MNRAS, 2015)
- Take into account **substructures:**
  - resolved (statistical) + unresolved: **boost**
  - vary distribution within host halo (antibiased, own profile,...)
  - Clumps within clumps: multiple levels of self-similar sub-subclustering (converges for ~2 levels)
- allow **triaxial distortion** of halo profile (semiaxis ratio a, b, c)



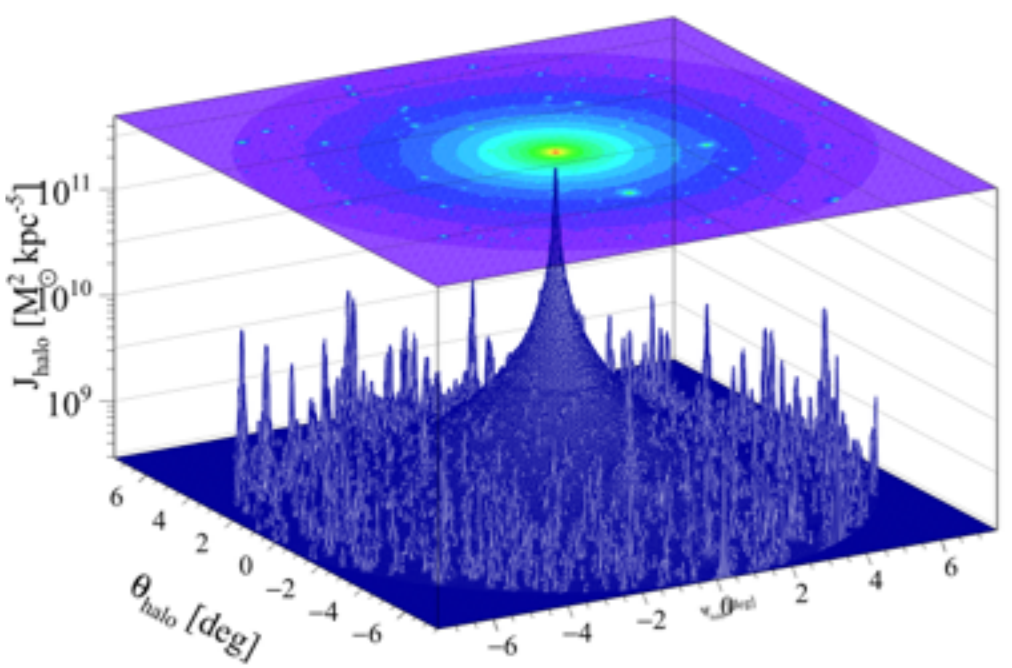
Comparison of classical dSph, brightest galaxy clusters, and galactic DM foreground
(Charbonnier et al., MNRAS, 2011; Nezri et al., MNRAS, 2012)



Despali et al. (2013)



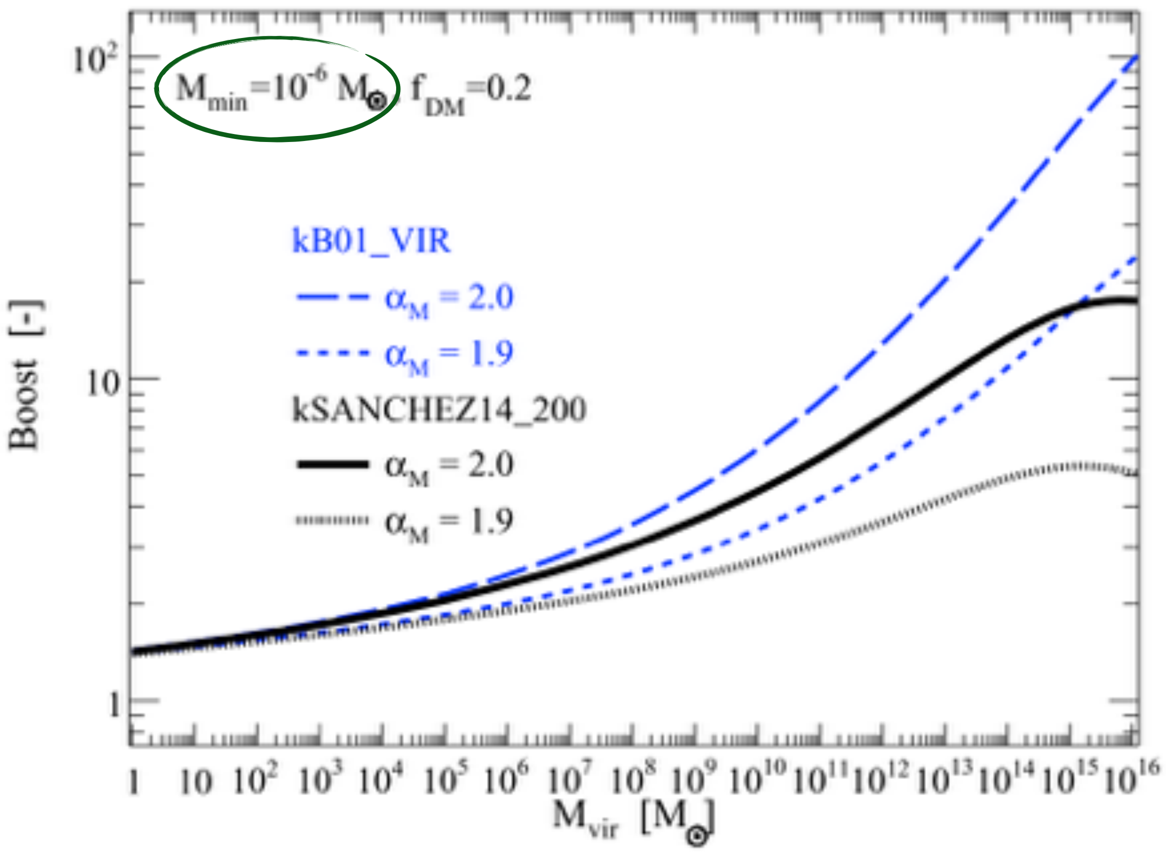
N-body simulations/kinematic analyses find triaxial halo shapes



