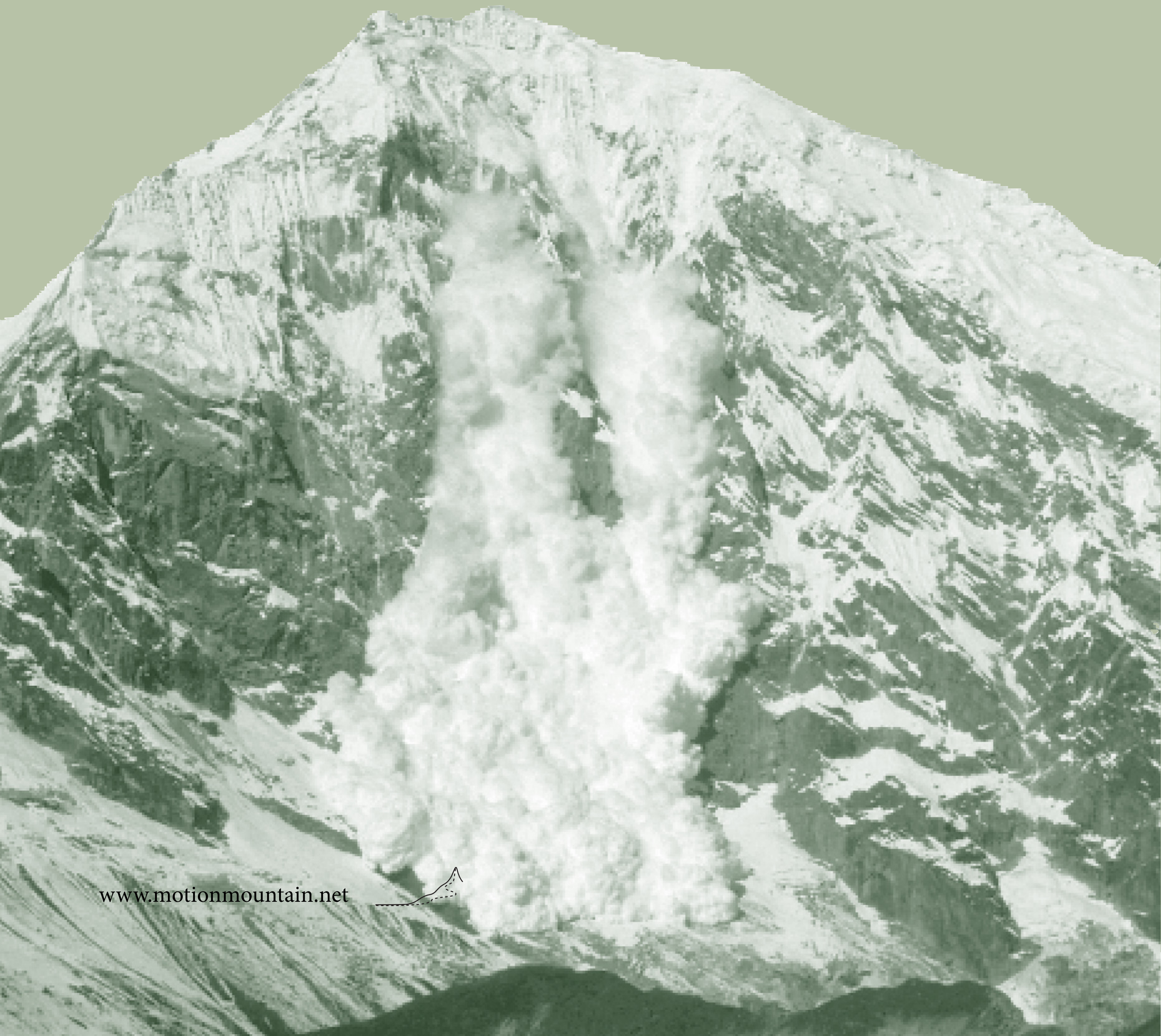
Christoph Schiller

MOTION MOUNTAIN

THE ADVENTURE OF PHYSICS – VOL.IV

QUANTUM THEORY: THE SMALLEST CHANGE



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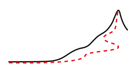

Christoph Schiller

Motion Mountain

The Adventure of Physics
Volume IV

Quantum Theory:
The Smallest Change

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Editio vicesima quarta.

Twenty-fourth edition.

To Britta, Esther and Justus Aaron

τῷ ἐμοὶ δαὶμονι

Die Menschen stärken, die Sachen klären.

PREFACE

> Primum movere, deinde docere.*
>
> Antiquity

This book is written for anybody who is curious about nature and motion. Have you ever asked: Why do people, animals, things, images and space move? The answer leads to many adventures; this volume presents those due the discovery that there is a smallest change in nature. This smallest change leads to what is called *quantum theory*. In the structure of modern physics, shown in Figure 1, quantum physics covers three points; this volume covers the introduction to the point in the lower right: the foundations of quantum theory.

The present introduction to quantum physics arose from a threefold aim I have pursued since 1990: to present the basics of motion in a way that is simple, up to date and captivating.

In order to be *simple*, the text focuses on concepts, while keeping mathematics to the necessary minimum. Understanding the concepts of physics is given precedence over using formulae in calculations. The whole text is within the reach of an undergraduate.

In order to be *up to date*, the text is enriched by the many gems – both theoretical and empirical – that are scattered throughout the scientific literature.

In order to be *captivating*, the text tries to startle the reader as much as possible. Reading a book on general physics should be like going to a magic show. We watch, we are astonished, we do not believe our eyes, we think, and finally we understand the trick. When we look at nature, we often have the same experience. Indeed, every page presents at least one surprise or provocation for the reader to think about. Numerous interesting challenges are proposed.

The motto of the text, *die Menschen stärken, die Sachen klären*, a famous statement by Hartmut von Hentig on pedagogy, translates as: 'To fortify people, to clarify things.' Clarifying things requires courage, as changing habits of thought produces fear, often hidden by anger. But by overcoming our fears we grow in strength. And we experience intense and beautiful emotions. All great adventures in life allow this, and exploring motion is one of them.

Munich, 24 June 2011.

* 'First move, then teach.' In modern languages, the mentioned type of *moving* (the heart) is called *motivating*; both terms go back to the same Latin root.

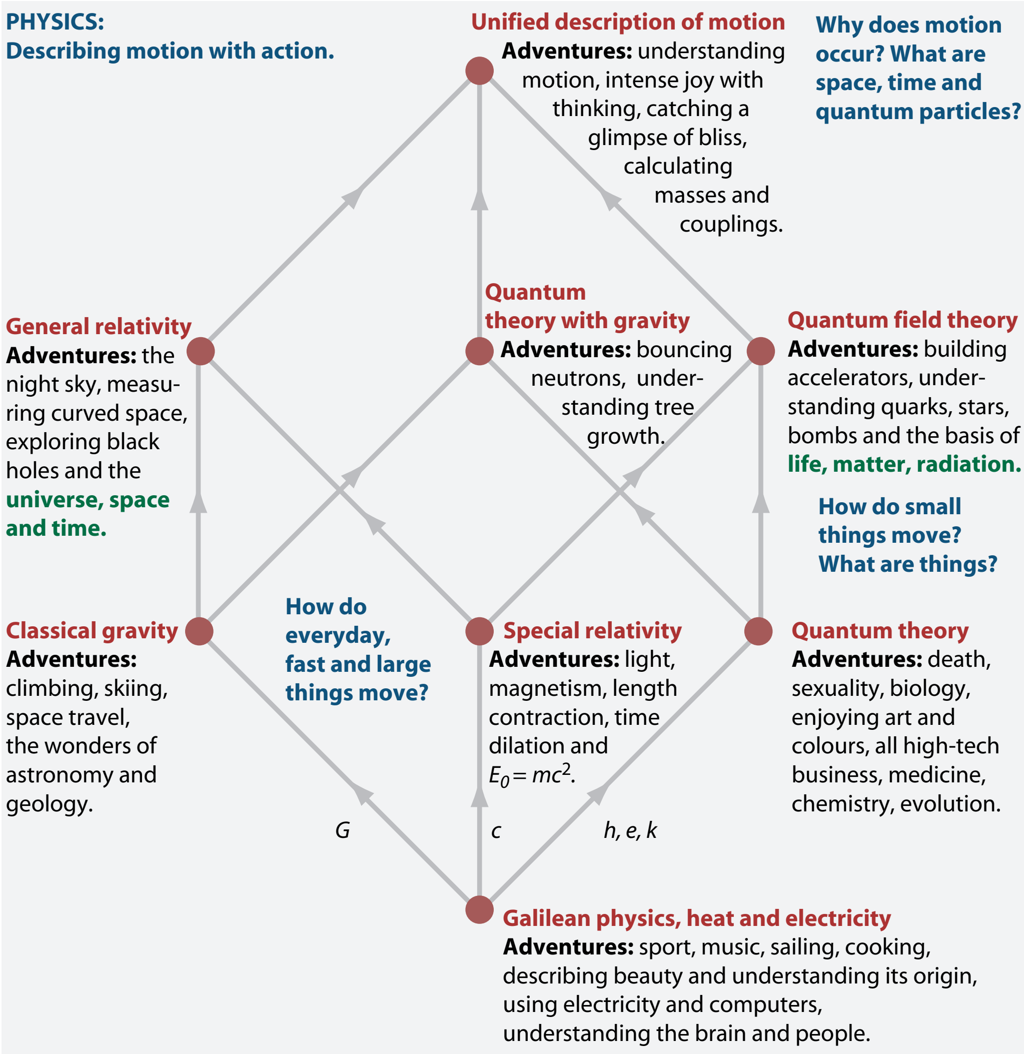

FIGURE 1 A complete map of physics: the connections are defined by the speed of light c, the gravitational constant G, the Planck constant h, the Boltzmann constant k and the elementary charge e.

Advice for learners

In my experience as a teacher, there was one learning method that never failed to transform unsuccessful pupils into successful ones: if you read a book for study, summarize every section you read, *in your own words, aloud*. If you are unable to do so, read the section again. Repeat this until you can clearly summarize what you read in your own words, aloud. You can do this alone in a room, or with friends, or while walking. If you do this with everything you read, you will reduce your learning and reading time significantly. In addition, you will enjoy learning from good texts much more and hate bad texts much less. Masters of the method can use it even while listening to a lecture, in a low voice, thus avoiding to ever take notes.

Using this book

Text in green, as found in many marginal notes, marks a link that can be clicked in a pdf reader. Such green links are either bibliographic references, footnotes, cross references to other pages, challenge solutions, or pointers to websites.

Solutions and hints for *challenges* are given in the appendix. Challenges are classified as research level (r), difficult (d), standard student level (s) and easy (e). Challenges of type r, d or s for which no solution has yet been included in the book are marked (ny).

Feedback and support

This text is and will remain free to download from the internet. I would be delighted to receive an email from you at fb@motionmountain.net, especially on the following issues:

Challenge 1 s

— What was unclear and should be improved?
— What story, topic, riddle, picture or movie did you miss?
— What should be corrected?

In order to simplify annotations, the pdf file allows adding yellow sticker notes in Adobe Reader.

Alternatively, you can provide feedback on www.motionmountain.net/wiki. Help on the specific points listed on the www.motionmountain.net/help.html web page would be particularly welcome. All feedback will be used to improve the next edition. On behalf of all readers, thank you in advance for your input. For a particularly useful contribution you will be mentioned – if you want – in the acknowledgements, receive a reward, or both.

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A paper edition of this book, printed on demand and delivered by mail to any address, can be ordered at stores.lulu.com/motionmountain. But above all, enjoy the reading!

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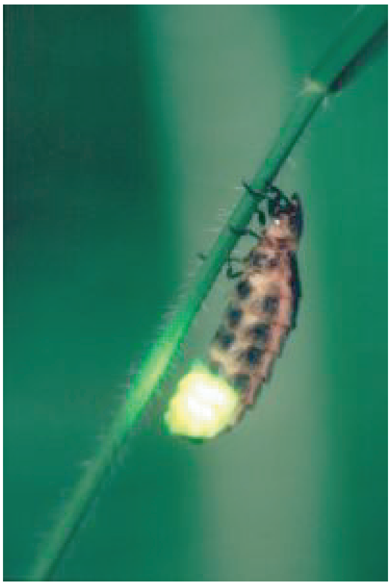

Quantum Theory:
The Smallest Change

In our quest to understand how things move,
we discover that there is a minimal change in nature,
implying that motion is fuzzy,
that boxes are never tight,
that matter is composed of elementary units,
and that light and interactions are streams of particles.
The minimal change explains why antimatter exists,
why particles are unlike gloves,
why copying machines do not exist,
that probabilities are reasonable,
and how all colours in nature are formed.

Chapter 1

MINIMUM ACTION – QUANTUM THEORY FOR POETS

> “Natura [in operationibus suis] non facit saltus.*”
> 15th century

Climbing Motion Mountain up to this point, we completed three legs. We came across Galileo’s mechanics (the description of motion for kids), then continued with Einstein’s relativity (the description of motion for science-fiction enthusiasts), and finally explored Maxwell’s electrodynamics (the description of motion for business people). These three classical descriptions of motion are impressive, beautiful and useful. However, they have a small problem: they are wrong. The reason is simple: none of them describes *life*.

