ANALYSIS OF LORENTZIAN SUB-PLANCKIAN COSMOLOGY VIA ASYMPTOTIC SAFETY

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Outline of the talk

- Quantum Gravity and Asymptotic Safety in Q.G.
- Brief description of Lorentian Asymptotic Safety
- ADM formalism with G and Λ variable.
- Cosmologies of the Sub-Planck era
- Bouncing and Emergent Universes.
- Conclusions.

QUANTUM GRAVITY

- Einstein General Relativity works quite well for distances $l \gg l_{Pl}$ (=Planck lenght).
- Singularity problem and the quantum mechanical behavour of matter-energy at small distance suggest a quantum mechanical behavour of the gravitational field (Quantum Gravity) at small distances (High Energy).
- Many different approaches to Quantum Gravity: String Theory, Loop Quantum Gravity, Non-commutative Geometry, CDT, Asymptotic Safety etc.
- General Relativity is considered an effective theory. It is not pertubatively renormalizable (the Newton constant G has a (lenght)$^{-2}$ dimension)

QUANTUM GRAVITY

- Fundamental theories (in Quantum Field Theory), in general, are believed to be perturbatively renormalizable. Their infinities can be absorbed by redefining their parameters (m,g,..etc).

- Perturbative non-renormalizability: the number of counter terms increase as the loops orders do, then there are infinitely many parameters and no-predictivity of the theory.

- There exist fundamental (=infinite cut off limit) theories which are not "perturbatively non renormalizable (along the line of Wilson theory of renormalization).
- They are constructed by taking the infinite-cut off limit (continuum limit) at a non-Gaussinan fixed point ($u_* \neq 0$, pert. Theories have trivial Gaussian point $u_* = 0$).

ASYMPTOTIC SAFETY

- The "Asymptotic safety" conjecture of Weinberg (1979) suggest to run the coupling constants of the theory, find a non (NGFP)-Gaussian fixed point in this space of parameters, define the Quantum theory at this point.

- d=2+ϵ: F. P. exists (Weinberg); d=4 NGFP in the Einstein-Hilbert truncation exists (Reuter and Sauressing 2002).

- The main idea is that if one has a classical action of Gravity with a_i constants coupled to operator O(x,g) as follow (Riemannian case of Quantum Gravity)

$$S(M,g) = \int_M \sqrt{g} d^4x \sum_0^\infty a_i O(x,g)$$

- The Renormaliztion Group (RG) equation is ($\tilde{a}$ is the dimensionless coupling constant)

$$k\partial_k \tilde{a}_i(k) = \beta_i(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \ldots)$$

- $\tilde{a}_\star \neq 0$ is a NGFP if $\beta_i(\tilde{a}_\star)=0$, $\forall\, i$.

ASYMPTOTIC SAFETY

- Linearize the previous equations and get:

$$k\partial_k \tilde{a}_i(k) = \sum_j B_{ij}(\tilde{a}_j - \tilde{a}_{\star j}) \qquad B_{ij} := \partial_j \beta_i(\tilde{a}_\star) \qquad B = B_{ij}$$

- The solution of the previous equation is

$$\tilde{a}_i(k) = \tilde{a}_{i\star} + \sum_I C_I V_i^I \left(\frac{k_0}{k}\right)^{\Theta_I}$$

where

$$BV^I = -\Theta_I V^I$$

Right-eigen vectors

Critical exponents

$\tilde{a}(k) \mapsto \tilde{a}_\star, k \mapsto \infty$ implies $C_I=0$, $\forall$ I when Re $\Theta_I < 0$

- **UV-critical hypersurface $\mathcal{S}_{uv}$**

$\mathcal{S}_{uv}$:{ RG trajectories hitting the FP as k tend to ∞ }

Δ_{UV}= **dim** $\mathcal{S}_{uv}$ =number of attractive directions

=number of Θ_I with Re Θ_I>0.

ASYMPTOTIC SAFETY

- The dimension of the UV-crirical surface is the number of independent trajectories that emanates from the fixed point.

- The quantum theory has Δ_{UV} free parameters, if this number is finite the quantum theory exists.

- Up to now, one has considered a perturbative RG. In general, one strats from a (non-perturbative) Wilson-type (coarse grained) free energy functional

$$\Gamma_k[g_{\mu\nu}]$$

- $\Gamma_k[g_{\mu\nu}]$ has an IR cutoff at k; Γ_k contains all the quatum fluctuaction for p>k, and not yet those for p<k.