LMC $dJ/d\Omega$ profile with resolved substructure model (analysis done by M. Castaño, São Paulo)

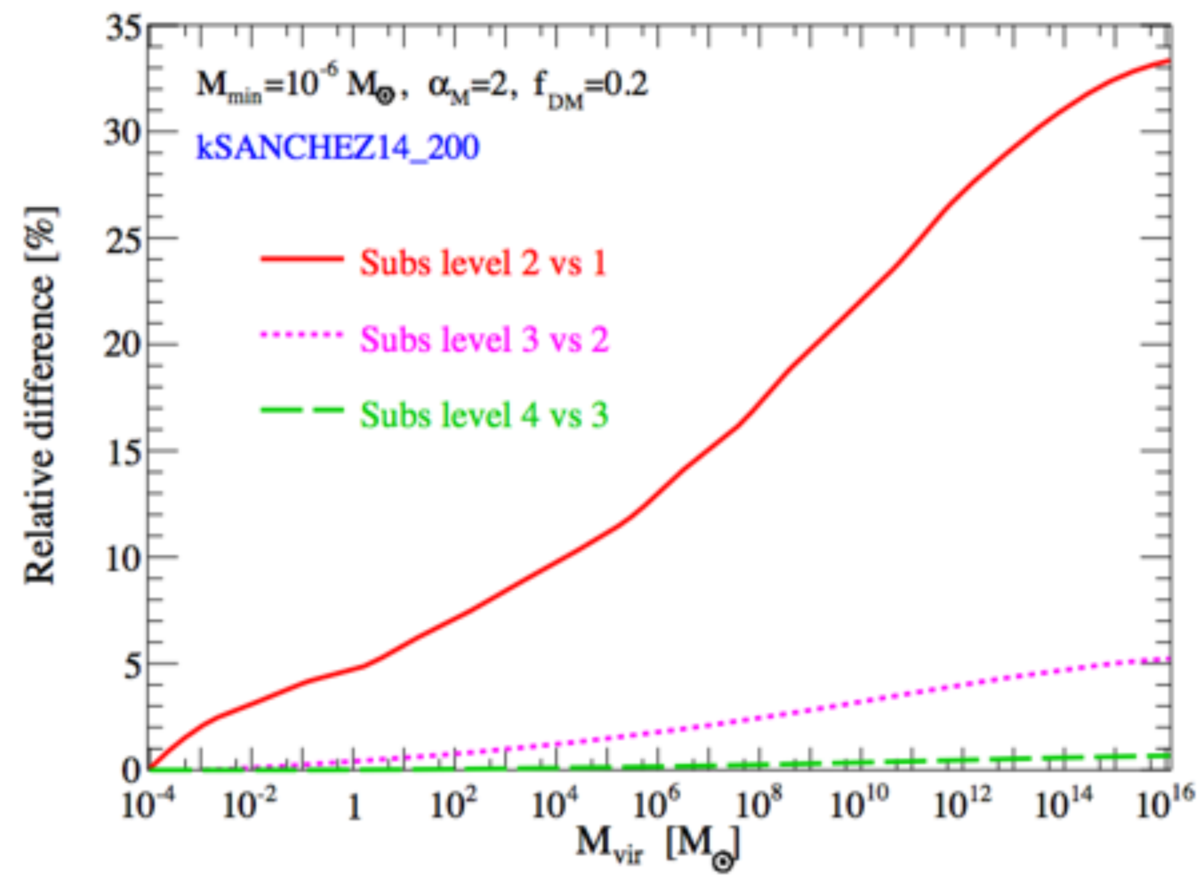
Some results: Boost uncertainty (Bonnivard et al., 1506.07628)

Boost from different $\left\langle \frac{\mathrm{d}J_{\mathrm{sub}}}{\mathrm{d}\Omega}(\theta) \right\rangle$ models



dSph: $\mathrm{boost}(\theta) \sim 1 - 2$
Galaxy cluster: $\mathrm{boost}(\theta) \sim 10 - 100$?

Also consider self-similar sub-subclustering

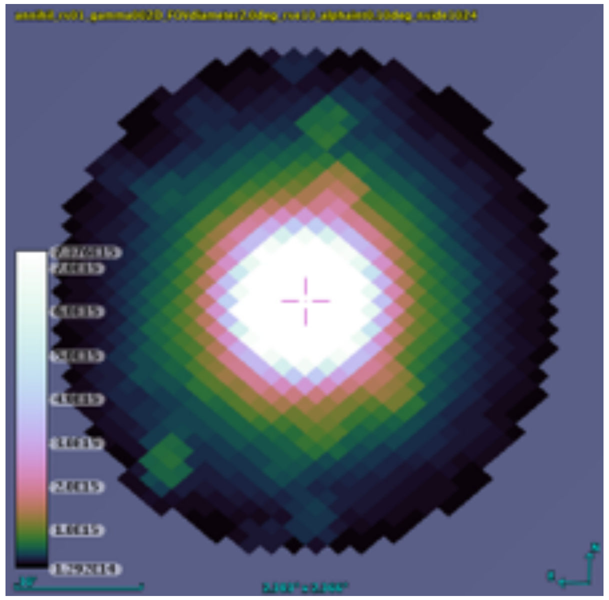


CLUMPY features (I): $\rho_{\rm sm} + \rho_{\rm subs} \rightarrow \mathcal{J}$-factor/flux