Whenever we observe a flower or a butterfly, such as those of Figure 2, we enjoy the bright colours, the motion, the wild smell, the soft and delicate shape or the fine details of their symmetries. None of the three classical descriptions of nature can explain any of these properties; neither do they explain the impression that the flower makes on our senses. Classical physics can describe certain aspects of the impression, but it cannot explain their origins. For such an explanation, we need *quantum theory*. In fact, we will discover that in life, *every type of pleasure* is an example of quantum motion. Take any example of a pleasant situation:** for example, a beautiful evening sky, a waterfall, a caress, or a happy child. Classical physics is not able to explain it: the colours, shapes and sizes involved remain mysterious.

Challenge 2 s

In the early days of physics, this limitation was not seen as a shortcoming, because neither senses nor material properties were thought to be related to motion – and pleasure was not considered a serious subject of investigation for a respectable researcher. However, we have since learned that our senses of touch, smell and sight are primarily *detectors of motion*. Without motion, there would be no senses. Furthermore, all detectors are made of matter. During the exploration on electromagnetism we began to understand that all properties of matter are due to motions of charged constituents. Density, stiffness, colour, and all other material properties result from the electromagnetic behaviour of the Lego bricks of matter: namely, the molecules, the atoms and the electrons. Thus, the properties of matter are also consequences of motion. Moreover, we saw that these tiny constituents are *not* correctly described by classical electrodynamics. We even found

Vol. I, page 315

Ref. 1

* ‘Nature [in its workings] makes no jumps.’
** The photograph on page 12 shows a female glow worm, *Lampyris noctiluca*, as found in the United Kingdom (© John Tyler, www.johntyler.co.uk/gwfacts.htm).

FIGURE 2 Some examples of quantum machines (© Linda de Volder).

Vol. III, page 118 that light itself does not behave classically. Therefore the inability of classical physics to describe matter, light and the senses is indeed due to its intrinsic limitations.

In fact, every failure of classical physics can be traced back to a single, fundamental Ref. 2 discovery made in 1899 by Max Planck:*

▷ *In nature, actions smaller than* $\hbar = 1.06 \cdot 10^{-34}\,\mathrm{Js}$ *are not observed.*

All attempts to observe actions smaller than this fail.** In other words, in nature – as in a good film – there is always some action. The existence of a minimal action – the so-called

* Max Planck (1858–1947), professor of physics in Berlin, was a central figure in thermostatics. He discovered and named *Boltzmann's constant* k and the *quantum of action* h, often called Planck's constant. His introduction of the quantum hypothesis gave birth to quantum theory. He also made the works of Einstein known in the physical community, and later organized a job for him in Berlin. He received the Nobel Prize for physics in 1918. He was an important figure in the German scientific establishment; he also was one of the very few who had the courage to tell Adolf Hitler *face to face* that it was a bad idea to fire Jewish professors. (He got an outburst of anger as answer.) Famously modest, with many tragedies in his personal life, he was esteemed by everybody who knew him.

** In fact, this story is a slight simplification: the constant originally introduced by Planck was the (unreduced) constant $h = 2\pi\hbar$. The factor 2π leading to the final quantum principle was found somewhat later, by other researchers.

This somewhat unconventional, but didactically useful, approach to quantum theory is due to Niels Bohr. Ref. 3, Ref. 4 Nowadays, it is hardly ever encountered in the literature, despite its simplicity.

Niels Bohr (b. 1885 Copenhagen, d. 1962 Copenhagen) was one of the great figures of modern physics. A daring thinker and a polite man, he made Copenhagen University into the new centre of development of

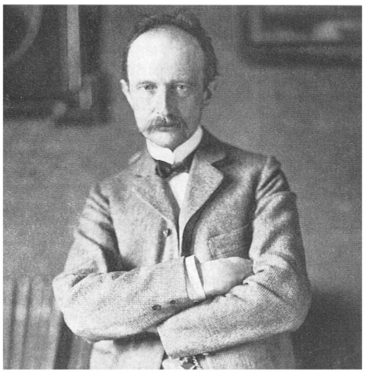

FIGURE 3 Max Planck (1858–1947)

FIGURE 4 Niels Bohr (1885–1962)

Challenge 3 s
quantum principle – is in complete contrast with classical physics. (Why?) However, it has passed an enormous number of experimental tests, many of which we will encounter in this part of our mountain ascent. Therefore, $\hbar$, which is pronounced 'aitch-bar', is called the *quantum of action*, or alternatively *Planck's constant*. Planck discovered the principle when studying the properties of incandescent light, i.e., light emanating from hot bodies. But the quantum principle also applies to motion of matter, and even, as we will see later, to motion of space-time.
Vol. III, page 118

The quantum principle states that no experiment can measure an action smaller than $\hbar$. For a long time, Einstein tried to devise experiments to overcome this limit. But he failed in all his attempts: nature does not allow it, as Bohr showed again and again. We recall that physical action is a measure for the *change* happening in a system. The quantum principle can thus rephrased as
Vol. I, page 199

▷ *In nature, a change smaller than* $\hbar = 1.06 \cdot 10^{-34}$ Js *cannot be observed.*

In physics – as in the theatre – action is a measure of the *change* occurring in a system. Therefore, a minimum action implies that *there is a minimum change in nature*. Thus the quantum of action would perhaps be better named the *quantum of change*. If we compare two observations, there will always be change between them.

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quantum theory, overshadowing Göttingen. He developed the description of the atom in terms of quantum theory, for which he received the 1922 Nobel Prize in Physics. He had to flee Denmark in 1943 after the German invasion, because of his Jewish background, but returned there after the war, continuing to attract the best physicists across the world.

TABLE 1 How to convince yourself and others that there is a minimum action, or minimum change $\hbar$ in nature. Compare this table with the two tables in volume II, that about maximum speed on page 23, and that about maximum force on page 97.

Issue	Method
The action value $\hbar$ is observer-invariant	check all observations
Local change or action values $< \hbar$ are not observed	check all observations
Change or action values $< \hbar$ are either non-local or not due to energy transport	check all observations
Local change or action values $< \hbar$ cannot be produced	check all attempts
Local change or action values $< \hbar$ cannot be imagined	solve all paradoxes
A smallest local change or action value $\hbar$ is consistent	1 – show that all consequences, however weird, are confirmed by observation
	2 – deduce quantum theory from it and check it

Can a minimum change really exist in nature? To accept the idea, we need to explore three points, detailed in Table 1. We need to show that no smaller change is *observed* in nature, that no smaller change *can* ever be observed, and show that *all consequences* of this smallest change, however weird they may be, apply to nature. In fact, this exploration constitues all of quantum physics. Therefore, these checks are all we do in the remaining of this part of our adventure. But before we explore some of the experiments that confirm the existence of a smallest change, we present some of its more surprising consequences.

The effects of the quantum of action on rest

Since action is a measure of change, a minimum observable action means that two successive observations of the same system always differ by at least $\hbar$. In every system, there is always *something* happening. As a consequence, in nature *there is no rest*. Everything moves, all the time, at least a little bit. *Natura facit saltus.** True, these jumps are tiny, as $\hbar$ is too small to be observable by any of our senses. But for example, the quantum of action implies that in a mountain – an archetypal 'system at rest' – all the atoms and electrons are continually buzzing around. Rest can be observed only macroscopically, and only as a long-time or many-particle average. Page 13

Since there is a minimum action for all observers, and since there is no rest, in nature *there is no perfectly straight or perfectly uniform motion*. Forget all you have learnt so far: Inertial motion is an approximation! An object can move in straight, uniform motion

* 'Nature makes jumps.'

only approximately, and only when observed over long distances or long times. We will see later that the more massive the object is, the better the approximation is. (Can you Challenge 4 s confirm this?) So *macroscopic* observers can still talk about space-time symmetries; and *special* relativity can thus be reconciled with quantum theory.

Also free fall, or motion along a geodesic, exists only as a long-time average. So *general* relativity, which is based on the existence of freely-falling observers, cannot be correct when actions of the order of $\hbar$ are involved. Indeed, the reconciliation of the quantum principle with general relativity – and thus with curved space – is a big challenge. (The solution is simple only for weak, everyday fields.) The issues involved are so mind-shattering that they form a separate, final, part of this mountain ascent. We thus explore situations without gravity first.

The consequences of the quantum of action for objects

Have you ever wondered why leaves are green? You probably know that they are green because they absorb blue (short-wavelength) and red (long-wavelength) light, while allowing green (medium-wavelength) light to be reflected. How can a system filter out the small and the large, and let the middle pass through? To do so, leaves must somehow *measure* the frequency. But we have seen that classical physics does not allow measurement of time (or length) intervals, as any measurement requires a measurement unit, and Vol. I, page 335 classical physics does not allow such units to be defined. On the other hand, it takes only a few lines to confirm that with the help of the quantum of action $\hbar$ (and the Boltzmann constant k, both of which Planck's discovered), fundamental units for *all* measurable quantities can be defined, including time and therefore frequency. (Can you find a combination of the speed of light c, the gravitational constant G and the quantum of action Challenge 5 s $\hbar$ that gives a time? It will only take a few minutes.)

When Planck saw that the quantum of action allowed to define all units in nature, he was as happy as a child; he knew straight away that he had made a fundamental discovery, even though (in 1899) quantum theory did not yet exist. He even told his seven-year-old Ref. 5 son Erwin about it, while walking with him through the woods around Berlin. Planck explained to his son that he had made a discovery as important as universal gravity. Indeed, Planck knew that he had found the key to understanding many of the effects that were then unexplained. In particular, without the quantum of action, colours could not exist: every colour is a quantum effect.*

Planck also realized that the quantum of action allows us to understand the *size* of Challenge 7 e all things. (Can you find the combination of c, G and $\hbar$ that yields a length?) With the quantum of action, it was finally possible to determine the maximum size of mountains, Vol. I, page 267 of trees and of humans. Planck knew that the quantum of action confirmed what Galileo had already deduced long before him: that sizes are due to fundamental, minimal scales in nature.

The size of objects is related to the size of atoms. In turn, the size of atoms is a direct consequence of the quantum of action. Can you derive an approximation for the size of atoms, knowing that it is given by the motion of electrons of mass m_e and charge e, Challenge 8 s constrained by the quantum of action? This connection, a simple formula, was discovered

* In fact, it is also possible to define all measurement units in terms of the speed of light c, the gravitational Challenge 6 s constant G and the electron charge e. Why is this not fully satisfactory?

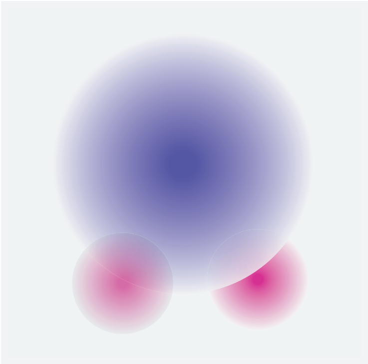

FIGURE 5 An artist's impression of a water molecule.

in 1910 by Arthur Erich Haas, 15 years before quantum theory was formulated. At the time, he was widely ridiculed. Nowadays, the formula is found in all textbooks.*

In determining the size of atoms, the quantum of action has another important consequence: Gulliver's travels are impossible. There are no tiny people and no giant ones. Classically, nothing speaks against the idea; but the quantum of action prevents it. Can you supply the detailed argument?

Challenge 9 s

But if rest does not exist, how can *shapes* exist? Any shape of everyday life, including that of a flower, is the result of body parts remaining *at rest* with respect to each other. Now, all shapes result from interactions between the constituents of matter, as shown most clearly in the shapes of molecules. But how can a molecule, such as the water molecule H_2O, shown in Figure 5, have a shape? In fact, a molecule does not have a *fixed* shape, but its shape fluctuates, as would be expected from the quantum of action. Despite the fluctuations, every molecule does have an *average* shape, because different angles and distances correspond to different energies. Again, these average length and angle values only exist because the quantum of action yields fundamental length scales in nature. Without the quantum of action, there would be *no* shapes in nature.

The *mass* of an object is also a consequence of the quantum of action, as we will see later on. Since all material properties – such as density, colour, stiffness or polarizability – are defined as combinations of length, time and mass units, we find that all material properties arise from the quantum of action.

In short, the quantum of action determines the size, shape, colour, mass, and all other properties of objects, from stones to whipped cream. Measurements are only possible at all because of the existence of the quantum of action.

Why 'quantum'?

Quantum effects surround us on all sides. However, since the quantum of action is so small, its effects on *motion* appear mostly, but not exclusively, in *microscopic* systems. The study of such systems was called *quantum mechanics* by Max Born, one of the major

* Before the discovery of $\hbar$, the only simple length scale for the electron was the combination $e^2/(4\pi\varepsilon_0 m_e c^2) \approx 3\,\mathrm{fm}$; this is ten thousand times smaller than an atom. We also note that any length scale containing e is a quantum effect, and not a classical length scale, because e is the quantum of electric charge.

TABLE 2 Some small systems in motion and the observed action values for their changes.