ASYMPTOTIC SAFETY

- Modes with p<k are suppressed in the path integral by a $(\text{mass})^2 = R_k(p^2)$,

- Γ_k interpolates between : $\Gamma_{k\mapsto\infty} = S$, classical (bare) action , and
  $\Gamma_{k\mapsto 0} = \Gamma$, standard effective action.

- Γ_k satisfies the RG equation, symbolically

$$k\partial_k\Gamma_k = \tfrac{1}{2}Tr\left[(\delta^2\Gamma_k + R_k)^{-1}k\partial_k R_k\right]$$

- Powerful nonperturbative approximation scheme: the space of the action functionals is "Truncate"

$$\Gamma_k[\cdot] = \sum_{i=0}^{N} g_i(k)k^{d_i}I_i[\cdot]$$

$I_i[\cdot]$ are given "local or non-local functionals" of the fields, $g_i(k)$ numerical parameters that carry the scale dependence.

ASYMPTOTIC SAFETY

- In the case of gravity the following truncation ansatz has been made

$$I_0[g] = \int dx\sqrt{g} \qquad I_1[g] = \int dx\sqrt{g}R(g) \qquad I_2[g] = \int dx\sqrt{g}R(g)^2$$

- The simplest truncation is the Einstein-Hilbert one

$$\Gamma_k = -\frac{1}{16\pi G_k}\int_M \sqrt{g}d^dx(R - 2\bar{\lambda}_k) \quad \text{+gf. +gh.}$$

- One has two running parameters G_k , dimensionless $g(k)=k^{d-2}\,G_k$, and $\overline{\lambda_k}$, which dimensionless is $\lambda(k) = \overline{\lambda_k}/k^2$.

- One inserts the previous ansatz into flow equation and expands

$$\mathrm{Tr}[\ldots]=(\ldots)\int\sqrt{g}\ +(\ldots)\int\sqrt{g}\,R$$

- Then, one derives the following flow equations

$$k\frac{dg(k)}{dk} = \beta_g(g,\lambda) \qquad\qquad k\frac{d\lambda(k)}{dk} = \beta_\lambda(g,\lambda)$$

Asymptotic Safety/Lorentian Case
(Manrique et al. 2011)

- One starts from Lorentian Manifolds with ADM metric decomposition

- Define a Path-Integral, in which the integration variables are the Lapse N and the Shifts N^i , as well as the three spatial matrics g_{ij}

- One define a regulator, $R_{k,}$ which cuts all modes f p<k. This regulator is function of the Laplacian on three dimensional surfaces.

- In case one consider, for example, an ADM decomposition in which the three-dimensional surface $M=S^3$ in the case of FLRW metric, the cut-off identification is the eigenvalue of the Laplacian on S^3, which happens to be the scale factor "a" of the Universe.

ADM FORMALISM WITH G AND Λ VARIABLE

- On the ADM decomposition M=$\mathcal{R}\times\Sigma$ the covariant metric tensor is

$$g = -(N^2 - N_i N^i)dt \otimes dt + N_i(dx^i \otimes dt + dt \otimes dx^i) + h_{ij}dx^i \otimes dx^j$$

- The extrinsic curvature K_{ij} and ${}^4R$

$$K_{ij} = \frac{1}{2}(-\frac{\partial h_{ij}}{\partial t} + \bar{\nabla}_i N_j + \bar{\nabla}_j N_i)$$, $\bar{\nabla}$ covariat derivative respect to h_{ij}

$$\sqrt{-g}\,{}^{(4)}R = N\sqrt{h}(K_{ij}K^{ij} - K^2 + {}^{(3)}R) - 2(K\sqrt{h})_{,0} + 2f^i{}_{,i} \qquad f^i \equiv \sqrt{h}\Big(KN^i - h^{ij}N_{,j}\Big)$$

- The Einstein-Hilbert action with the York boundary term term is:

$$S_{ADM}[h_{ij}, N, N^i] = \frac{1}{16\pi}\int_{R\times\Sigma} dtd^3x\sqrt{h}N\frac{1}{G(t,x)}\left({}^4R - 2\Lambda(t,x)\right) + \frac{1}{8\pi}\int_{\partial M}\frac{K\sqrt{h}}{G(t,x)}d^3x$$

the York term, as it is well, know is added in order to have a “differentiable action”.