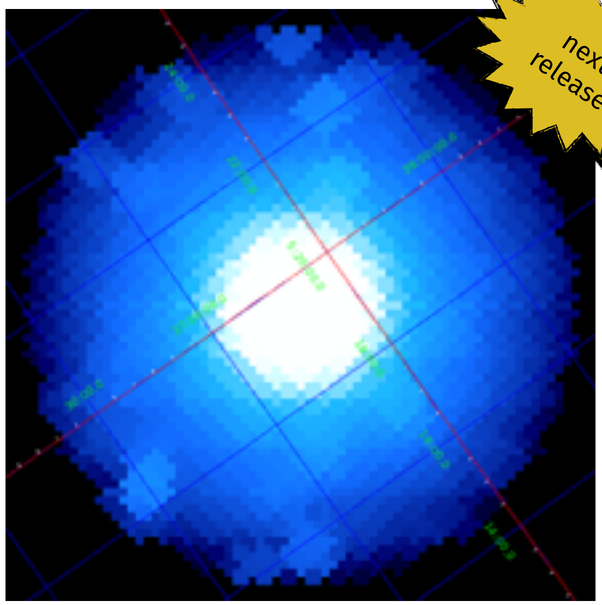
$\mathcal{J}$-factors/fluxes of individual objects (e.g. dSph's) from **pre-defined DM profiles**

- Directly **compute differential/integral fluxes** (1D and 2D), relying on PPPC4DMID (Cirelli et al., 2010)
- ROOT **pop-up graphics** (1D and 2D)
- **Choose output format:** ROOT, HEALPix FITS (2D), ASCII
- FITS images **interfaceable with gammalib/ctools**
- **Correct cosmology** (line-of-sight and angular diameter distance) + **EBL flux absorption** for extragalactic objects

next release

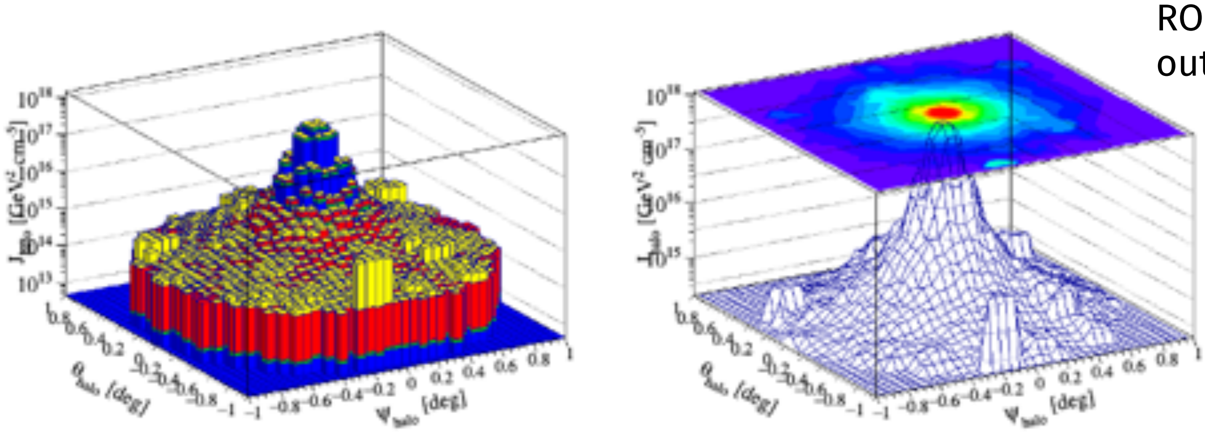


HEALPix pixelization (FITS format)