System and change	Action	Motion
Light		
Smallest amount of light absorbed by a coloured surface	$1\,\hbar$	quantum
Smallest impact when light reflects from mirror	$2\,\hbar$	quantum
Smallest consciously visible amount of light	*c.* $5\,\hbar$	quantum
Smallest amount of light absorbed in flower petal	$1\,\hbar$	quantum
Blackening of photographic film	*c.* $3\,\hbar$	quantum
Photographic flash	*c.* $10^{17}\,\hbar$	classical
Electricity		
Electron ejected from atom or molecule	*c.* 1–$2\,\hbar$	quantum
Electron extracted from metal	*c.* 1–$2\,\hbar$	quantum
Electron motion inside microprocessor	*c.* 2–$6\,\hbar$	quantum
Signal transport in nerves, from one molecule to the next	*c.* $5\,\hbar$	quantum
Current flow in lightning bolt	*c.* $10^{38}\,\hbar$	classical
Materials		
Tearing apart two neighbouring iron atoms	*c.* 1–$2\,\hbar$	quantum
Breaking a steel bar	*c.* $10^{35}\,\hbar$	classical
Basic process in superconductivity	$1\,\hbar$	quantum
Basic process in transistors	$1\,\hbar$	quantum
Basic magnetization process	$1\,\hbar$	quantum
Chemistry		
Atom collision in liquid at room temperature	$1\,\hbar$	quantum
Shape oscillation of water molecule	*c.* $1 - 5\,\hbar$	quantum
Shape change of molecule, e.g. in chemical reaction	*c.* $1 - 5\,\hbar$	quantum
Single chemical reaction curling a hair	*c.* $2 - 6\,\hbar$	quantum
Tearing apart two mozzarella molecules	*c.* $300\,\hbar$	quantum
Smelling one molecule	*c.* $10\,\hbar$	quantum
Burning fuel in a cylinder in an average car engine explosion	*c.* $10^{37}\,\hbar$	classical
Life		
Air molecule hitting eardrum	*c.* $2\,\hbar$	quantum
Smallest sound signal detectable by the ear	Challenge 10 ny	
Single DNA duplication step during cell division	*c.* $100\,\hbar$	quantum
Ovule fertilization	*c.* $10^{14}\,\hbar$	classical
Smallest step in molecular motor	*c.* $5\,\hbar$	quantum
Sperm motion by one cell length	*c.* $10^{15}\,\hbar$	classical
Cell division	*c.* $10^{19}\,\hbar$	classical
Fruit fly's wing beat	*c.* $10^{24}\,\hbar$	classical
Person walking one body length	*c.* $2 \cdot 10^{36}\,\hbar$	classical
Nuclei and stars		
Nuclear fusion reaction in star	*c.* $1 - 5\,\hbar$	quantum
Explosion of gamma-ray burster	*c.* $10^{80}\,\hbar$	classical

FIGURE 6 Max Born (1882–1970)

contributors to the field.* Later, the term *quantum theory* became more popular.

Quantum theory arises from the existence of *minimum* measurable values in nature, generalizing the idea that Galileo had in the seventeenth century. As discussed in detail Vol. I, page 265 earlier on, it was Galileo's insistence on 'piccolissimi quanti' – smallest quanta – of matter that got him into trouble. We will soon discover that the idea of a smallest change is necessary for a precise and accurate description of matter and of nature as a whole. Therefore Born adopted Galileo's term for the new branch of physics. The English language adopted the Latin singular 'quantum' instead of the Italian plural 'quanti' or the German plural 'Quanten'.

Note that the term 'quantum' does *not* imply that all measurement values are *multiples* of a smallest one: this is so only in a few cases.

Quantum theory is the description of microscopic motion. But when is quantum theory necessary? Table 2 shows that all processes on atomic and molecular scales, including biological and chemical processes, involve actions with values that are near the quantum of action. So do processes of light emission and absorption. These phenomena can *only* be described with quantum theory.

Table 2 also shows that the term 'microscopic' has a different meaning for a physicist and for a biologist. For a biologist, a system is microscopic if it requires a microscope for its observation. For a physicist, a system is microscopic if its characteristic action is of the order of the quantum of action. In other words, for a physicist a system is microscopic if it is *not* visible in a (light) microscope. To increase the confusion, some quantum physicists nowadays call their own class of microscopic systems 'mesoscopic', while others call their

* Max Born (b. 1882 Breslau, d. 1970 Göttingen) first studied mathematics, then turned to physics. A professor at Göttingen University, he made the city one of the world centres of physics. He developed quantum mechanics with his assistants Werner Heisenberg and Pascual Jordan, and then applied it to scattering, solid-state physics, optics and liquids. He was the first to understood that the state function describes a Ref. 6 probability amplitude. Born and Wolf together wrote what is still the main textbook on optics.

Born attracted to Göttingen the most brilliant talents of the time, receiving as visitors Hund, Pauli, Nordheim, Oppenheimer, Goeppert-Mayer, Condon, Pauling, Fock, Frenkel, Tamm, Dirac, Mott, Klein, Heitler, London, von Neumann, Teller, Wigner, and dozens of others. Being Jewish, Born lost his job in 1933; he emigrated, and became professor in Edinburgh, where he stayed for 20 years. Physics at Göttingen never recovered from this loss. For his elucidation of the meaning of the wave function he received the 1954 Nobel Prize in Physics.

FIGURE 7 Werner Heisenberg (1901–1976)

systems 'nanoscopic'. Both terms were introduced only to attract attention and funding: they are useless.

The effect of the quantum of action on motion

There is another way to characterize the difference between a microscopic, or quantum, system and a macroscopic, or classical, one. A minimum action implies that the difference between the action values S of two successive observations of the same system, a time Δt apart, is limited. One has

$$S(t+\Delta t) - S(t) = (E+\Delta E)(t+\Delta t) - Et = E\Delta t + t\Delta E + \Delta E\Delta t \geqslant \frac{\hbar}{2}\,. \tag{1}$$

The factor 1/2 arises from averaging. Now the values of the energy E and time t – but not of ΔE or Δt – can be set to zero if we choose a suitable observer. Thus, the existence of a quantum of action implies that in any system the evolution is constrained by

$$\Delta E\Delta t \geqslant \frac{\hbar}{2}\,, \tag{2}$$

where E is the energy of the system and t is its age, so that ΔE is the change of energy and Δt is the time between two successive observations.

Challenge 11 e

By a similar reasoning , we find that for any system the position and momentum are constrained by

$$\Delta x\Delta p \geqslant \frac{\hbar}{2}\,, \tag{3}$$

where Δx is the indeterminacy in position and Δp is the indeterminacy in momentum. These two famous relations were called *indeterminacy relations* by their discoverer, Werner Heisenberg.* In English they are often called 'uncertainty relations'; however,

* It is often said that the indeterminacy relation for energy and time has a different weight from that for momentum and position. This is a wrong idea, propagated by the older generation of physicists, which has survived through many textbooks for over 70 years. Just forget it. It is essential to remember that all four quantities appearing in the inequalities describe the *internal* properties of the system. In particular, t is a

this term is incorrect. The quantities are not uncertain, but *undetermined*: because of the quantum of action, system observables have *no* definite value. There is no way to ascribe a precise value to momentum, position, or any other observable of a quantum system.

Any system whose indeterminacy is of the order of $\hbar$ is a quantum system; if the indeterminacy product is much larger, the system is classical, and classical physics is sufficient for its description. So even though classical physics assumes that there are *no* measurement indeterminacies in nature, a system is *classical* only if its indeterminacies are *large* compared to the minimum possible ones!

In short, quantum theory is necessary whenever we try to measure some quantity as precisely as possible. In fact, every measurement is itself a quantum process. Therefore, measurement precision is limited. The quantum of action, through the indeterminacy relations, shows that *motion cannot be observed to infinite precision*. In other words, the microscopic world is *fuzzy*. This fact has many important consequences and many strange ones. For example, if motion cannot be observed with infinite precision, the very concept of motion needs to be handled with great care, as it cannot be applied in certain situations. In a sense, the rest of our quest is just an exploration of the implications of this result.

In fact, as long as space-time is *flat*, it turns out that we *can* retain the concept of motion to describe observations, provided we remain aware of the limitations implied by the quantum principle.

Quantum limits and surprises

The quantum of action, with its implied fuzziness, implies the existence of short-time deviations from energy, momentum and angular-momentum conservation in microscopic systems. Vol. I, page 191 In the first part of our mountain ascent, we realized that any type of nonconservation implies the existence of *surprises* in nature. Well, here are some of them.

Since precisely uniform motion does not exist, a system moving in one dimension only – such as the hand of a clock – always has the possibility of moving a bit in the opposite direction, thus leading to incorrect readings. Indeed, quantum theory predicts that clocks have essential limitations, and that *perfect clocks do not exist*.

It is also impossible to avoid that an object makes small displacement sideways. In fact, quantum theory implies that, strictly speaking, *neither uniform nor one-dimensional motion exists*.

Quantum limitations apply also to metre rules. It is impossible to ensure that the rule is completely at rest with respect to the object being measured. Thus the quantum of

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time variable deduced from changes observed *inside* the system, and not the time coordinate measured by an outside clock; similarly, the position x is not the external space coordinate, but the position characterizing the system. Ref. 7

Werner Heisenberg (1901–1976) was an important German theoretical physicist and an excellent table-tennis and tennis player. In 1925, as a young man, he developed, with some help from Max Born and Pascual Jordan, the first version of quantum theory; from it he deduced the indeterminacy relations. For these achievements he received the Nobel Prize for physics in 1932. He also worked on nuclear physics and on turbulence. During the Second World War, he worked on the German nuclear-fission programme. After the war, he published several successful books on philosophical questions in physics, slowly turned into a crank, and tried unsuccessfully – with some half-hearted help from Wolfgang Pauli – to find a unified description of nature based on quantum theory, the 'world formula'.

action implies again, on the one hand, that measurements are possible, and on the other hand, that their *accuracy is limited*.

It also follows from the quantum of action that any inertial or freely-falling observer must be *large*, as only large systems approximate inertial motion. *An observer cannot be microscopic.* If humans were not macroscopic, they could neither observe nor study motion.

Because of the finite accuracy with which microscopic motion can be observed, faster-than-light motion is possible in the microscopic domain! Quantum theory thus predicts *tachyons*, at least over short time intervals. For the same reason, *motion backwards in time* is possible over microscopic times and distances. In short, a quantum of action implies the existence of microscopic time travel. However, this remains impossible in the macroscopic domain, such as everyday life.

But there is more: the quantum of action implies that *there is no permanence in nature*. Imagine a moving car suddenly disappearing for good. In such a situation, neither momentum nor energy would be conserved. The action change for such a disappearance is large compared to $\hbar$, so that its observation would contradict even classical physics – as you may wish to check. However, the quantum of action allows a *microscopic* particle, such as an electron, to disappear for a *short* time, provided it reappears afterwards.

Challenge 12 s

The quantum of action also implies that *the vacuum is not empty*. If one looks at empty space twice, the two observations being separated by a tiny time interval, some energy will be observed the second time. If the time interval is short enough, then because of the quantum of action, matter particles will be observed. Indeed, particles can appear anywhere from nowhere, and disappear just afterwards: the action limit requires it. In summary, nature exhibits short-term appearance and disappearance of matter. In other words, the classical idea of an *empty* vacuum is correct only when the vacuum is observed over a *long* time.

The quantum of action implies that compass needles cannot work. If we look twice in quick succession at a compass needle, or even at a house, we usually observe that it stays oriented in the same direction. But since physical action has the same dimensions as angular momentum, a minimum value for action implies a minimum value for angular momentum. Even a macroscopic object has a minimum value for its rotation. In other words, quantum theory predicts that *everything rotates*. Something can be non-rotating only approximately, when observations are separated by long time intervals.

Challenge 13 e

For microscopic systems, the quantum limits on rotation have specific effects. If the rotation angle can be observed – as for molecules – the system behaves like a macroscopic object: its position and orientation are fuzzy. But for a system whose rotation angle cannot be observed, the quantum of action limits the angular momentum to multiples of $\hbar/2$. In particular, all microscopic bound systems – such as molecules, atoms, or nuclei – contain rotational motion and rotating components.

Transformation, life and Democritus

At the beginning of our adventure, we mentioned that the Greeks distinguished three types of changes: transport, growth, and transformation. We also mentioned that Democritus had deduced that all these types of changes – including life and death – were in fact the same, and due to the motion of atoms. The quantum of action makes exactly this

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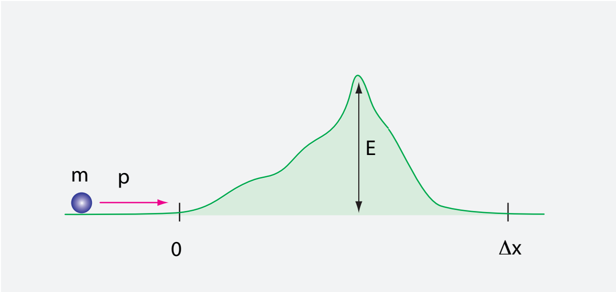

FIGURE 8 Hills are never high enough.

point.