ADM FORMALISM WITH G AND Λ VARIABLE

- Assuming that Σ has no boundary, that is $\partial\Sigma=0$, the ADM action S_{ADM} becomes

$$S_{ADM}[h_{ij},N,N^i]=\frac{1}{16\pi}\int_{R\times\Sigma}\underbrace{\left[\frac{N\sqrt{h}}{G}(K_{ij}K^{ij}-K^2+{}^{(3)}R-2\Lambda)-2\frac{G_{,0}}{G^2}K\sqrt{h}+2\frac{G_{,i}f^i}{G^2}\right]}_{\mathcal{L}_{ADM}}dtd^3x$$

- The associated momenta π_{ij} to h^{ij} are:

$$\pi_{ij}=\frac{\delta\mathcal{L}_{ADM}}{\delta\dot{h}_{ij}}=-\frac{\sqrt{h}}{16\pi G}\left(K_{ij}-h_{ij}K\right)+\frac{\sqrt{h}\,h_{ij}}{16\pi NG^2}\left(G_{,0}-G_{,k}N^k\right)$$

- Their form suggests that one can define a new momentum variable $\tilde{\pi}_{ij}$

$$\tilde{\pi}_{ij}=\pi_{ij}-\frac{\sqrt{h}\,h_{ij}}{16\pi NG^2}\left(G_{,0}-G_{,k}N^k\right)=-\frac{\sqrt{h}}{16\pi G}\left(K_{ij}-h_{ij}K\right)$$

ADM FORMALISM WITH G AND Λ VARIABLE

- At this point, one can ask if the following change of variables

$$\left(N, N^i, h^{ij}, \pi_N, \pi_{N^i}, \pi_{ij}\right) \mapsto \left(N, N^i, h^{ij}, \pi_N, \pi_{N^i}, \tilde{\pi}_{ij}\right)$$

is canonical, in Hamiltonian sense, that is it preserves the symplectic two form $\Omega = dq \wedge dp$. The implementation of the previous condition becomes

$$F \equiv \frac{\partial(q_1, \ldots q_n, p_1, \ldots, p_n)}{\partial(Q_1, \ldots, Q_n, P_1, \ldots, P_n)} \qquad J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$F^T J F = J$$

- It can be easily verified that the variable transformation above is canonical

ADM FORMALISM WITH G AND Λ VARIABLE

- The Hamiltonian density $\mathcal{H}_{ADM}$ (relative to h_{ab}) is

$$\mathcal{H}_{ADM} \equiv \pi^{ab}\dot{h}_{ab} - \mathcal{L}_{ADM}$$

- Implementing all substitutions, one gets

$$\mathcal{H}_{ADM} = N\left((16\pi G)G_{abcd}\tilde{\pi}^{ab}\tilde{\pi}^{cd} - \frac{\sqrt{h}({}^{3}R - 2\Lambda)}{16\pi G}\right) - 2\tilde{\pi}^{ab}\bar{\nabla}_a N_b + \frac{\sqrt{h}(G_{,0} - G_{,k}N^k)\bar{\nabla}_a N^a}{8\pi G^2 N} + \frac{G_{,i}\sqrt{h}h^{ij}}{8\pi G^2}N_{,j}$$

$G_{abcd} = \frac{1}{2\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd})$ is the DeWitt supermetric

- In order, at first stage, to make contact with standard Hamiltonian general Relativity, one imposes the following gauge on the lapse functions N^a

$$\bar{\nabla}_a N^a = 0$$

ADM FORMALISM WITH G AND Λ VARIABLE

- Boundaries integrations, under the previous assumptions, allows to write

$$\mathcal{H} = (16\pi G) G_{abcd} \tilde{\pi}^{ab} \tilde{\pi}^{cd} - \frac{\sqrt{h}({}^3R - 2\Lambda)}{16\pi G} - \sqrt{h} \bar{\nabla}_j \left(\frac{G_{,i} h^{ij}}{8\pi G^2} \right)$$ Hamiltonian Constraint

$$\mathcal{H}^a = -2\bar{\nabla}_b \tilde{\pi}^{ba}$$ Momentum constraint

- Therefore the the ADM Hamiltonian density $\mathcal{H}_{ADM}$ can be written

$$\mathcal{H}_{ADM} = N\mathcal{H} + N^a \mathcal{H}_a$$

as in ADM Hamiltonian General Relativity!