next release

projected FITS image

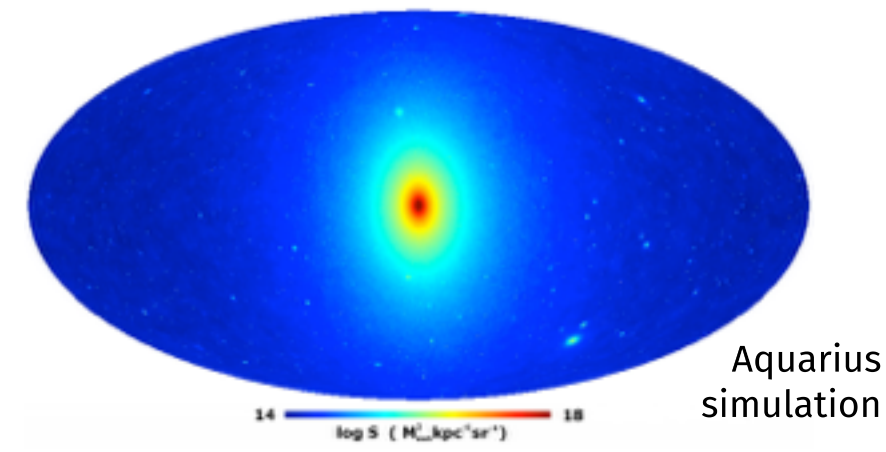


ROOT plots and output format

CLUMPY features (II): Full-sky MW analysis with subhalos

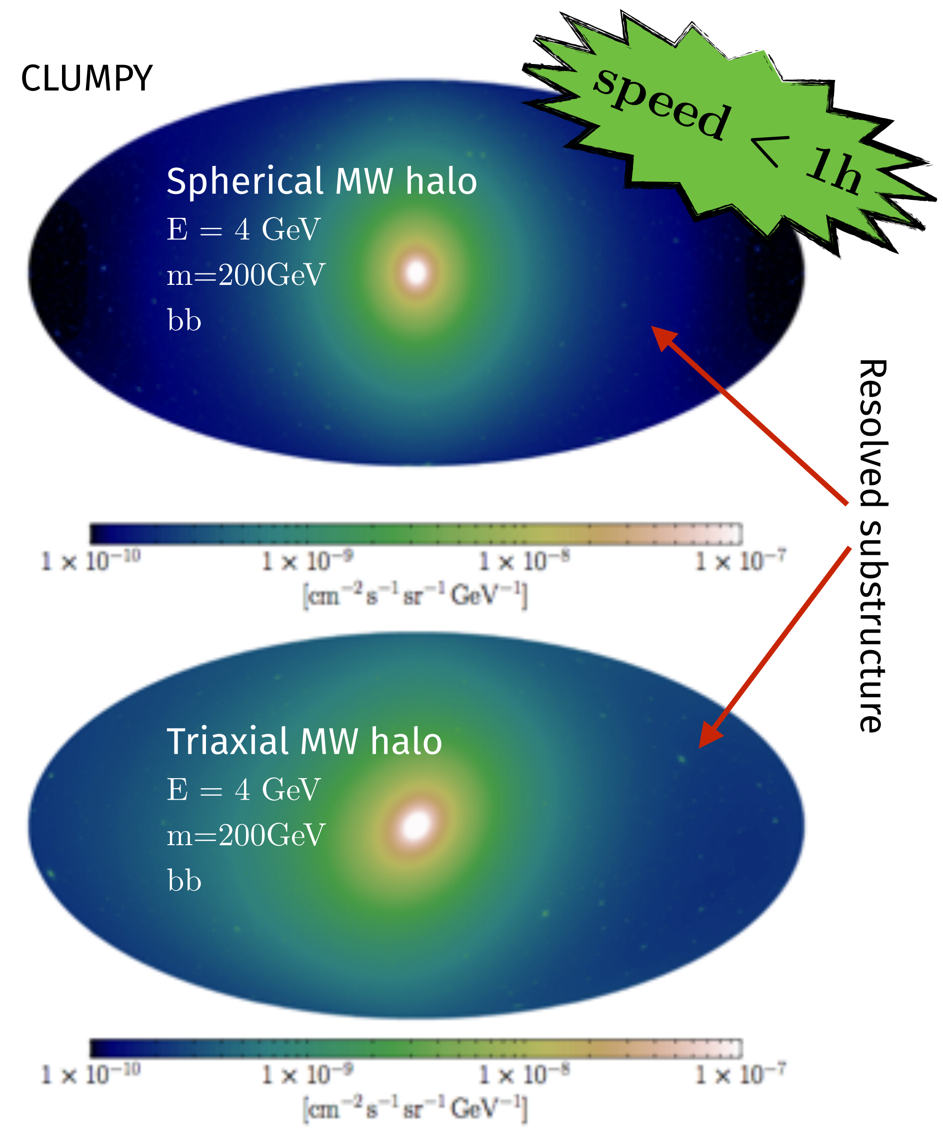
Skymaps of full or partial J-factor sky from DM in the Milky Way halo

- Fast realistic synthetic skymaps at any instrumental resolution
  - check that we recover N-body simulation end-products from a handful of parameters
  - extend N-body simulation results by varying key parameters to study impact on halo/substructure brightness
- Resolved substructure
  - Smartly pre-select brightest subhalos for speed (e.g., reduce 10^{15} total subhalos in the MW to $\sim 10^4$ at a precision of 2% and $\theta_{\mathrm{int}} = 0.2°$).
  - allows to do statistical assessment of MW substructure properties (average mass, distance, luminosity,...)
  - Application in the context of CTA and sensitivity to dark halo searches in Hütten et al., JCAP (2016)



Aquarius simulation

CLUMPY



DESY

CLUMPY features (III): Jeans analysis module

From stellar kinematics to DM profile

Data

- Light profile & velocity dispersion

$I(R)$ — de-projection → ν stellar density

$\sigma_p^2(R)$ ← projection — $\bar{v}_r^2$ radial velocity dispersion

stellar density & radial velocity dispersion

Gravitation

- Spherical Jeans equation: solve for $\bar{v}_r^2$

$$\frac{1}{\nu}\frac{\mathrm{d}\left(\nu\bar{v}_r^2\right)}{\mathrm{d}r} + \frac{2\,\beta_{\rm ani}\,\bar{v}_r^2}{r} = -\frac{G\,M(r)}{r^2}$$