First of all, a minimum action implies that cages in zoos are dangerous and banks are not safe. A cage is a feature that it needs a lot of energy to overcome. Mathematically, the wall of a cage is an energy hill, like the one shown in Figure 8. Imagine that a particle with momentum p approaches one side of the hill, which is assumed to have width Δx.

In everyday life – and thus in classical physics – the particle will never be observed on the other side of the hill if its kinetic energy $p^2/2m$ is *less* than the height E of the hill. But imagine that the missing momentum to overcome the hill, $\Delta p = \sqrt{2mE} - p$, satisfies $\Delta x \Delta p \leqslant \hbar/2$. The particle will have the possibility to overcome the hill, despite its insufficient energy. The quantum of action thus implies that a hill of width

$$\Delta x \leqslant \frac{\hbar/2}{\sqrt{2mE} - p} \qquad (4)$$

is *not* an obstacle to a particle of mass m. But this is not all. Since the value of the particle momentum p is itself uncertain, a particle can overcome the hill even if the hill is *wider* than the value (4) – although the broader it is, the lower the probability will be. So any particle can overcome *any* obstacle. This is called the *tunnelling effect*, for obvious reasons.

In short, the minimum-action principle implies that there are no tight boxes in nature. Thanks to the tunnelling effect, *matter is not impenetrable*, in contrast to everyday, classical observation. Can you explain why lion cages work *despite* the quantum of action?

Challenge 14 s

By the way, the quantum of action also implies that a particle with a kinetic energy greater than the energy height of a hill can be reflected by the hill. Classically, this is impossible. In quantum theory, the feat is possible, because the wave function does not vanish at the location of the hill; sloppily speaking, the wave function is non-zero inside the hill. It thus will be also non-zero behind the hill. As a result, quantum systems can penetrate or 'tunnel' through hills.

The minimum-action principle also implies that bookshelves are dangerous. Why? Shelves are obstacles to motion. A book on a shelf is in the same situation as the mass in Figure 9: the mass is surrounded by energy hills hindering its escape to the outer, lower-energy world. But thanks to the tunnelling effect, escape is always possible. The same picture applies to a branch of a tree, a nail in a wall, or anything attached to anything else. Things can never be permanently fixed together. In particular, we will discover that

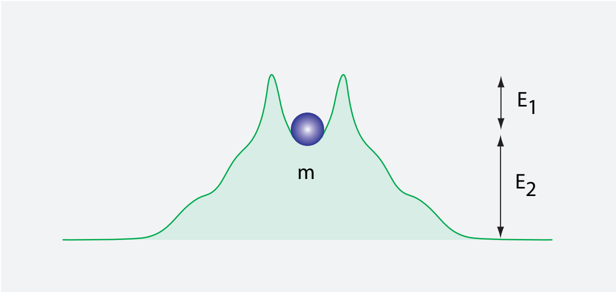

FIGURE 9 Leaving enclosures.

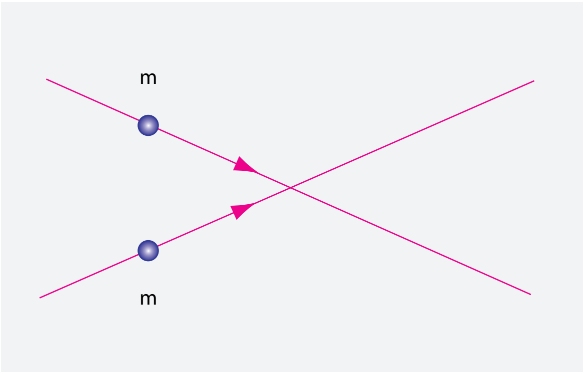

FIGURE 10 Identical objects with crossing paths.

every example of light emission – even radioactivity – results from this effect. The quantum of action thus implies that *decay is part of nature*. Note that decay often appears in everyday life, under a different name: *breaking*. In fact, all breakages require the quantum Ref. 8 of action for their description. Obviously, the cause of breaking is often classical, but the *mechanism* of breaking is always quantum. Only objects that obey quantum theory can break. In short, there are no stable excited systems in nature. For the same reason, by the Challenge 15 ny way, no memory can be perfect. (Can you confirm this?)

Taking a more general view, *ageing* and *death* also result from the quantum of action. Death, like ageing, is a composition of breaking processes. Breaking is a form of decay, and is due to tunnelling. Death is thus a quantum process. Classically, death does not Challenge 16 s exist. Might this be the reason why so many people believe in immortality or eternal youth?

We will also discover that the quantum of action is the reason for the importance of the action observable in classical physics. In fact, the existence of a *minimal* action is the reason for the *least*-action principle of classical physics.

A minimum action also implies that matter cannot be continuous, but must be composed of smallest entities. Indeed, any flow of a truly continuous material would contra- Challenge 17 s dict the quantum principle. Can you give the precise argument? Of course, at this point in our adventure, the non-continuity of matter is no longer a surprise. But the quantum of action implies that even *radiation* cannot be continuous. As Albert Einstein was the first to state clearly, light is made of quantum particles.

More generally, the quantum of action implies that in nature *all flows and all waves are*

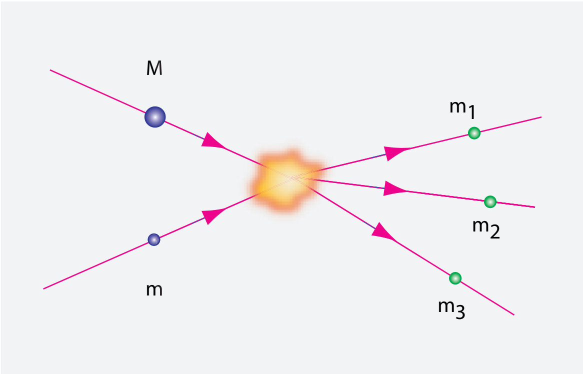

FIGURE 11 Transformation through reaction.

made of microscopic particles. The term 'microscopic' (or 'quantum') is essential, as such particles do *not* behave like little stones. We have already encountered several differences, and we will encounter others shortly. For these reasons, there should be a special name for microscopic particles; but so far all proposals, of which *quanton* is the most popular, have failed to catch on.

The quantum of action has several strange consequences for microscopic particles. Take two such particles with the same mass and composition. Imagine that their paths cross, and that at the crossing they approach each other very closely, as shown in Figure 10. A minimum action implies that in such a situation, if the distance becomes small enough, the two particles can switch roles, without anybody being able to avoid, or notice, it. Thus, in a volume of gas it is *impossible* – thanks to the quantum of action – to follow particles moving around and to say which particle is which. Can you confirm this deduction, and specify the conditions, using the indeterminacy relations? (Challenge 18 s) In summary, in nature *it is impossible to distinguish between identical particles*. Can you guess what happens in the case of light? (Challenge 19 s)

But matter deserves still more attention. Imagine again two particles – even two different ones – approaching each other very closely, as shown in Figure 11. We know that if the approach distance gets small, things get fuzzy. Now, the minimum-action principle makes it possible for something to happen in that small domain as long as resulting outgoing products have the same total linear momentum, angular momentum and energy as the incoming ones. Indeed, ruling out such processes would imply that arbitrarily small actions could be observed, thus eliminating nature's fuzziness, as you may wish to check for yourself. (Challenge 20 e) In short, *a minimum action allows transformation of matter*. One also says that the quantum of action allows particle *reactions*. In fact, we will discover that *all* kinds of reactions in nature, including breathing, digestion, and all other chemical and nuclear reactions, are due just to the existence of the quantum of action.

One type of process that is especially dear to us is *growth*. The quantum of action implies that all growth happens in small steps. Indeed, all growth processes in nature are quantum processes.

Above all, as mentioned already, the quantum of action explains life. Only the quantum of action makes reproduction and heredity possible. In short, birth, sexuality and death are consequences of the quantum of action.

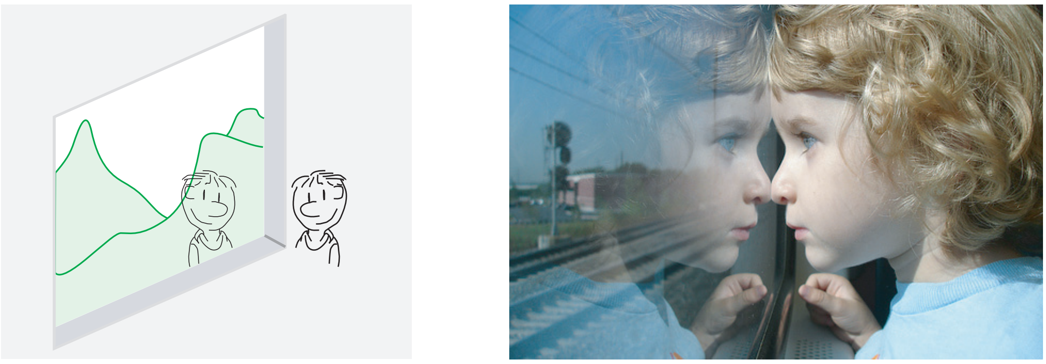

FIGURE 12 A famous quantum effect: how do train windows manage to show two superimposed images? (photo © Greta Mansour).

So Democritus was both right and wrong. He was right in deducing fundamental constituents for matter and radiation. He was right in unifying all change in nature – from transport to transformation and growth – as motion of particles. But he was wrong in assuming that the small particles behave like stones.

Randomness – a consequence of the quantum of action

What happens if we try to measure a change smaller than the quantum of action? Nature has a simple answer: we get *random results*. If we build an experiment that tries to produce a change or action of the size of a quarter of the quantum of action, the experiment will produce a change of *one* quantum of action in a quarter of the cases, and *no* change in three quarters of the cases, thus giving an *average* of one quarter of $\hbar$.

The quantum of action leads to randomness at microscopic level. This can be seen also in the following way. Because of the indeterminacy relations, it is impossible to obtain definite values for both the momentum and the position of a particle. Obviously, this is also impossible for the individual components of an experimental set-up or an observer. Therefore, initial conditions – both for a system and for an experimental set-up – cannot be exactly duplicated. A minimum action thus implies that whenever an experiment on a microscopic system is performed twice, the outcomes will (usually) be different. The outcomes could only be the same if both the system and the observer were in exactly the same configuration each time. However, because of the second principle of thermodynamics and because of the quantum of action, this is impossible. Therefore, *microscopic systems behave randomly*. Obviously, there will be some *average* outcome; but in all cases, microscopic observations are probabilistic. Many find this conclusion of quantum theory the most difficult to swallow. The quantum of action implies that the behaviour of quantum systems is strikingly different from that of classical systems. But the conclusion is unavoidable: nature behaves randomly.

Can we observe randomness in everyday life? Yes. Every *window* proves that nature behaves randomly on a microscopic scale. Everybody knows that one can use a train window either to look at the *outside* landscape or, by concentrating on the reflected image, to observe some interesting person *inside* the carriage. In other words, glass reflects some of the light particles and lets some others pass through. More precisely, glass reflects a

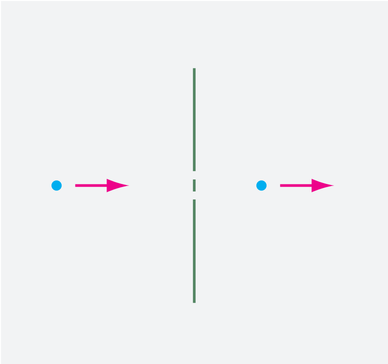

FIGURE 13 A particle and a screen with two nearby slits.

random selection of light particles; yet the average proportion is constant. Partial reflection is thus similar to the tunnelling effect. Indeed, the partial reflection of photons in glass is a result of the quantum of action. Again, the situation can be *described* by classical physics, but the precise amount of reflection cannot be *explained* without quantum theory. Without the quantum of action, train journeys would be much more boring.

Waves – a consequence of the quantum of action

The quantum of action implies an important result about the *paths* of particles. If a particle travels from one point to another, there is no way to say which path it has taken in between. Indeed, in order to distinguish between two possible, but slightly different, paths, actions smaller than $\hbar$ would have to be measured reliably. In particular, if a particle is sent through a screen with two sufficiently close slits, as illustrated in Figure 13, it is impossible to say which slit the particle passed through. This impossibility is fundamental.

We already know phenomena of motion for which it is not possible to say with precision how something moves or which path is taken behind two slits: *waves* behave in this way. All waves are subject to the indeterminacy relations

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$$\Delta\omega\Delta t \geqslant \frac{1}{2} \quad \text{and} \quad \Delta k \Delta x \geqslant \frac{1}{2} \,. \tag{5}$$

A wave is a type of motion described by a *phase* that changes over space and time. This turns out to hold for all motion. In particular, this holds for matter.