ADM FORMALISM WITH G AND Λ VARIABLE

- This is a system with Dirac's constraints, in fact the primary constraints are

$$\pi = \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{N}} \approx 0 \qquad \pi_i = \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{N}^i} \approx 0$$

- The total Hamiltonian H_T is then

$$H_T = \int d^3x \lambda \pi + \int d^3x \lambda^i \pi_i + \int d^3x \left(N\mathcal{H} + N^i \mathcal{H}_i \right)$$

λ and λ_i being, following Dirac's constraint theory, Lagrange multipliers.

- The Poisson brackets are so defined

$$\{A, B\} = \int d^3x \left(\frac{\delta A}{\delta h^{ij}} \frac{\delta B}{\delta \tilde{\pi}_{ij}} - \frac{\delta A}{\delta \tilde{\pi}_{ij}} \frac{\delta B}{\delta h^{ij}} \right)$$

ADM FORMALISM WITH G AND Λ VARIABLE

- Preserving primary constaints $\pi \approx 0$ $\pi_i \approx 0$, one gets the secondary constaints: the Hamiltonian constraint $\mathcal{H}$ and the and the momenta constraints constraints $\mathcal{H}_i$

$$\dot{\pi} = \{\pi, H_T\} = \mathcal{H} \approx 0 \qquad \dot{\pi}_i = \{\pi_i, H_T\} = \mathcal{H}_i \approx 0$$

- One can verify that there are no further constraints, and all the constraints are first class on the constraint manifold.

- Implementing the gauge condition $\bar{\nabla}_i N^i \approx 0$ as further constraints, one gets

$$\{\bar{\nabla}_i N^i, H_T\} = \{\bar{\nabla}_i N^i, \int d^3x \lambda^k \pi_k\} = \bar{\nabla}_j \lambda^j \approx 0$$

- Therefore this gauge condition makes the primary constraints $\pi_i \approx 0$ second class and fixes a condition on the Lagrange multipliers λ^i

COSMOLOGIES OF THE SUB-PLANCK ERA

- Consider the previous E-H action with matter

$$S = \int_M d^4x\sqrt{-g}\left\{\frac{R-2\Lambda(k)}{16\pi G(k)} + \mathcal{L}_m\right\} + \frac{1}{8\pi}\int_{\partial M}\frac{K\sqrt{h}}{G(k)}d^3x$$

- One starts from a FLRW metric, in which the shifts $N^i = 0$

$$ds^2 = -N(t)^2dt^2 + \frac{a(t)^2}{1-Kr^2}dr^2 + a(t)^2(r^2d\theta^2 + r^2\sin\theta d\phi^2)$$

- Perfect fluid, with density ρ and pressure p and equation of state $p = w\rho$, w is a constant. Imposing the conservation of matter stress energy tensor $T^{\mu\nu}{}_{;\nu} = 0$ one get $\rho(a) = ma^{-3-3w}$ with m and intergation constant, and $\mathcal{L}_m = -mNa^{-3w}$

- Following Manrique et al., $k \sim \frac{1}{a}$ so that the

$$\mathcal{L}_g = \frac{3\,a\dot{a}^2}{8\pi N(t)G(a)} + \frac{3\,a^2\dot{a}^2G'(a)}{8\pi NG^2(a)} + \frac{3\,aNK}{8\pi G(a)} - \frac{a^3N\Lambda(a)}{8\pi G(a)} - \frac{2Nm}{a^{3w}} + \frac{d}{dt}\left(\frac{3a^2\dot{a}^2}{8\pi NG(a)}\right)$$

where the total derivative is the York-term.

COSMOLOGIES OF THE SUB-PLANCK ERA

- Repeating the Dirac's constraint analysis as above, one get, as constraints, the momentum π conjucated to the Lapse $N(t)$ and the Hamiltonian constraint $\mathcal{H}$

$$\mathcal{H} = -\frac{2\pi G^2(a) p_a^2}{3a(G(a) - aG'(a))} - \frac{3aK}{8\pi G(a)} + \frac{a^3 \Lambda(a) N}{\pi G(a)} + \frac{2m}{a^{3w}}$$

- The total Hamiltonian H_T is

$$H_T = N\mathcal{H} + \lambda\pi$$

- Imposing the gauge $N = 1$ as a constraint $N - 1 \approx 0$, one gets that π becomes second class constraint and $\lambda = 0$

$$\{N - 1, \pi\} = 1 \qquad \frac{d}{dt}(N - 1) = \{N - 1, H_T\} = \lambda = 0$$

COSMOLOGIES OF THE SUB-PLANCK ERA

- From the Hamiltonian constraint, one gets the quantum-Friedman

$$\frac{K}{a^2H^2} - \frac{8\pi G(a)\,\rho + \Lambda(a)}{3H^2} + \eta(a) + 1 = 0$$

in which $\eta(a) = -\frac{\partial \log G(a)}{\partial \log a}$

- This implies an equation of evolution for $a(t)$

$$\dot{a}^2 = -\tilde{V}_K(a) \equiv -\frac{K + V(a)}{\eta(a) + 1} \qquad V(a) = \frac{a^2}{3}(8\pi G(a)\,\rho + \Lambda(a))$$

- Notice the allowed regions for the dynamical evolution are $\tilde{V}_K(a) \leq 0$.