$\beta_{\rm ani} = 1 - \bar{v}_\theta^2/\bar{v}_r^2$: anisotropy

$M(r) = 4\pi\int_0^r \rho(r')\,r'^2\,\mathrm{d}r'$ enclosed mass

DM model

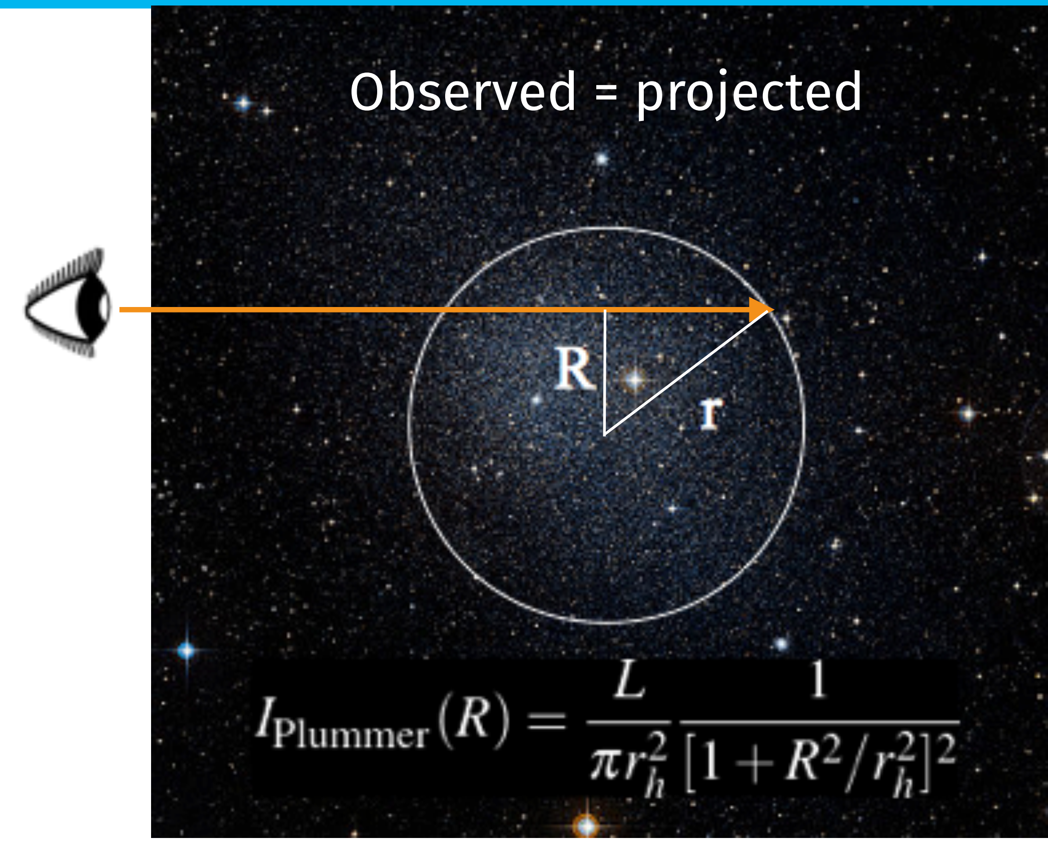
- Dark matter profile

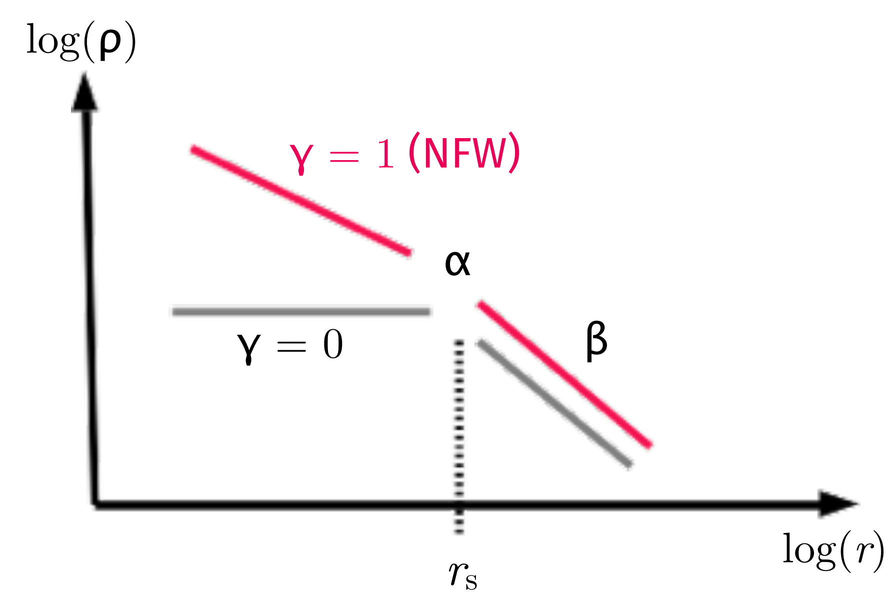
$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)^{\gamma}\left[1+\left(\frac{r}{r_s}\right)^{\alpha}\right]^{(\beta-\gamma)/\alpha}}$$

Fit

- χ^2 or MCMC analysis to extract DM parameters

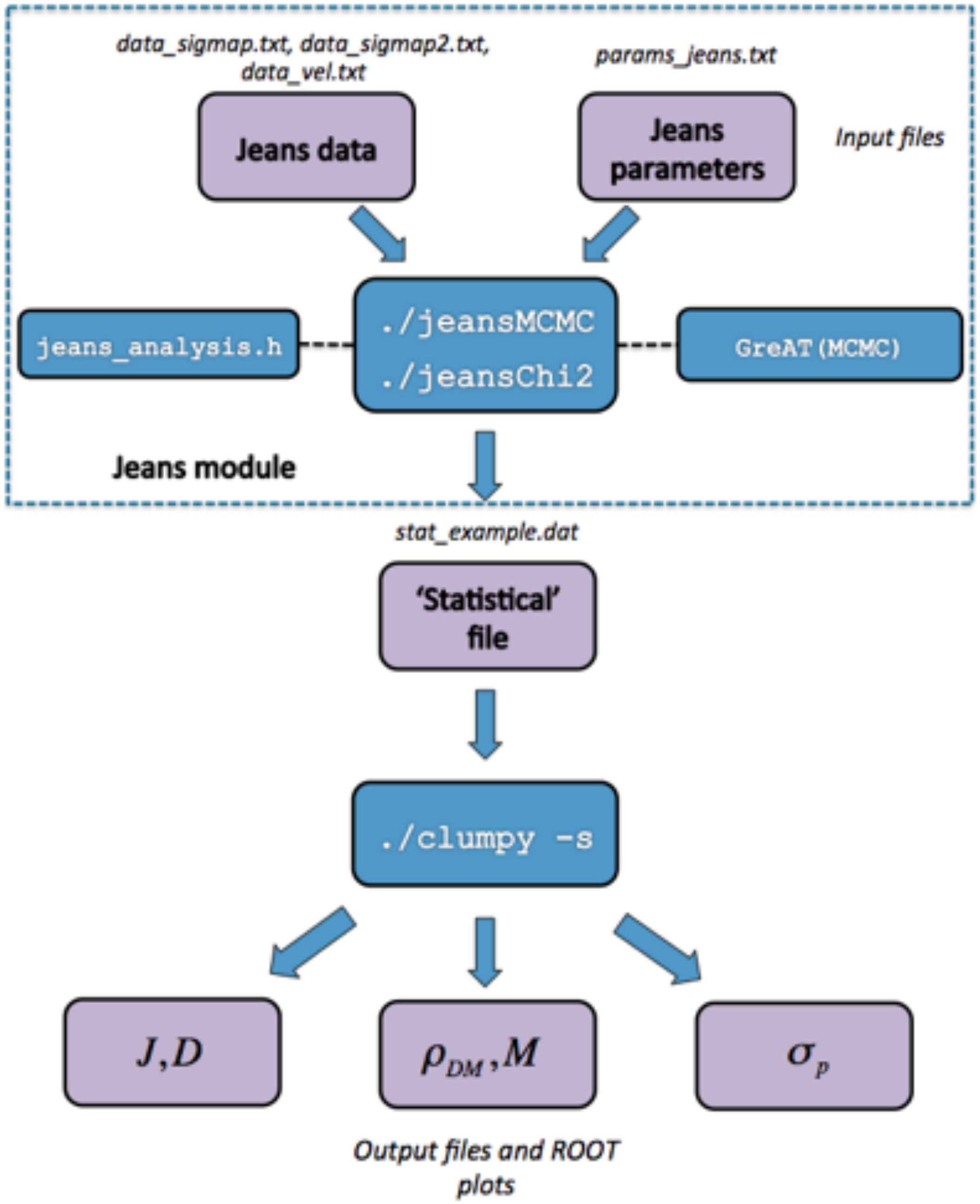
ρ_s, r_s, α, β, γ, and $\beta_{\rm ani}$





