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Observables and Dispersion Relations in κ-Minkowski and κ-FRW noncommutative spacetimes

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-Deformation of symmetries to quantum symmetries, and of spaces to non-commutative spaces.

-Quantum Lie algebra as the algebra of infinitesimal symmetry transformations on a noncommutative space.

-Wave equations in noncommutative spacetimes canonically constructed within NC differential geometry.

-Application: Quantum Lie algebra of energy and momentum, i.e. of translation generators in k-Minkowski spacetime. Wave equations and dispersion relations.

-Turning on a cuved metric background.

Some physical motivations for NC spacetimes

Classical Mechanics$\longrightarrow$ Quantum Mechanics
functions (observables) on phase space become noncommutative (phase space noncommutativity)

General Relativity $\longrightarrow$ Quantum Gravity
Spacetime structure itself becomes noncommutative.

This expectation is supported by Gedanken experiments –e.g. localization of test particles within Planck scale accuracy– suggesting that spacetime structure is not necessarily that of a smooth manifold (a continuum of points).

Below Planck scales it is then natural to conceive a more general spacetime structure. A cell-like or lattice like structure, or a noncommutative one where (as in quantum mechanics phase-space) uncertainty relations and discretization naturally arise.

Space and time are then described by a *Noncommutative Geometry*

NC spacetimes features, like generalized uncertainty relations (GUP) involving space and time coordinates, or minimal area and volume elements arise also in String Theory and Loop Quantum Gravity.

Yet another motivation to study deformed (noncommutative) spacetimes comes from *quantum gravity phenomenology*. Even if quantum gravity effects should become relevant at Planck scale, some quantum gravity signatures could be detected at lower scales. This is the case for cumulative phenomena.
Example: light travelling in a quantum spacetime could have a velocity dependent on the photons energy. Even a tiny modification of the usual dispersion relations could then be detected due to the cumulative effect of light travelling long distances.
A natural setting for this study is that of Gamma Ray Bursts (GRB) from distant galaxies [Miguejio, Smolin, Amelino-Camelia, Ellis, Mavromatos, Nanopulos].
Another possibility is that of high precision (quantum) optics experiments based on interferometry techniques [Hogan] [Genovese].

Summary

-Gedanken experiment: Localize at Planck scale is a way to test spacetime structure.

- Spacetime structure might however be detected from deformed dispersion relations of light, whose effects cumulate in time.

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A propotypical example of NC spacetime that appears in both the mathematics and the phenomenology literatures is k-Minkowski spacetime:

$$x^0 \star x^i - x^i \star x^0 = \frac{i}{\kappa} x^i \ .$$

The general construction we outline will be applied to k-Minkowski spacetime.

The deformation quantization method we consider is the Drinfeld twist method used to deform spaces via a symmetry group.

Other approaches use the κ-Poincaré group that is not a Drinfeld twist of the Poincaré group. Therefore, despite the κ-Minkwski space is the same, the symmetries and the differential geometry will differ form those considered in [Lukierski, Nowicki, Ruegg, Majid, Amelino-Camelia, et al.]

Drinfeld Twist $\mathcal{F}$. A general method to obtain a wide class of NC spaces and their symmetry Hopf algebras.

Key idea: First deform the symmetry group of a spacetime (e.g. conformal group, Poincaré group) to a quantum group and then deform the algebra of functions on spacetime so that it has this quantum group symmetry.

- G acts on M (G is a Lie group of transformations on spacetime M).

- g acts on $Fun(M)$ (g is Lie algebra of infinitesimal transformations)

  $t \in g$ acts as a derivation on $Fun(M)$:

  $$t(fh) = t(f)h + ft(h) \quad .$$

  for short we write this Leibnirz rule as

  $$\Delta(t) = t \otimes 1 + 1 \otimes t$$

Examples of twists $\mathcal{F}$

For the general notion see [Drinfeld '83, '85]

Example 1: Moyal-Weyl deformation.
Let g Poincaré Lie algebra or conformal Lie algebra.

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}$$

($\theta^{\mu\nu}$ constant antisymmetric matrix).

$$\mathcal{F} = 1 \otimes 1 - \frac{1}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu - \frac{1}{8}\theta^{\mu\nu}\theta^{\rho\sigma}P_\mu P_\rho \otimes P_\nu P_\sigma + \ldots\ldots$$

Notation:

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha \,, \qquad \mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha \ .$$

Let $A = Fun(M)$ then $f \star h = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(h)$, i.e.,

$$\begin{aligned} f \star h &= fh - \frac{i}{2}\theta^{\mu\nu}P_\mu(f)P_\nu(h) + \ldots. \\ &= fh + \frac{i}{2}\theta^{\mu\nu}\partial_\mu(f)\partial_\nu(h) + \ldots. \end{aligned} \tag{1}$$

In particular $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$.