We saw above that quantum systems are subject to

$$\Delta E \Delta t \geqslant \frac{\hbar}{2} \quad \text{and} \quad \Delta p \Delta x \geqslant \frac{\hbar}{2} \,. \tag{6}$$

We are thus led to ascribe a frequency and a wavelength to a quantum system:

$$E = \hbar\omega \quad \text{and} \quad p = \hbar k = \hbar\frac{2\pi}{\lambda} \,. \tag{7}$$

The energy–frequency relation was deduced by Albert Einstein in 1905; it is found to be valid in particular for all examples of emission and absorption of *light*. In 1923 and 1924, Louis de Broglie* predicted that the relation should hold also for all quantum *matter* particles. The experimental confirmation came a few years later. (This is thus another example of a discovery that was made about 20 years too late.) In short, the quantum of action implies that *matter particles behave like waves*. In particular, the quantum of action implies the existence of interference for matter.

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Particles – a consequence of the quantum of action

The quantum of action, the smallest change, implies that flows cannot be arbitrary weak. This applies to *all* flows: in particular, it applies to rivers, solid matter flows, gas flows, light beams, energy flows, entropy flows, momentum flows, angular momentum flows, probability flows, signals of all kind, electrical charge flows, colour charge flows and weak charge flows.

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Water flows in rivers, like any other matter flow, cannot be arbitrary small: the quantum of action implies that there is a smallest matter flow in nature. Depending on the situation, the smallest matter flow is a molecule, an atom or a smaller particle. Indeed, the quantum of action is also at the origin of the observation of a smallest charge in electric current. Since all matter can flow, the quantum of action implies that *all matter has particle aspects*.

In the same way, the quantum of action, the smallest change, implies that light cannot be arbitrarily faint. There is a smallest illumination in nature; it is called a *photon* or a *light quantum*. Now, light is a wave, and the argument can be made for any other wave as well. In short, the quantum of action thus implies that *all waves have particle aspects*. This has been proved for light waves, water waves, X rays, sound waves, plasma waves, fluid whirls and any other wave type that has ever been observed. (Gravitational waves have not yet been observed; it is expected that their particle-like aspects, the *gravitons*, exist also in this case.)

In summary, the quantum of action states: *if it moves, it is made of quantum particles, or quantons*. Later on we will explore and specify the exact differences between a quantum particle and a small stone or a grain of sand. We will discover that matter quantons move differently, behave differently under rotation, and behave differently under exchange.

Quantum information

In computer science, the smallest unit of change is called a 'bit change'. The existence of a smallest change in nature implies that computer science – or information science – can be used to describe nature, and in particular quantum theory. This analogy has

* Louis de Broglie (b. 1892 Dieppe, d. 1987 Paris), French physicist and professor at the Sorbonne. The energy–frequency relation for light had earned Albert Einstein his Nobel Prize already in 1921. De Broglie expanded it to predict the wave nature of the electron (and of all other quantum matter particles): this was the essence of his doctoral thesis. The prediction was confirmed experimentally a few years later, in 1927. For the prediction of the wave nature of matter, de Broglie received the Nobel Prize for physics in 1929. Being an aristocrat, he did no more research after that. For example, it was Schrödinger who then wrote down the wave equation, even though de Broglie could equally have done so.

attracted much research in the past decades, and explored many interesting questions: Is there unlimited information storage possible? Can information be read out and copied completely? Can information be transmitted while keeping it secret? Can information transmission and storage be performed independently of noise? Can quantum physics be used to make new types of computers? So far, the answer to all these questions is negative; but the hope to change the situation is not dead yet.

The analogy between quantum theory and information science has limitations: information science can describe only the 'software' side of devices. For a physicist, the 'hardware' side of nature is central. The hardware of nature enters the description whenever the actual value $\hbar$ of the quantum of action must be introduced.

As we explore the similarities and differences between nature and information science, we will discover that the quantum of action implies that macroscopic physical systems cannot be copied – or 'cloned', as quantum theorists like to say. Nature does not allow copies of macroscopic objects. In other words, perfect copying machines do not exist. The quantum of action makes it impossible to gather and use all information in a way that allows production of a perfect copy.

The exploration of copying machines will remind us again that the precise order in which measurements are performed in an experiment matters. When the order of measurements can be reversed without affecting the net result, physicists speak of 'commutation'. The quantum of action implies that physical observables do not commute.

We will also find that the quantum of action implies that systems are not always independent, but can be *entangled*. This term, introduced by Erwin Schrödinger, describes one of the most absurd consequences of quantum theory. Entanglement makes everything in nature connected to everything else. Entanglement produces effects that *seem* (but are not) faster than light. Entanglement produces a (fake) form of non-locality. Entanglement also implies that trustworthy communication cannot exist.

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Ref. 9

We will also discover that *decoherence* is an ubiquitous process in nature that influences all quantum systems; it allows measurements on the one hand and makes quantum computers impossible on the other.

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Curiosities and fun challenges about the quantum of action

Even if we accept that no experiment performed so far contradicts the minimum action, we still have to check that the minimum action does not contradict reason. In particular, the minimum action must also be consistent with all *imagined* experiments. This is not self-evident.

* *

Angular momentum has the same dimensions as action. A smallest action implies that there is a smallest angular momentum in nature. How can this be, given that some particles have spin zero, i.e., have no angular momentum?

Challenge 21 s

* *

Could we have started the whole discussion of quantum theory by stating that there is a minimum angular momentum instead of a minimum action?

Challenge 22 s

* *

Niels Bohr, besides propagating the idea of a minimum action, was also an enthusiast of the so-called *complementarity principle*. This is the idea that certain pairs of observables of a system – such as position and momentum – have linked *precision*: if one of the pair is known to high precision, the other is necessarily known with low precision. Can you deduce this principle from the minimum action?

Challenge 23 s

* *

When electromagnetic fields come into play, the value of the action (usually) depends on the choice of the vector potential, and thus on the choice of gauge. We saw in the section on electrodynamics that a suitable choice of gauge can change the value of the action by adding or subtracting any desired amount. Nevertheless, there is a smallest action in nature. This is possible, because in quantum theory, physical gauge changes cannot add or subtract any amount, but only multiples of *twice* the minimum value. Thus they do not allow us to go below the minimum action.

* *

Adult plants stop growing in the dark. Without light, the reactions necessary for growth cease. Can you show that this is a quantum effect, not explainable by classical physics?

Challenge 24 s

* *

Most quantum processes in everyday life are electromagnetic. Can you show that the quantum of action must also hold for nuclear processes, i.e., for processes that are not electromagnetic?

Challenge 25 s

* *

Challenge 26 s

Is the quantum of action independent of the observer, even near the speed of light? This question was the reason why Planck contacted the young Einstein, inviting him to Berlin, thus introducing him to the international physics community.

* *

The quantum of action implies that tiny people, such as *Tom Thumb*, cannot exist. The quantum of action implies that fractals cannot exist in nature. The quantum of action implies that '*Moore's law*' of semiconductor electronics, which states that the number of transistors on a chip doubles every two years, cannot be correct. Why not?

Challenge 27 ny

* *

Take a horseshoe. The distance between the two ends is not fixed, since otherwise their position and velocity would be known at the same time, contradicting the indeterminacy relation. Of course, this reasoning is also valid for any other solid object. In short, both quantum mechanics and special relativity show that rigid bodies do not exist, albeit for different reasons.

The dangers of buying a can of beans

Another way to show the absurd consequences of quantum theory is given by the ultimate product warning, which according to certain well-informed lawyers should be printed on every can of beans and on every product package. It shows in detail how deeply our human condition fools us.
Ref. 10

Warning: care should be taken when **looking** at this product:

- It emits heat radiation.
- Bright light has the effect to compress this product.

Warning: care should be taken when **touching** this product:

- Part of it could heat up while another part cools down, causing severe burns.

Warning: care should be taken when **handling** this product:

- This product consists of at least 99.999 999 999 999 % empty space.
- This product contains particles moving with speeds higher than one million kilometres per hour.
- Every kilogram of this product contains the same amount of energy as liberated by about one hundred nuclear bombs.*
- In case this product is brought in contact with antimatter, a catastrophic explosion will occur.
- In case this product is rotated, it will emit gravitational radiation.

Warning: care should be taken when **transporting** this product:

- The force needed depends on its velocity, as does its weight.
- This product will emit additional radiation when accelerated.
- This product attracts, with a force that increases with decreasing distance, every other object around, including its purchaser's kids.

Warning: care should be taken when **storing** this product:

- It is impossible to keep this product in a specific place and at rest at the same time.
- Except when stored underground at a depth of several kilometres, over time cosmic radiation will render this product radioactive.
- This product may disintegrate in the next 10^{35} years.
- It could cool down and lift itself into the air.
- This product warps space and time in its vicinity, including the storage container.
- Even if stored in a closed container, this product is influenced and influences all other objects in the universe, including your parents in law.
- This product can disappear from its present location and reappear at any random place in the universe, including your neighbour's garage.

* A standard nuclear warhead has an explosive yield of about 0.2 megatons (implied is the standard explosive trinitrotoluene or TNT), about thirteen times the yield of the Hiroshima bomb, which was 15 kilotonne. A megatonne is defined as 1 Pcal=4.2 PJ, even though TNT delivers about 5 % slightly less energy than this value. In other words, a megaton is the energy content of about 47 g of matter. That is less than a handful for most solids or liquids.
Ref. 11

Warning: care should be taken when **travelling away from** this product:

- It will arrive at the expiration date before the purchaser does so.

Warning: care should be taken when **using** this product:

- Any use whatsoever will increase the entropy of the universe.
- The constituents of this product are exactly the same as those of any other object in the universe, including those of rotten fish.

The impression of a certain paranoid side to quantum physics is purely coincidental.

A summary: quantum physics, the law and indoctrination

Don't all the deductions from the quantum of action presented so far look wrong, or at least crazy? In fact, if you or your lawyer made some of the statements on quantum physics in court, maybe even under oath, you might end up in prison! However, all the above statements are correct: they are all confirmed by experiment. And there are many more surprises to come. You may have noticed that, in the preceding examples, we have made no explicit reference to electricity, to the nuclear interactions, or to gravity. In these domains the surprises are even more astonishing. Observation of antimatter, electric current without resistance, the motion inside muscles, vacuum energy, nuclear reactions in stars, and maybe soon the boiling of empty space, will fascinate you as much as they have fascinated, and still fascinate, thousands of researchers.

In particular, the consequences of the quantum of action for the early universe are mind-boggling. Just try to explore for yourself its consequences for the big bang. Challenge 28 d Together, all these topics will lead us a long way towards the top of Motion Mountain. The consequences of the quantum of action are so strange, so incredible, and so numerous, that quantum physics can rightly be called the description of motion for *crazy* scientists. In a sense, this generalizes our previous definition of quantum physics as the description of motion related to pleasure.

Unfortunately, it is sometimes said that 'nobody understands quantum theory'. This is wrong. In fact, it is worse than wrong: it is indoctrination and disinformation. Dictatorships use indoctrination and disinformation to prevent people from making up their own mind and from enjoying life. But the consequences of the quantum of action can be understood and enjoyed by everybody. In order to do so, our first task on our way towards the top of Motion Mountain will be to use the quantum of action to study of our classical standard of motion: the motion of light.

> “Nie und nirgends hat es Materie ohne
> Bewegung gegeben, oder kann es sie geben.”
> Friedrich Engels, *Anti-Dühring*.*

Ref. 12 * 'Never and nowhere has matter existed, nor can it exist, without motion.' Friedrich Engels (1820–1895) was one of the theoreticians of Marxism, often also called Communism.

Chapter 2

LIGHT – THE STRANGE CONSEQUENCES OF THE QUANTUM OF ACTION

> “Alle Wesen leben vom Lichte,
> jedes glückliche Geschöpfe.”
> Friedrich Schiller, *Wilhelm Tell.**

What is the faintest lamp?

Since all the colours of materials are quantum effects, it becomes mandatory to study the properties of light itself. If there is a smallest change, then there should also be a smallest illumination in nature. This was already predicted in ancient Greece, for example by Epicurus (341–271 bce). He stated that light is a stream of little particles, so that the smallest possible illumination would be that due to a single light particle. Today, the particles are called *light quanta* or *photons*. Incredibly, Epicurus himself could have checked his prediction with an experiment. Ref. 13

Ref. 14 In the 1930s Brumberg and Vavilov found a beautiful way to check the existence of photons using the naked eye and a lamp. Our eyes do not allow us to *consciously* detect single photons, but Brumberg and Vavilov found a way to circumvent this limitation. In fact, the experiment is so simple that it could have been performed many centuries earlier; but nobody had had a sufficiently daring imagination to try it.

Brumberg and Vavilov constructed a small shutter that could be opened for time intervals of 0.1 s. From the other side, in a completely dark room, they illuminated the opening with extremely weak green light: about 200 aW at 505 nm, as shown in Figure 14. At that intensity, whenever the shutter opens, on average about 50 photons can pass. This is just the sensitivity threshold of the eye. To perform the experiment, they repeatedly looked into the open shutter. The result was simple but surprising. Sometimes they observed light, and sometimes they did not. Whether they did or did not was completely random. Brumberg and Vavilov gave the simple explanation that at low lamp powers, because of fluctuations, the number of photons is above the eye threshold half the time, and below it the other half. The fluctuations are random, and so the conscious detection of light is as well. This would *not* happen if light were a continuous stream: in that case, the eye would detect light at every opening of the shutter. (At higher light intensities, the percentage of non-observations quickly decreases, in accordance with the explanation given.) In short, light is made of photons. Nobody knows how the theory of light would have developed if this simple experiment had been performed 100 or even 2000 years earlier.