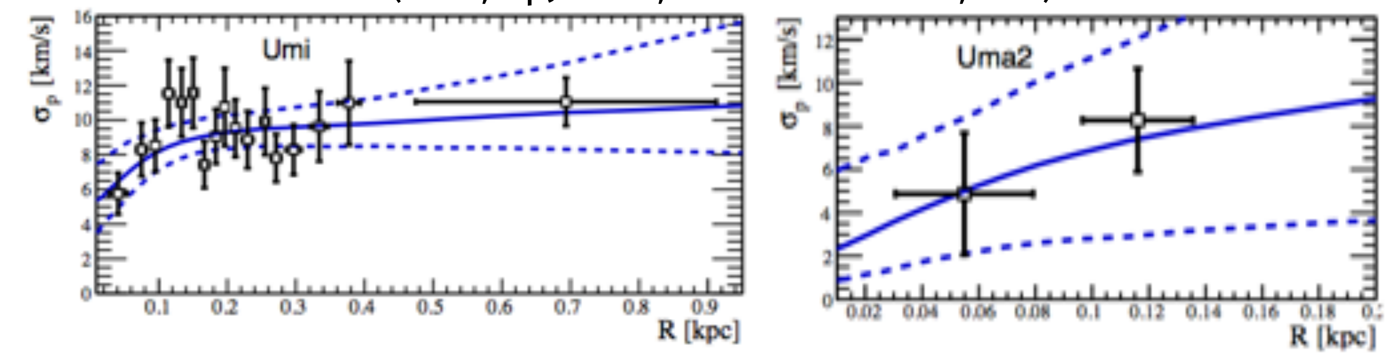
CLUMPY features (III): Jeans analysis module

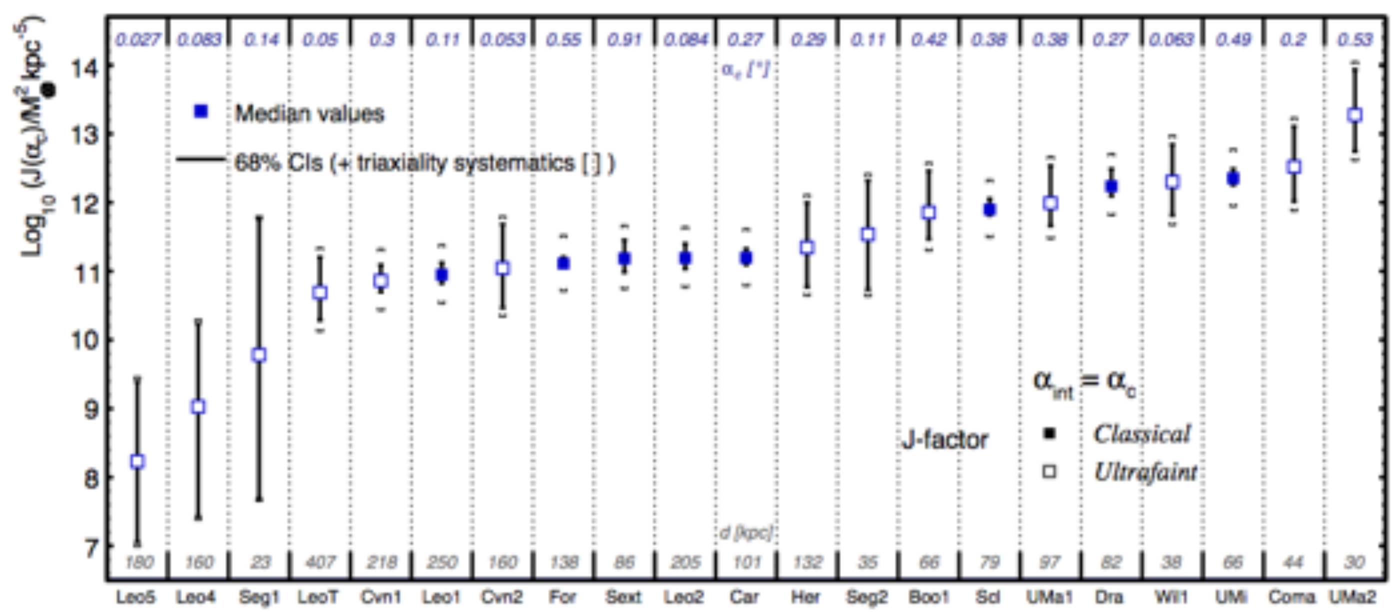
Dsph galaxy analysis: ranking and/or credible intervals