Example 2: Jordanian deformations, c.f. [Ogievetsky, '93], [Borowiec, Pachol '09]

$$\mathcal{F} = e^{iD\otimes\sigma} \quad , \qquad \sigma = \ln(1 + \frac{1}{\kappa}P_0) \; , \qquad\qquad [D, P_0] = P_0$$

where $g = span\{P_\mu, M_{\mu\nu}, D\}$ is the Poincaré-Weyl algebra (Poincaré plus dilatation D). Can be also extended to the conformal algebra.

g acts on $Fun(M)$ via $D(f) = ix^\mu\partial_\mu(f)$, $P_0(f) = i\partial_0 f$.

In this way $Fun(M)_\star$ becomes the algebra of κ**-Minkowski spacetime**

$$x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa}x^k \quad , \qquad x^i \star x^k - x^k \star x^i = 0 \qquad (2)$$

Leibniz rule is deformed: $P_\mu(f \star g) = P_\mu(f) \star e^\sigma(g) + f \star P_\mu(g)$, i.e.,

$$\Delta^{\mathcal{F}}(P_\mu) = P_\mu \otimes e^\sigma + 1 \otimes P_\mu \; .$$

The twist $\mathcal{F}$ deforms the Poincaré-Weyl group in a quantum Poincaré-Weyl group. Are $P_\mu, M_{\mu\nu}, D$ elements of a quantum Lie algebra $g^{\mathcal{F}}$? No!

Quantum Lie algebra $g^{\mathcal{F}}$ [P.A., Dimitrijevic, Meyer, Wess, 06]

Canonical construction see also LNP 774 Springer §7

Given a twist $\mathcal{F}$ consider the quantization operator $\mathcal{D}$ defined by, for all $t \in g$,

$$\mathcal{D}(t) = \ell_{\mathsf{f}^\alpha(t)}\, \bar{\mathsf{f}}_\alpha \tag{3}$$

where $\ell_{t't''} = \ell_{t'}\ell_{t''}$ product of Lie derivatives. Then

$$g^{\mathcal{F}} := \{\mathcal{D}(t),\ \text{with } t \in g\}\ . \tag{4}$$

Theorem $g^{\mathcal{F}}$ is a quantum Lie algebra in the sense of Woronowicz:

i) Minimally deformed Leibniz rule

$$\Delta^{\mathcal{F}}(\mathcal{D}(t)) = \mathcal{D}(t) \otimes 1 + \bar{\mathsf{R}}^\alpha \otimes \mathcal{D}(\bar{\mathsf{R}}_\alpha(t)) \tag{5}$$

$$\bar{\mathsf{R}}^\alpha \otimes \bar{\mathsf{R}}_\alpha = \mathcal{F}\mathcal{F}_{21}^{-1}.$$

ii) $\ell^{\mathcal{F}}_{\mathcal{D}(t)}\mathcal{D}(t') \in g^{\mathcal{F}}$ closure under adjoint action (Lie derivative)

Moreover, set $t^{\mathcal{F}} = \mathcal{D}(t)$, $t'^{\mathcal{F}} = \mathcal{D}(t')$, and define

$$[t^{\mathcal{F}}, t'^{\mathcal{F}}]_{\mathcal{F}} := \ell^{\mathcal{F}}_{t^{\mathcal{F}}} t'^{\mathcal{F}}$$

then

$$[t^{\mathcal{F}}, t'^{\mathcal{F}}]_{\mathcal{F}} = t^{\mathcal{F}} t'^{\mathcal{F}} - (\bar{R}^{\alpha}(t))^{\mathcal{F}} (\bar{R}_{\alpha}(t'))^{\mathcal{F}} \ .$$

Furthermore this deformed Lie bracket satisfies the braided-antisymmetry property and the braided-Jacobi identity

$$[t^{\mathcal{F}}, t'^{\mathcal{F}}]_{\mathcal{F}} = -[\bar{R}^{\alpha}(t'^{\mathcal{F}}), \bar{R}_{\alpha}(t^{\mathcal{F}})]_{\mathcal{F}}$$
$$[t^{\mathcal{F}}, [t'^{\mathcal{F}}, t''^{\mathcal{F}}]_{\mathcal{F}}]_{\mathcal{F}} = [[t^{\mathcal{F}}, t'^{\mathcal{F}}]_{\mathcal{F}}, t''^{\mathcal{F}}]_{\mathcal{F}} + [\bar{R}^{\alpha}(t'^{\mathcal{F}}), [\bar{R}_{\alpha}(t^{\mathcal{F}}), t''^{\mathcal{F}}]_{\mathcal{F}}]_{\mathcal{F}} \ .$$