- Close to NGFP, using cut off $k \sim \frac{1}{a}$, the follwing approximate solution for RG-equation are deduced (Biemans et al. 2017)

$$G(a) \approx G_0\left(1 + G_0/g_* a^{-2}\right)^{-1}$$

$$\Lambda(a) \approx \lambda_* a^{-2} + \lambda_0$$

(λ_*, g_*) NGFP,

λ_0 fixes the IR trajectories of $\Lambda(a)$

COSMOLOGIES OF THE SUB-PLANCK ERA

- Regions for evolution, already noticed, $\tilde{V}_K(a) \leq 0$, and bouncing cosmological models have $\tilde{V}_K(a) = 0$ for some $a = a_b$

$$\tilde{V}_K(a) = \frac{\lambda_0 \left[\left(a^2 + a_0^2\right)\left(a^2 + l_k\right) + l_g\, a^{1-3w}\right]}{3\left(a_0^2 - a^2\right)} = 0$$

where $a_0 \equiv \sqrt{g_*/G_0}$ and $l_g = \dfrac{m\, g_*\, a_0^2}{\lambda_0}$ $\quad l_k = \dfrac{\lambda_* - 3\,K}{\lambda_0}$

- In the case $w \geq -\dfrac{1}{3}$ and $K = 0$, the pervious equation has two accetable positive solutions and the bouncing cosmologies are determined by the existence of the solution