* ‘From light all beings live, each fair-created thing.’ Friedrich Schiller (b. 1759 Marbach, d. 1805 Weimar), German poet, playwright and historian.

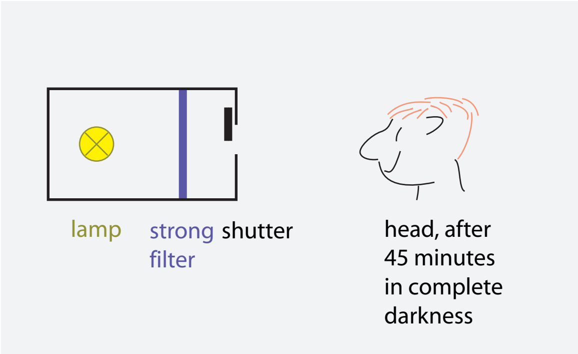


FIGURE 14 How to experience single photon effects (see text).

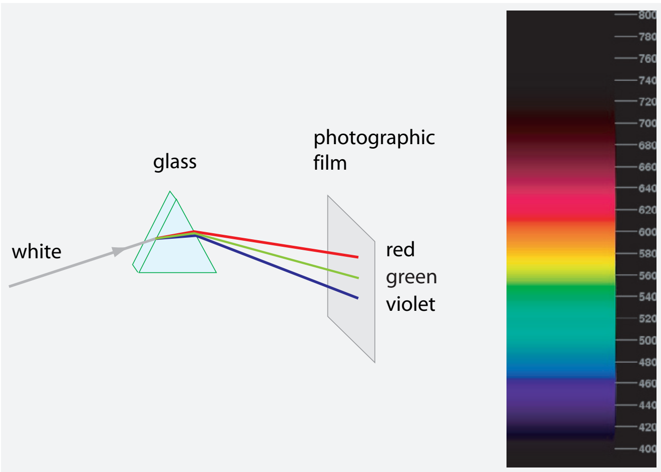


FIGURE 15 How does a white-light spectrum appear at extremely long screen distances? (The short-screen-distance spectrum shown, © Andrew Young, is optimized for CRT display, not for colour printing, as explained on mintaka.sdsu.edu/GF/explain/optics/rendering.html.)

The detection of photons becomes clearer if we use devices to help us. A simple way is to start with a screen behind a prism illuminated with white light, as shown in Figure 15. The light is split into colours. As the screen is placed further and further away, the illumination intensity cannot become arbitrarily small, as that would contradict the quantum of action. To check this prediction, we only need some black-and-white photographic film. Film is blackened by daylight of any colour; it becomes dark grey at medium intensities and light grey at lower intensities. Looking at an extremely light grey film under the microscope, we discover that, even under uniform illumination, the grey shade is actually composed of black spots, arranged more or less densely. All these spots have the same size, as shown in Figure 16. This regular size suggests that a photographic film reacts to single photons. Detailed research confirms this conjecture; in the twentieth century, the producers of photographic films have elucidated the underlying mechanism in all its details.

Single photons can be detected most elegantly with electronic devices. Such devices can be photomultipliers, photodiodes, multichannel plates or rod cells in the eye; a selection is shown in Figure 17. Also these detectors show that low-intensity light does *not* produce a homogeneous colour: on the contrary, low-intensity produces a random pat-

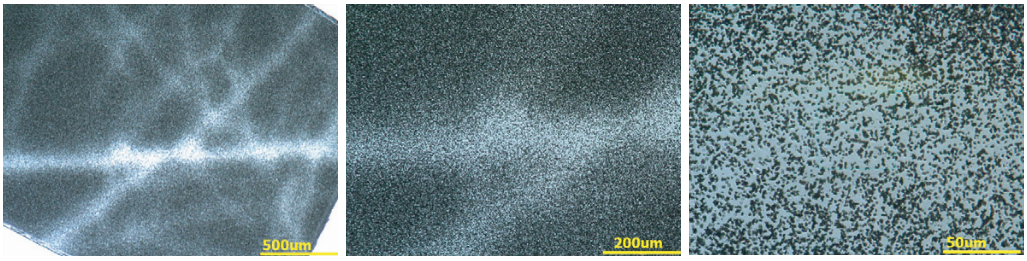


FIGURE 16 Exposed photographic film at increasing magnification (© Rich Evans).

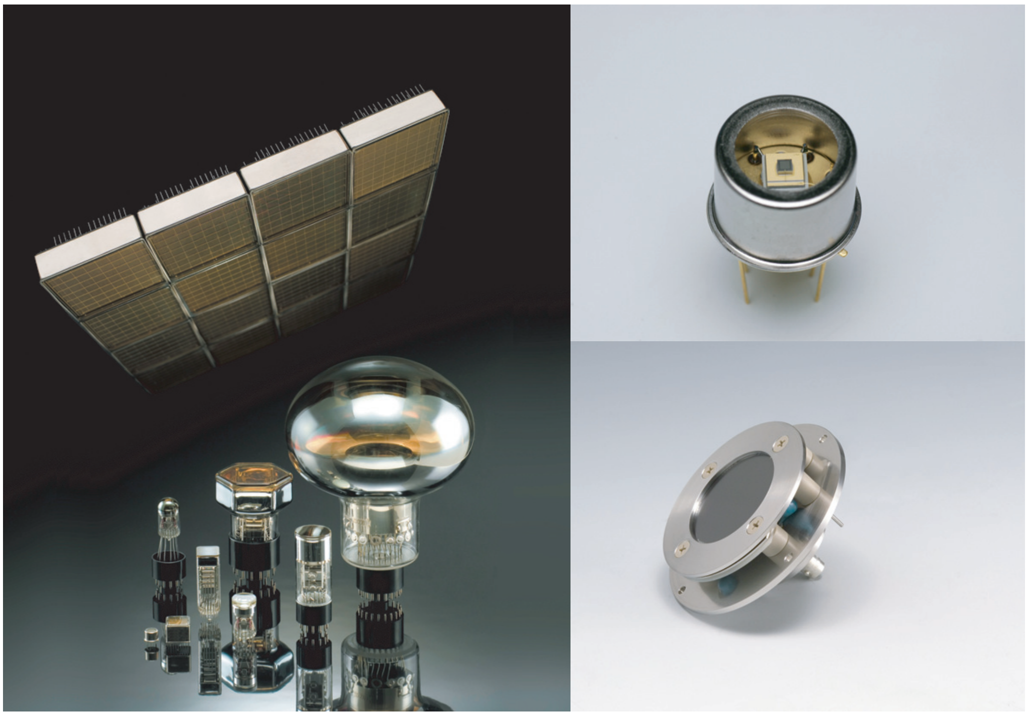


FIGURE 17 Detectors that allow photon counting: photomultiplier tubes (left), an avalanche photodiode (top right, *c.* 1 cm) and a multichannel plate (bottom right, *c.* 10 cm) (© Hamamatsu Photonics).

tern of equal spots, even when observing typical wave phenomena such as interference patterns, as shown in Figure 18.

The observation is general: whenever sensitive light detectors are constructed with the aim of 'seeing' as accurately as possible (and thus in environments as dark as possible), one finds that light manifests as a stream of *light quanta*. Nowadays they are usually called *photons*, a term that appeared in 1926. Light of low or high intensity corresponds to a stream with a small or large number of photons.

A particularly interesting example of a low-intensity source of light is a single atom. Atoms are tiny spheres. When atoms radiate light or X-rays, the radiation should be emitted as a spherical wave. But in all experiments, the light emitted by an atom is *never* found to form a spherical wave, in contrast to what we might expect from everyday physics.

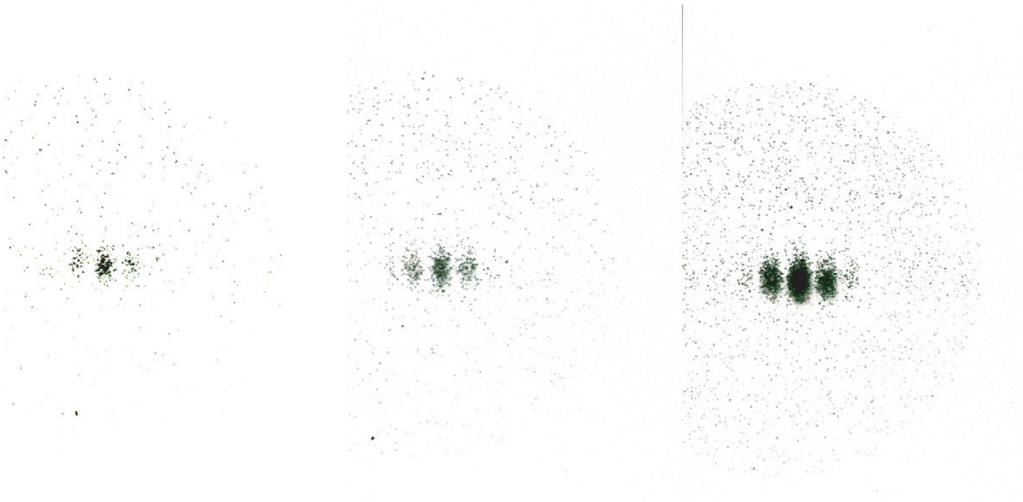


FIGURE 18 Light waves are made of particles: observation of photons – black spots in these negatives – in a low intensity double slit experiment, with exposure times of 1, 2 and 5 s, using an image intensifier (© Delft University of Technology).

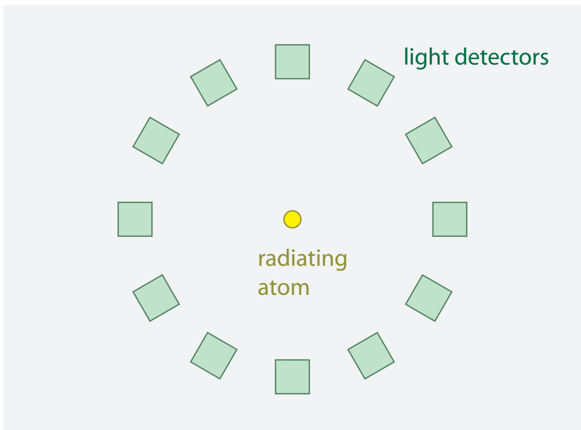


FIGURE 19 An atom radiating one photon triggers only one detector and recoils in only one direction.

Whenever a radiating atom is surrounded by many detectors, only a *single* detector is triggered. Only the average over many emissions and detections yields a spherical shape. The experiments shows clearly that partial photons cannot be detected.

All experiments in dim light show that the continuum description of light is incorrect. All these experiments thus prove directly that light is a stream of particles, as Epicurus had proposed in ancient Greece. More precise measurements confirm the role of the quantum of action: every photon leads to the *same* amount of change. All photons of the same frequency blacken a film or trigger a scintillation screen in the same way. The amount of change induced by a single photon is indeed the *minimal* amount of change that light can produce.

If there were no minimum action, light could be packaged into arbitrarily small amounts. But this is not possible. In other words, the classical description of light by

Vol. III, page 73 a *continuous* vector potential $A(t, x)$, or electromagnetic field $F(t, x)$, whose evolution is described by a principle of least action, is *wrong*. Continuous functions do not describe the observed particle effects. A modified description is required. The modification has to be significant only at low light intensities, since at high intensities the classical Lagrangian accurately describes all experimental observations.*

At which intensities does light cease to behave as a continuous wave? Human eyesight does not allow us to consciously distinguish single photons, although experiments show that the hardware of the eye is in principle able to do so. (Ref. 15) The faintest stars that can be seen at night produce a light intensity of about 0.6 nW/m^2. Since the pupil of the eye is small, and we are not able to see individual photons, photons must have energies smaller than 100 aJ. Brumberg and Vavilov's experiment yields an upper limit of around 20 aJ.

An exact value for the quantum of action found in light must be deduced from laboratory experiment. Today, recording and counting individual photons is a standard procedure. Photon counters are part of many spectroscopy set-ups, such as those used to measure tiny concentrations of materials. For example, they are used to detect drugs in human hair.

Photons

In general, all experiments show that a beam of light of frequency f or angular frequency ω, which determines its colour, is accurately described as a stream of photons, each with the same energy E given by

$$E = \hbar\, 2\pi f = \hbar\, \omega\ . \tag{8}$$

This relation was first deduced by Max Planck in 1899. He showed that for light, the smallest measurable action is given by the quantum of action $\hbar$. In summary, *colour* is a property of photons. A coloured light beam is a hailstorm of corresponding photons.