The Jordanian twist gives the quantum Poincaré-Weyl Lie algebra

$g^{\mathcal{F}} = span\{P^{\mathcal{F}}, M^{\mathcal{F}}, D^{\mathcal{F}}\}$ with

$$P^{\mathcal{F}}_{\mu} = P_{\mu} \frac{1}{1 + \frac{1}{\kappa} P_0} \quad , \quad M^{\mathcal{F}}_{\mu\nu} = M_{\mu\nu} \quad , \quad D^{\mathcal{F}} = D \ .$$

Twisted commutators:

$$\begin{aligned} [M^{\mathcal{F}}_{\mu\nu}, P^{\mathcal{F}}_{\rho}]_{\mathcal{F}} &= M^{\mathcal{F}}_{\mu\nu} P^{\mathcal{F}} - P^{\mathcal{F}}_{\rho} M^{\mathcal{F}}_{\mu\nu} - P^{\mathcal{F}}_{\rho} \frac{1}{\kappa} ([P_0, M_{\mu\nu}])^{\mathcal{F}} \ , \\ [D^{\mathcal{F}}, P^{\mathcal{F}}_{\mu}]_{\mathcal{F}} &= D^{\mathcal{F}} P^{\mathcal{F}}_{\mu} - P^{\mathcal{F}}_{\mu} D^{\mathcal{F}} + P^{\mathcal{F}} \frac{i}{\kappa} P_0 \ , \end{aligned} \tag{6}$$

while the remainig twisted commutators are usual commutators.

The deformed Leibnitz rule

$$\Delta^{\mathcal{F}}\left(P_\mu^{\mathcal{F}}\right) = P_\mu^{\mathcal{F}} \otimes 1 \;+\; (1 - \frac{1}{\kappa}P_0) \otimes P_\mu^{\mathcal{F}}$$

gives addition of momenta:

$$p_\mu^{\mathcal{F}\,tot} = p_\mu^{\mathcal{F}} + p_\mu'^{\mathcal{F}} - \frac{1}{\kappa} p_0^{\mathcal{F}} p_\mu'^{\mathcal{F}} \ .$$

The deviation from the usual addition law is quadratic in the momenta (and not exponential like in κ-Poincaré in the usual basis of generators). The total energy is invariant under the usual permutation of the two particles.

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We identify the physical Energy and momentum operators on κ-Minkowski spacetime with $P_\mu^{\mathcal{F}}$ because

-they are the quantum Lie algebra uniquely associated with the twist deformation $\mathcal{F}$ (in particular in the classical limit $P_\mu^{\mathcal{F}} \to P_\mu$).

-they are dual to the one forms dx^μ in the sense that

$$df = dx^\mu \star iP_\mu^{\mathcal{F}}(f) \ ,$$

where $P_\mu^{\mathcal{F}} = \frac{P_\mu}{1+\frac{1}{\kappa}P_0} = \frac{-i\partial_\mu}{1-\frac{i}{\kappa}\partial_0}$
and d is the usual differential that satisfies $d(f \star g) = df \star g + f \star dg$.

Field equation The d'Alembert operator $\square^{\mathcal{F}}$ on κ-Minkowski spacetime can be defined:

(1) as the quadratic Casimir $\eta^{\mu\nu} P^{\mathcal{F}}_{\mu} P^{\mathcal{F}}_{\nu}$

(2) via the Hodge $*^{\mathcal{F}}$-operator, as the Laplace-Beltrami operator $*^{\mathcal{F}} d *^{\mathcal{F}} d$, where $*^{\mathcal{F}} = \mathcal{D}(*)$.

These definitions coincide. Furthermore, the d'Alembert operator is invariant under infinitesimal translations, rotations and boosts, and transforms as in the commutative case under dilatations:

$$\left[P^{\mathcal{F}}_{\mu}, \square^{\mathcal{F}}\right]_{\mathcal{F}} = 0 \;, \quad \left[M^{\mathcal{F}}_{\mu\nu}, \square^{\mathcal{F}}\right]_{\mathcal{F}} = 0 \;, \quad \left[D^{\mathcal{F}}, \square^{\mathcal{F}}\right]_{\mathcal{F}} = -2i\square^{\mathcal{F}} \;.$$

NC wave equation For a scalar field φ we have the wave equation

$$\square^{\mathcal{F}}\varphi = P_\mu^{\mathcal{F}} P^{\mu\,\mathcal{F}} = \frac{1}{(1-\frac{i}{\kappa}\partial_0)^2}\square\varphi = -m^2\varphi\ , \tag{7}$$

where $\square = \partial_\mu \partial^\mu$.