CLUMPY modules I + III together

Bonnivard et al. (2015, ApJL 808, 36 + MNRAS 453, 849)





Many new MW satellite galaxies just discovered (DES) & expected (e.g., LSST): CLUMPY can be used as soon as spectroscopic data is available

Towards the next (3rd) release:

- **Extragalactic module IV**: Compute γ-ray and ν fluxes from various extragalactic sources (total isotropic flux + variation)
- Improved output tailored to use CLUMPY with CTA tools (gammalib/ctools)
- Improved user input options (parameter file or command line options)
- Code compilation optimization for easier installation and platform-compatibility.
- Moved to git

▸ **To be released within the next ~ 6 months (Hütten et al., in prep.)**

Summary

CLUMPY: multi-purpose code for indirect DM detection modeling and analysis

- Code distribution and usage:
  - Open-source: reproducible and comparable J-factor calculations
  - User-friendly, fully documented using Doxygen, lots of examples & tests to run
  - All runs from single parameter file or command line (profiles, concentration, spectra...)
- Fast computation of:
  - Annihilation or decay astrophysical factors using any DM profile
  - Boost from substructures and its uncertainty
  - Integrated/differential fluxes in γ-rays and neutrinos, mixing user-defined branching ratios
- Four main modules / physics cases:
  - I. DM emission from list of objects (dSph galaxies, galaxy clusters)
  - II. Full-sky map mode for Galactic DM emission with substructure + additional objects from list
  - III. Jeans module: full analysis from kinematic data to J-factors for dSph
  - IV. Full-sky map mode for extragalactic DM emission (coming with the 3rd release)

Growing use in the community for state-of-the-art DM studies for many targets (dSphs, cluster, dark clumps...) and by several collaborations (ANTARES, HAWC, CTA)

Download **https://lpsc.in2p3.fr/clumpy/ + stay tuned for 3rd release!**

Thanks for your attention

All parameters controlled from parameter file

Name	Definition
Cosmological parameters (updated from Planck results)	
gCOSMO_HUBBLE	Hubble expansion rate $h = H_0/(100\ \mathrm{km\ s^{-1}\ Mpc^{-1}})$ [-]
gCOSMO_RHO0_C	Critical density of the universe [$M_\odot$ kpc^{-3}]
gCOSMO_OMEGA0_M	Present-day pressure-less matter density
gCOSMO_OMEGA0_LAMBDA	Present-day dark energy density
Dark matter parameters	
gDM_FLAG_CVIR_DIST	Distribution around $\bar{c}(M)$ from which concentrations are drawn: {kLOGNORM, kDIRAC}
gDM_LOGCVIR_STDDEV	Width of log-normal $c(M)$ distribution (if gDM_FLAG_CVIR_DIST=kLOGNORM)
gDM_SUBS_NUMBEROFLEVELS	Number of levels for subhaloes
gDM_MMIN_SUBS	Minimal mass of DM haloes [$M_\odot$]
gDM_MMAXFRAC_SUBS	Defines the maximal mass of clump in host halo: $M_{\rm max}$ = gDM_MMAXFRAC_SUBS × $M_{\rm host}$
gDM_RHOSAT	Saturation density for DM [$M_\odot$ kpc^{-3}]
Generic (sub-)halo structural parameters (TYPE = DSPH, GALAXY or CLUSTER)	
gTYPE_CLUMPS_{FLAG_PROFILE, ...}	Description of subhaloes for host TYPE: $c(M)$, inner profile, shape parameters
gTYPE_DPDM_SLOPE	Slope of the clump mass function
gTYPE_DPDV_{FLAG_PROFILE, RSCALE, ...}	Spatial distribution of substructures in object TYPE
gTYPE_SUBS_MASSFRACTION	Mass fraction of the host halo in clumps
Milky-Way DM (sub-)halo structural parameters	
gGAL_CLUMPS_{FLAG_PROFILE, ...}	Description of Milky-way DM subhaloes
gGAL_DPDM_SLOPE	Slope of clump mass function
gGAL_DPDV_{FLAG_PROFILE, RSCALE, ...}	Spatial distribution of substructures in object TYPE
gGAL_SUBS_{M1, M2, N_INM1M2}	Number of Milky-Way subhaloes in [M_1, M_2]
gGAL_{RHOSOL, RSOL, RVIR}	Local DM density [GeV cm^{-3}], distance GC–Sun [kpc], virial radius [kpc]
gGAL_TOT_{FLAG_PROFILE, RSCALE, ...}	Description of the total DM profile
gGAL_TRIAXIAL_AXES[0-3]	Dimensionless major (a), intermediate (b), and minor (c) axes (see Eq. (18))
gGAL_TRIAXIAL_ROTANGLES[0-3]	Euler rotation angles for triaxial Milky-Way halo [deg]
gGAL_TRIAXIAL_IS	Switch-on or off triaxiality calculation (i.e., use or not the 2 parameters above)