The value of *Planck's constant* can be determined from measurements of black bodies or other light sources. Vol. III, page 118 All such measurements coincide and yield

$$\hbar = 1.1 \cdot 10^{-34}\ \mathrm{Js}\ , \tag{9}$$

a value so small that we can understand why photons go unnoticed by humans. For example, a green photon with a wavelength of 555 nm has an energy of 0.37 aJ. (Challenge 29 e) Indeed, in normal light conditions the photons are so numerous that the continuum approximation for the electromagnetic field is highly accurate. In the dark, the insensitivity of the signal processing of the human eye – in particular the slowness of the light receptors – makes photon counting impossible. (Ref. 15) However, the eye is not far from the maximum possible sensitivity. From the numbers given above about dim stars, we can estimate that humans are able to see *consciously*, under ideal conditions, flashes of about half a dozen photons. (Challenge 30 ny)

Let us explore the other properties of photons. Above all, photons have no measurable (rest) mass and no measurable electric charge. Can you confirm this? (Challenge 31 s) In fact, experiments

* The transition from the classical case to the quantum case used to be called *quantization*. This concept, and the ideas behind it, are only of historical interest today.

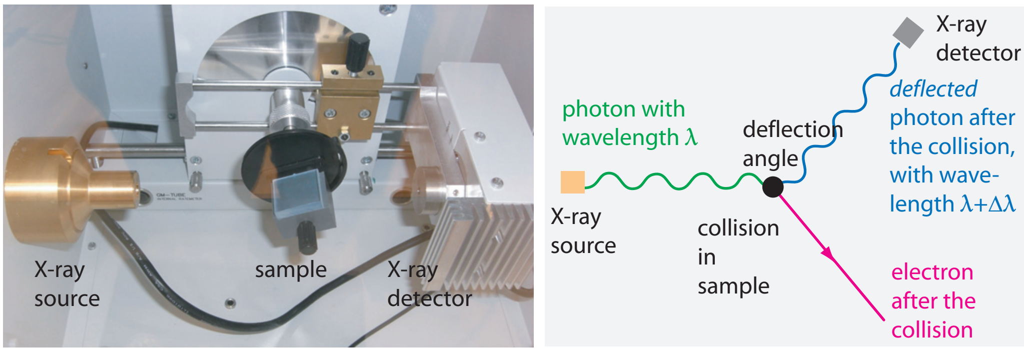


FIGURE 20 A modern version of Compton's experiment fits on a table. The experiment shows that photons have momentum: X-rays – and thus the photons they consist of – change frequency when they hit the electrons in matter in exactly the same way as predicted from colliding particles (© Helene Hoffmann).

can only give an upper limit for both quantities. The present experimental upper limit for the (rest) mass of a photon is 10^{-52} kg, and for the charge is $5 \cdot 10^{-30}$ times the electron charge. These limits are so small that we can safely say that both the mass and the charge of the photon vanish. Ref. 16

We know that light can *hit* objects. Since the energy, the lack of mass and the speed of photons are known, we deduce that the photon momentum is given by Challenge 32 e

$$p = \frac{E}{c} = \hbar \frac{2\pi}{\lambda} \quad \text{or} \quad \boldsymbol{p} = \hbar \boldsymbol{k} \, . \tag{10}$$

In other words, if light is made of particles, we should be able to play billiard with them. This is indeed possible, as Arthur Compton showed in a famous experiment in 1923. He directed X-rays, which are high-energy photons, onto graphite, a material in which electrons move almost freely. He found that whenever the electrons in the material are hit by the X-ray photons, the deflected X-rays change colour. As expected, the strength of the hit is related to the deflection angle of the photon, as deduced from Figure 20. From the colour change and the reflection angle, Compton confirmed that the photon momentum indeed satisfies the expression $\boldsymbol{p} = \hbar \boldsymbol{k}$. Ref. 17

All other experiments agree that photons have momentum. For example, when an atom *emits* light, the atom feels a *recoil*. The momentum again turns out to be given by the expression $\boldsymbol{p} = \hbar \boldsymbol{k}$. In short, the quantum of action determines the momentum of the photon.

The value of a photon's momentum respects the indeterminacy principle. Just as it is impossible to measure exactly both the wavelength of a wave and the position of its crest, so it is impossible to measure both the momentum and the position of a photon. Can you confirm this? In other words, the value of the photon momentum is a direct consequence of the quantum of action. Challenge 33 ny

From our study of classical physics, we know that light has a property beyond its colour: light can be *polarized*. That is only a complicated way to say that light can *turn* the objects that it shines on. In other words, light has an angular momentum oriented Vol. III, page 102 along the axis of propagation. What about photons? Measurements consistently find that each light quantum carries an *angular momentum* given by $L = \hbar$. It is called its *helicity*. The quantity is similar to one found for massive particles: one therefore also speaks of the *spin* of a photon. In short, photons somehow 'turn' – in a direction either parallel or antiparallel to the direction of motion. Again, the magnitude of the photon helicity, or spin, is no surprise; it confirms the classical relation $L = E/\omega$ between energy and angular momentum that we found in the section on classical electrodynamics. Note that, counterintuitively, the angular momentum of a single photon is fixed, and thus independent of its energy. Even the most energetic photons have $L = \hbar$. Of course, the value of the helicity also respects the limit given by the quantum of action. The many consequences of the helicity value $\hbar$ will become clear shortly.

Challenge 34 ny
Vol. III, page 102

What is light?

> La lumière est un mouvement luminaire de corps lumineux.
>
> Blaise Pascal*

In the seventeenth century, Blaise Pascal used the above statement about light to make fun of certain physicists, ridiculing the blatant use of a circular definition. Of course, he was right: in his time, the definition was indeed circular, as no meaning could be given to any of the terms. But whenever physicists study an observation with care, philosophers lose out. All those originally undefined terms now have a definite meaning and the circular definition is resolved. Light is indeed a type of motion; this motion can rightly be called 'luminary' because, in contrast to the motion of material bodies, it has the unique property $v = c$; the luminous bodies, today called photons, are characterized, and differentiated from all other particles, by their dispersion relation $E = cp$, their energy $E = \hbar\omega$, their spin $L = \hbar$, the vanishing of all other quantum numbers, and the property of being the quanta of the electromagnetic field.

In short, *light is a stream of photons*. It is indeed a 'luminary movement of luminous bodies'. Photons provide our first example of a general property of the world on small scales: *all waves and all flows in nature are made of quantum particles*. Large numbers of (coherent) quantum particles – or *quantons* – behave and form as waves. We will see shortly that this is the case even for matter. Quantons are the fundamental constituents of *all* waves and *all* flows, without exception. Thus, the everyday continuum description of light is similar in many respects to the description of water as a continuous fluid: photons are the atoms of light, and continuity is an approximation valid for large numbers of particles. Single quantons often behave like classical particles.

Physics books used to discuss at length a so-called *wave–particle duality*. Let us be clear from the start: quantons, or quantum particles, are *neither* classical waves *nor* classical particles. In the microscopic world, quantons are the fundamental objects.

However, there is much that is still unclear. Where, inside matter, do these monochromatic photons come from? Even more interestingly, if light is made of quantons, all electromagnetic fields, even static ones, must be made of photons as well. However, in static

* 'Light is the luminary movement of luminous bodies.' Blaise Pascal (b. 1623 Clermont, d. 1662 Paris), important French mathematician and physicist up to the age of 26, after which he became a theologian and philosopher.

fields nothing is flowing. How is this apparent contradiction resolved? And what implications does the particle aspect have for these static fields? What is the difference between quantons and classical particles? The properties of photons require more careful study. Let us go on.

The size of photons

First of all, we might ask: what are these photons made of? All experiments so far, performed down to the present limit of about 10^{-20} m, give the same answer: 'we can't find anything'. This is consistent with both a vanishing mass and a vanishing size of photons. Indeed, we would intuitively expect a body with a finite size to have a finite mass. Thus, although experiments can give only an upper limit, it is consistent to claim that a photon has *zero size*.

A particle with zero size cannot have any constituents. Thus a photon cannot be divided into smaller entities: photons are not composite. For this reason, they are called *elementary* particles. We will soon give some further strong arguments for this result. (Can you find one?) Nevertheless, the conclusion is strange. How can a photon have vanishing size, have no constituents, and still be *something*? This is a hard question; the answer will appear only in the last volume of our adventure. At the moment we simply have to accept the situation as it is. We therefore turn to an easier question.

Challenge 35 s

Are photons countable? – Squeezed light

> Also gibt es sie doch.
>
> Max Planck*

We saw above that the simplest way to count photons is to distribute them across a large screen and then to absorb them. But this method is not entirely satisfactory, as it destroys the photons. How can we count photons without destroying them?

One way is to reflect photons in a mirror and measure the recoil of the mirror. It seems almost unbelievable, but nowadays this effect is becoming measurable even for small numbers of photons. For example, it has to be taken into account in relation to the laser mirrors used in gravitational wave detectors, whose position has to be measured with high precision.

Vol. II, page 158

Another way of counting photons without destroying them involves the use of special high-quality laser cavities. It is possible to count photons by the effect they have on atoms cleverly placed inside such a cavity.

In other words, light intensity can indeed be measured without absorption. These measurement show an important issue: even the best light beams, from the most sophisticated lasers, *fluctuate* in intensity. There are *no* steady beams. This comes as no surprise: if a light beam did *not* fluctuate, observing it twice would yield a vanishing value for the action. However, there is a minimum action in nature, namely $\hbar$. Thus any beam and any flow in nature *must* fluctuate. But there is more.

* 'Thus they do exist after all.' Max Planck, in his later years, said this after standing silently, for a long time, in front of an apparatus that counted single photons by producing a click for each photon it detected. For a large part of his life, Planck was sceptical of the photon concept, even though his own experiments and conclusions were the starting point for its introduction.

A light beam is described by its intensity and its phase. The change – or action – that occurs while a beam moves is given by the variation in the product of intensity and phase. Experiments confirm the obvious deduction: the intensity and phase of a beam behave like the momentum and position of a particle, in that they obey an indeterminacy relation. You can deduce it yourself, in the same way as we deduced Heisenberg's relations. Using as characteristic intensity $I = E/\omega$, the energy divided by the angular frequency, and calling the phase φ, we get*

$$\Delta I \, \Delta\varphi \geqslant \frac{\hbar}{2} \,. \qquad (12)$$

For light emitted from an ordinary lamp, the product on the left-hand side of the above inequality is much larger than the quantum of action. On the other hand, laser beams can (almost) reach the limit. Laser light in which the two indeterminacies differ greatly from each other is called *non-classical light* or *squeezed light*; it is used in many modern research applications. Such light beams have to be treated carefully, as the smallest disturbances transform them back into ordinary laser beams, in which the two indeterminacies have the same value. Extreme examples of non-classical light are beams with a given, fixed, photon number, and thus with an extremely high phase indeterminacy.

The observation of non-classical light highlights a strange fact, valid even for classical light: the number of photons in a light beam is not a well-defined quantity. In general, it is undetermined, and it *fluctuates*. The number of photons at the beginning of a beam is not necessarily the same as the number at the end of the beam. Photons, unlike stones, cannot be counted precisely – as long as they are moving and not absorbed. In flight, it is only possible to determine an approximate number, within the limits set by indeterminacy.

One extreme example, shown in Figure 21, is a light beam with an (almost) fixed phase. In such a beam, the photon number fluctuates from zero to infinity. In other words, in order to produce *coherent light*, such as a laser beam, that approximates a pure sine wave as perfectly as possible, we must build a source in which the photon number is as undetermined as possible.

At the other extreme is a beam with a fixed number of photons: in such a beam of *non-classical light*, the phase fluctuates erratically. In contrast, the *thermal light* that we encounter in most everyday situations – such as the light from an incandescent lamps – lies somewhere in between, the phase and intensity indeterminacies being of similar magnitude.

As an aside, it turns out that even in deep, dark intergalactic space, far from any star, there are about 400 photons per cubic centimetre. This number, like the number of photons in a light beam, also has a measurement indeterminacy. Can you estimate it? Challenge 36 ny

In summary, unlike pebbles, *photons are countable, but their number is not fixed.* And this it not the only difference between photons and pebbles.

* A large photon number is assumed in the expression. This is obvious, as $\Delta\varphi$ cannot grow beyond all bounds. The exact relations are

$$\Delta I \, \Delta \cos\varphi \geqslant \frac{\hbar}{2} |\langle \sin\varphi \rangle|$$
$$\Delta I \, \Delta \sin\varphi \geqslant \frac{\hbar}{2} |\langle \cos\varphi \rangle| \qquad (11)$$

where $\langle x \rangle$ denotes the expectation value of the observable x.