This wave equation was proposed on the basis of phenomenological considerations in [Maguejio, Smolin '01].

Rmk. For massless fields this wave equation is equivalent to the undeformed one $\square\varphi = 0$, (indeed the differential operator $\frac{1}{(1-\frac{i}{\kappa}\partial_0)^2}$ is invertible. This result agrees with the well known one for free fields in noncommutative Moyal-Weyl space). Therefore no modified dispersion relations for massless fields.

Solutions $e^{i(kx-\omega t)}$ of the wave equation satisfy however the modified Einstein-Planck and de Broglie relations (we set Planck energy to be $E_P = -c\hbar\kappa$):

$$E^{\mathcal{F}} = \frac{\hbar\omega}{1+\frac{\hbar\omega}{E_p}}\ ,\quad P_i^{\mathcal{F}} = \frac{\hbar p_i}{1+\frac{\hbar\omega}{E_p}}\quad ; \tag{8}$$

hence maximum energy is Planck energy.

Rmk. For massive fields we still have $E^{\mathcal{F}2} = (P_i^{\mathcal{F}})^2 + m^2$.
The wave eq. is solved by $\varphi = e^{(ikx-wt)}$ that gives the group velocity

$$v_g = \frac{dw}{dk} = c^2\frac{k}{\omega}(1+\frac{m^2c^3}{\hbar^2\omega\kappa}) + \mathcal{O}(\frac{1}{\kappa^2})$$

Theorem: The diffeomorphism in momentum space considered in [Maguejio Smolin] (so-called Deformed or Doubly special relativity paper)

$$p_\mu \to U(p_\mu) = \frac{p_\mu}{1 - \lambda p_0} \,,$$

coincides with the quantization map $\mathcal{D}$:

$$P_\mu \to P_\mu^{\mathcal{F}} = \mathcal{D}(P_\mu) = \frac{P_\mu}{1 + \frac{1}{\kappa} P_0}$$

This holds more in general for functions of momenta.
The nonlinear realization of the Lorentz group in momentum space considered in [Maguejio Smolin]

$$M'_{\mu\nu} = U M_{\mu\nu} U^{-1}$$

were the usual Lorentz generators are $M_{\mu\nu} = p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}$ coincides with the quantum generators:

$$M'_{\mu\nu} = M^{\mathcal{F}}_{\mu\nu} = \mathcal{D}(M_\mu{}^\nu) \,.$$

Rmk. Since $\mathcal{D}$ is defined on the abstract Lie algebra generators, and therefore it is defined also in position space, this extends to position space the proposal of [Maguejio Smolin]. Notice however that we obtain different energy momentum generators.

With this paper we have therefore contributed to fill a gap between theory and phenomenology. We have considered a top down approach where the NC differential geometry constraints the formalims and points out the physical generators as quantum Lie algebra elements.

Phenomenologically motivated deformed special relativity theories are obtained and analized with these tools.

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So far we have considered deformations of flat Minkowski spacetime. What happens if we turn on a nontrivial backgroud metric besides nontrivial commutativity?

The Laplace-Beltrami operator $*^{\mathcal{F}} d *^{\mathcal{F}} d$ is a natural candidate for a scalar field wave equation.

Toy model: 2-dim Friedman-Robertson-Walker-Lemaitre (FRWL) universe.

Metric

$$ds^2 = dt^2 + a^2(t)dx^2$$

$a(t)$ - scale factor.
κ-noncommutativity: $t \star x - x \star t = \frac{i}{\kappa}x$, we are in κ-FRWL spacetime.

Assume the solution of the form:

$$\varphi = \lambda(t) \star e^{-ikx}$$

The NC wave equation

$$*^{\mathcal{F}} d *^{\mathcal{F}} d\varphi = 0$$

reduces to

$$a \star \partial_0^2 \lambda + \partial_0(a) \star e^{\sigma} \partial_0 \lambda + a^{-1} \star k^2 \lambda = 0 \tag{9}$$

-Expand the star product at first order in the NC deformation $\frac{1}{\kappa}$.

-Change coordinates to conformal time η (as usually done in commutative case as well).

-Since the frequency ω is much higher than H_0 the equation can be solved. The variation of the speed of light v_{ph} with respect to the usual one c (of photons in flat spacetime, or of low energetic photons) turns out to be linear in the frequency ω.

Conclusions

-We have considered a NC spacetime that in the flat case has a deformed Poincaré Weyl symmetry (conformal symmetry). Equations of motion for massless fields are derived from deformed symmetry principles and the differential geometry of Drinfeld twist.

Turning on a curved metric we obtain nontrivial dispersion relations due to the combined effect of curvature and noncommutativity.