All parameters controlled from parameter file

Particle physics ingredients (for γ-ray and ν flux calculation)	
`gPP_BR[gN_PP_BR]`	List of comma-separated values of branching ratios for the 28 channels
`gPP_DM_ANNIHIL_DELTA`	For annihilating DM, factor 2 in calculation if Majorana, 4 if Dirac
`gPP_DM_ANNIHIL_SIGMAV_CM3PERS`	For annihilating DM, velocity averaged cross-section $\langle\sigma v\rangle_0$ [cm^3 s^{-1}]
`gPP_DM_DECAY_LIFETIME_S`	For decaying DM, lifetime $\tau_{\rm DM}$ of DM candidate [s]
`gPP_DM_IS_ANNIHIL_OR_DECAY`	Switch for annihilating or decaying DM (*replace deprecated* ~~`gSIMU_IS_ANNIHIL_OR_DECAY`~~)
`gPP_DM_MASS_GEV`	Mass $m_{\rm DM}$ of the DM candidate [GeV]
`gPP_FLAG_SPECTRUMMODEL`	Model to calculate final state (*replace deprecated* ~~`gDM_GAMMARAY_FLAG_SPECTRUM`~~)
`gPP_NUMIXING_THETA{12, 13, 23}_DEG`	Neutrino mixing angles [deg]
Simulation parameters/outputs (for a given CLUMPY run)	
`gLIST_HALOES`	DM haloes considered in J-factor calculations [default=`data/list_generic.txt`]
`gLIST_HALOES_JEANS`	Objects considered in Jeans's analysis [default=`data/list_generic_jeans.txt`]
`gSIMU_ALPHAINT_DEG`	Integration angle $\alpha_{\rm int}$ [deg] (if `gSIMU_HEALPIX_NSIDE` not -1, use HEALPix resolution)
`gSIMU_EPS`	Precision used for any operation requiring one (numerical integration, . . .)
`gSIMU_SEED`	Seed of random number generator to draw clumps (if 0, from computer clock)
`gSIMU_FLAG_NUFLAVOUR`	Choice of neutrino flavour (`kNUE, kNUMU, kNUTAU`)
`gSIMU_FLUX_AT_E_GEV`	Energy (GeV) at which to calculate fluxes
`gSIMU_FLUX_E_MIN`	Lower energy bound (GeV) for the integrated flux calculation
`gSIMU_FLUX_E_MAX`	Upper energy bound (GeV) for the integrated flux calculation
`gSIMU_GAUSSBEAM_GAMMA_FWHM_DEG`	Gaussian beam [deg] for γ-ray detector for skymaps smoothing (no smoothing if set to -1)
`gSIMU_GAUSSBEAM_NEUTRINO_FWHM_DEG`	Gaussian beam [deg] for ν detector for skymaps smoothing (no smoothing if set to -1)
`gSIMU_HEALPIX_NSIDE`	$N_{\rm side}$ of HEALPix maps (if -1, set to be as close as possible to $\alpha_{\rm int}$)
`gSIMU_HEALPIX_RING_WEIGHTS_DIR`	Ring weights directory for improved quadrature (optional)
`gSIMU_IS_ASTRO_OR_PP_UNITS`	Outputs (plots and files) in astro ($M_\odot$ and kpc) or particle physics (GeV and cm) units.
`gSIMU_IS_WRITE_FLUXMAPS`	For 2D skymaps, whether to save or not γ-ray and ν fluxes (the J factor is always saved)
`gSIMU_IS_WRITE_FLUXMAPS_INTEG_OR_DIFF`	If `gSIMU_IS_WRITE_FLUXMAPS` is true, whether to save integrated or differential fluxes
`gSIMU_IS_WRITE_GALPOWERSPECTRUM`	Whether to calculate (and save) or not the DM power-spectrum for the Milky-Way
`gSIMU_IS_WRITE_ROOTFILES`	Whether to save or not `.root` files even if option -p is used (not enabled for skymaps and 'stat')
`gSIMU_OUTPUT_DIR`	Output directory to select other than local run (directory is `output/` if set to -1)

J_{tot} integration, substructures, and boost factor

$$J = \int_0^{\Delta\Omega} \int_{l_{\text{min}}}^{l_{\text{max}}} \frac{1}{l^2} \left(\rho_{\text{sm}} + \sum_i \rho_{\text{cl}}^i \right)^2 l^2 dl d\Omega$$

up to 20% of J_{tot} in some config.

$$J_{\text{sm}} \equiv \int_0^{\Delta\Omega} \int_{l_{\text{min}}}^{l_{\text{max}}} \rho_{\text{sm}}^2 dl d\Omega$$

$$J_{\text{cross-prod}} \equiv 2 \int_0^{\Delta\Omega} \int_{l_{\text{min}}}^{l_{\text{max}}} \rho_{\text{sm}} \sum_i \rho_{\text{cl}}^i dl d\Omega$$

$$J_{\text{subs}} \equiv \int_0^{\Delta\Omega} \int_{l_{\text{min}}}^{l_{\text{max}}} \left(\sum_i \rho_{\text{cl}}^i \right)^2 dl d\Omega$$

exact realisation (mass and position) of DM distribution unknown

Average description using mass and spatial distribution from simulations

$$\langle J_{\text{cross-prod}} \rangle = 2 \int_0^{\Delta\Omega} \int_{l_{\text{min}}}^{l_{\text{max}}} \rho_{\text{sm}} \langle \rho_{\text{subs}} \rangle dl d\Omega$$

$$\langle J_{\text{subs}} \rangle = N_{\text{tot}} \int_0^{\Delta\Omega} \int_{l_{\text{min}}}^{l_{\text{max}}} \frac{d\mathcal{P}_V}{dV} dl d\Omega \int_{M_{\text{min}}}^{M_{\text{max}}} \mathcal{L}(M) \frac{\mathcal{P}_M}{dM} dM$$

$$\mathcal{L}(M) \equiv \int_{V_{\text{cl}}} (\rho_{\text{cl}})^2 \, dV$$