BOUNCING COSMOLOGIES

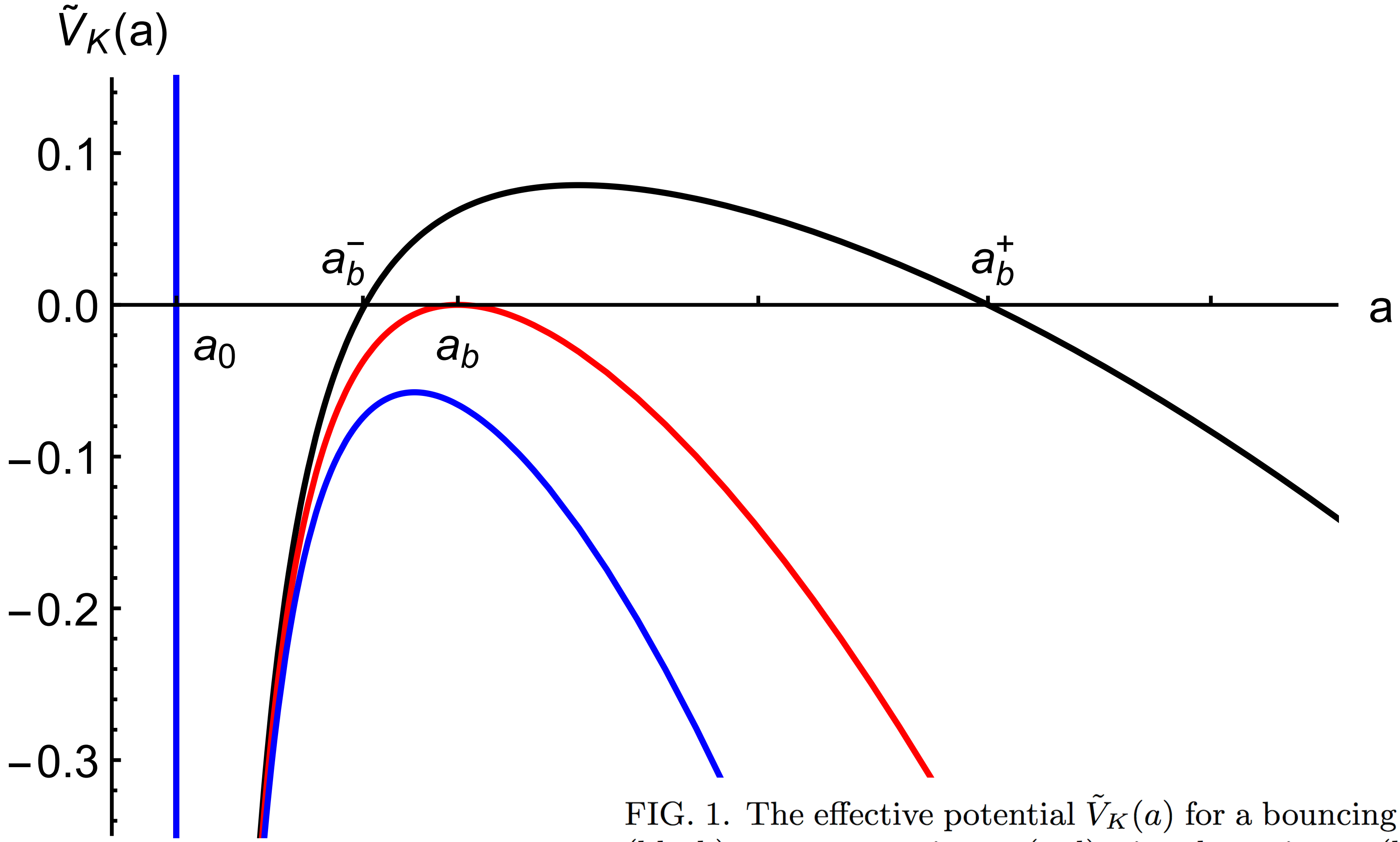

FIG. 1. The effective potential $\tilde{V}_K(a)$ for a bouncing universe (black), emergent universe (red), singular universe (blue), for $K = 0$, $w = 1/3$, $g_* = 0.1$, $\lambda_* = -0.5$, $\xi = 1$ and $m = 3$. Black, red and blue correspond to $\lambda_0 = 2 \times 10^{-4}$, $\lambda_0 = 8.3 \times 10^{-4}$ and $\lambda_0 = 1.5 \times 10^{-3}$ respectively.

EMERGENT UNIVERSES

- The most interesting case is when the two bouncing solutions coincide

$a_b^{\pm} = a_b = a_{max} > a_0$. In this case, red line, this point is a point of maximum, a_{max} then one has $\tilde{V}_K(a_{max}) = 0$ $\dot{a} = \ddot{a} = 0$

- This is the condition for Emergent universe. A point at right of a_b will spend an infinite time to reach a_b An Emergent Universe is an inflationary Universe without singularity which will emerge from a minimal radius universe (no singularity!) .

- Asymptotic Safety predicts $\lambda_* \neq 0$. Some cases of Asymptotic Safety coupled with matter ((Biemans et al. 2017) show that λ_* is negative. This is sufficient in order to have wide ranges of λ_* for which there exists bounces for every kind of topology of spatial three-dimensional slices with $K = -1, 0, 1$

EMERGENT UNIVERSES

- Around the bounce one finds that the scale factor inflates

$$a(t) \sim a_b + \epsilon \, e^{\sqrt{\beta} t} \qquad \beta \equiv \frac{4\,\lambda_0\,(a_0^2 + l_k)}{3\,(3\,a_0^2 + l_k)}$$

- It can be verified that there are ranges of λ_0 such that β is positive.

- In classical GR, Ellis and Maartens (2003) have highlighted that Emerging Universes are possible only in the case $K = 1$. For example, there exists a simple model with $w = \frac{1}{3}$ and $K = 1$ such that

$$a(t) = a_i \left[1 + exp\left(\frac{\sqrt{2}t}{a_i} \right) \right]^{\frac{1}{2}}$$

a_i being the minimal radius.

CONCLUSIONS

- An ADM analysis of Einstein General Relativity with Cosmological constant has been performed in the case in which G and Λ are variable. One finds analogies with the constraint analysis of Einstein Classical General Relativity.

- Sub-Planckian cosmological models derived by Asymptotic Safety techniques have been studied. They exhibit Emergent Universes also in the case K=0 and K=-1, that are impossible to draw from Classical General Relativity.

- The ADM analysis has been performed choosing a special "gauge" in order to simplify the calculations. A future line of research could be to study this ADM formalsism in the general case.

- The RG improved Einstein Equations has shown that there exists bouncing cosmologies and emergent universes. At the moment, this is still a preliminary analysis. Much work need to be done to study the dependence of these solutions from the the Asymptotic Safety parameters.