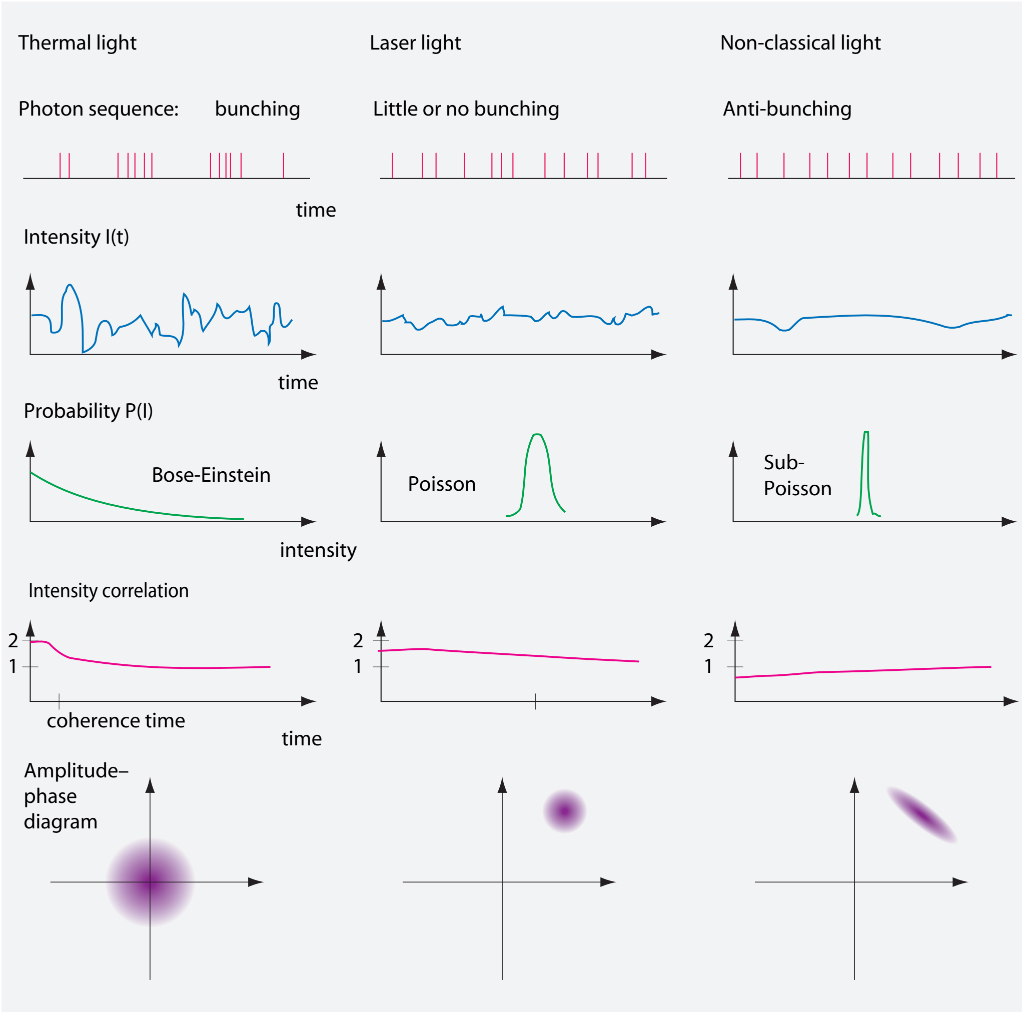


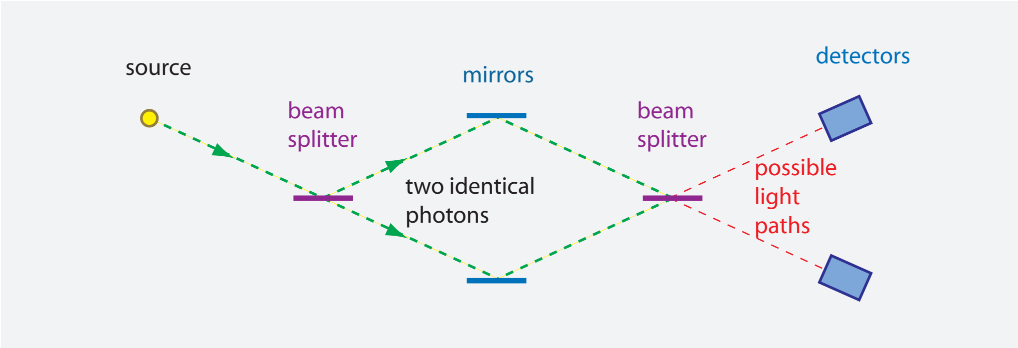
FIGURE 21 Three types of light: thermal light, laser light and squeezed light.

The positions of photons

Where is a photon when it moves in a beam of light? Quantum theory gives a simple answer: nowhere in particular. This is proved most spectacularly by experiments with interferometers, such as the basic interferometer shown in Figure 22. Interferometers show that even a beam made of a *single* photon can be split, led along two different paths, and then recombined. The resulting interference shows that the single photon cannot be said to have taken either of the two paths. If one of the two paths is blocked, the pattern on the screen changes. In other words, somehow the photon must have taken both paths at the same time. Photons cannot be localized: they have no position.*

This impossibility of localizing photons can be quantified. It is impossible to localize photons in the direction *transverse* to the motion. It is less difficult to localize photons

* One cannot avoid this conclusion by saying that photons are *split* at the beam splitter: if a detector is placed in each arm, one finds that they never detect a photon at the same time. Photons cannot be divided.



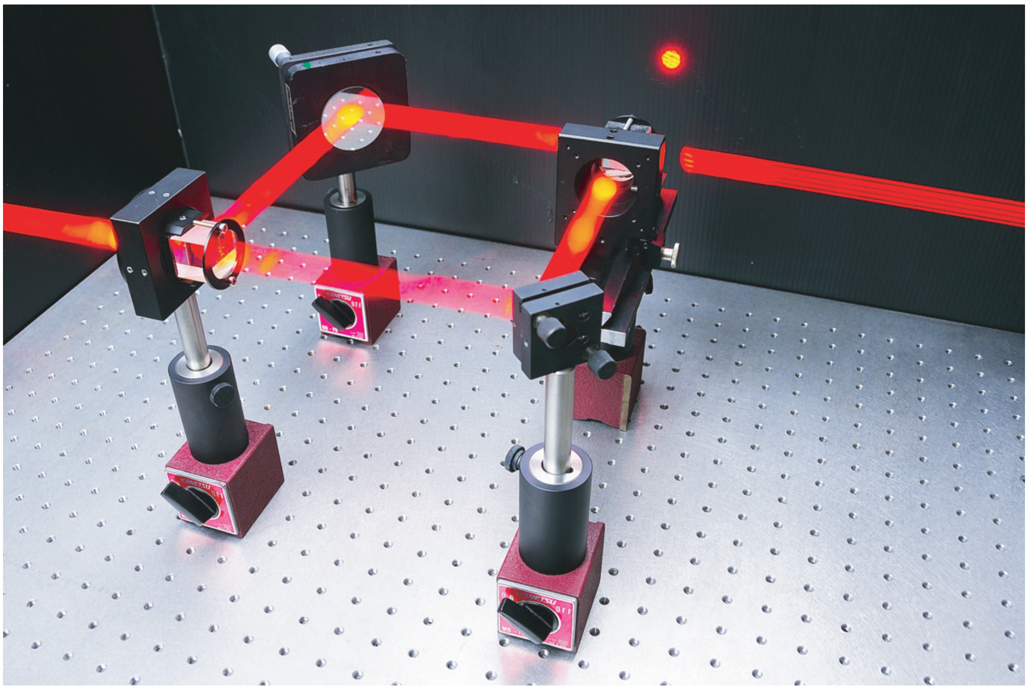


FIGURE 22 The Mach–Zehnder interferometer and a practical realization, about 0.5 m in size (© Félix Dieu and Gaël Osowiecki).

along the direction of motion. In the latter case, the quantum of action implies that the indeterminacy in the longitudinal position is given by the wavelength of the light. Can you confirm this? Challenge 37 ny

In particular, this means that photons *cannot* be simply visualized as short wave trains. Photons are truly *unlocalizable* entities, specific to the quantum world.

Now, if photons can *almost* be localized along their direction of motion, we can ask how photons are lined up in a light beam. Of course, we have just seen that it does not make sense to speak of their *precise* position. But do photons in a perfect beam arrive at almost-regular intervals?

To the shame of physicists, the study of this question was initiated by two astronomers, Robert Hanbury Brown and Richard Twiss, in 1956. They used a simple method to mea- Ref. 18

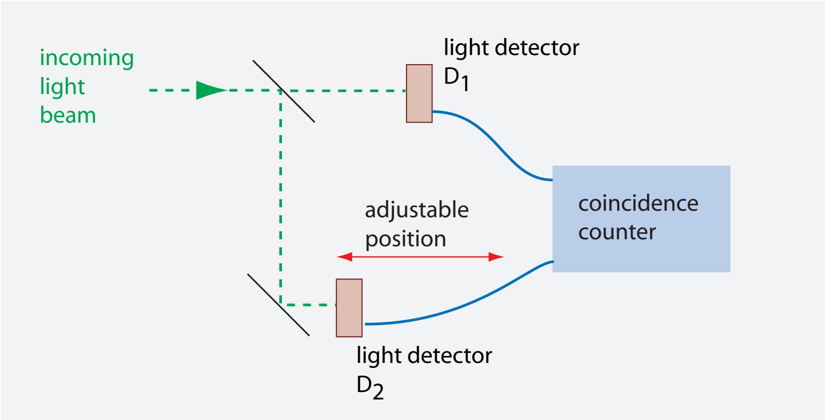


FIGURE 23 How to measure photon statistics with an electronic coincidence counter, the variation being measured by varying the position of a detector.

sure the probability that the second photon in a light beam arrives at a given time after the first one. They simply split the beam, put one detector in the first branch, and varied the position of a second detector in the other branch.

Hanbury Brown and Twiss found that, for coherent light, the clicks in the two counters – and thus the photons themselves – are *correlated*. This result is completely contrary to classical electrodynamics. The result is one of the many that show that photons are indeed necessary to describe light. To be more precise, their experiment showed that whenever the first photon hits, the second one is most likely to hit just afterwards. Thus, photons in beams are *bunched*. (As we will see below, this also implies that photons are bosons.) Page 53

Every light beam has an upper time limit for bunching, called the *coherence time*. For times longer than the coherence time, the probability for bunching is low, and independent of the time interval, as shown in Figure 23. The coherence time characterizes every light beam, or rather every light source. In fact, it is often easier to think in terms of the *coherence length* of a light beam. For thermal lamps, the coherence length is only a few micrometres: a small multiple of the wavelength. The largest coherence lengths, of over 100 000 km, are obtained with research lasers. Interestingly, coherent light is even found in nature: several special stars have been found to emit it. Ref. 19

Although the intensity of a good laser beam is almost constant, the photons do not arrive at regular intervals. Even the best laser light shows bunching, though with different statistics and to a lesser degree than lamp light. Light whose photons arrive regularly, thus exhibiting so-called *(photon) anti-bunching*, is obviously non-classical in the sense defined above; such light can be produced only by special experimental arrangements. Extreme examples of this phenomenon are being investigated at present by several research groups aiming to construct light sources that emit one photon at a time, at regular time intervals, as reliably as possible.

In summary, experiments force us to conclude that light is made of photons, but also that photons cannot be localized in light beams. It makes no sense to talk about the position of a photon in general; the idea makes sense only in some special situations, and then only approximately and as a statistical average.

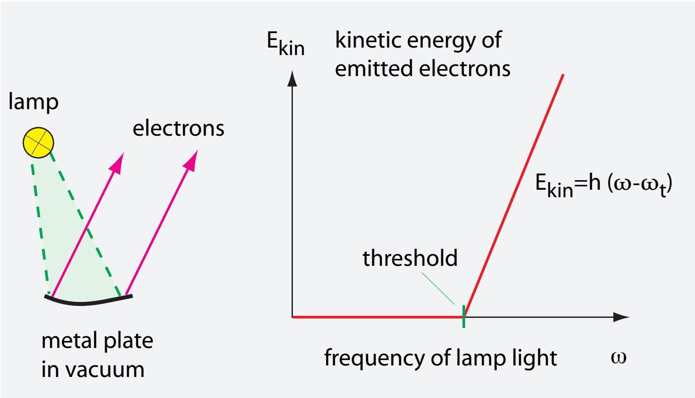


FIGURE 24 The kinetic energy of electrons emitted in the photoelectric effect.

Are photons necessary?

In light of the results uncovered so far, the answer to the above question is obvious. But the issue is tricky. In textbooks, the *photoelectric effect* is usually cited as the first and most obvious experimental proof of the existence of photons. In 1887, Heinrich Hertz observed that for certain metals, such as lithium or caesium, incident ultraviolet light leads to charging of the metal. Later studies of the effect showed that the light causes emission of electrons, and that the energy of the ejected electrons does not depend on the intensity of the light, but only on the difference between $\hbar$ times its frequency and a material-dependent threshold energy. Figure 24 summarizes the experiment and the measurements.

In classical physics, the photoelectric effect is difficult to explain. But in 1905, Albert Einstein deduced the measurements from the assumption that light is made of photons of energy $E = \hbar\omega$. He imagined that this energy is used partly to take the electron over the threshold, and partly to give it kinetic energy. More photons only lead to more electrons, not to faster ones. In 1921, Einstein received the Nobel Prize for the explanation of the photoelectric effect. But Einstein was a *genius*: he deduced the correct result by a somewhat incorrect reasoning. The (small) mistake was the assumption that a classical, continuous light beam would produce a different effect. In fact, it is easy to see that a classical, continuous electromagnetic field interacting with discrete matter, made of discrete atoms containing discrete electrons, would lead to exactly the same result, as long as the motion of electrons is described by quantum theory. Several researchers confirmed this early in the twentieth century. The photoelectric effect by itself does *not* imply the existence of photons.

Ref. 20

Ref. 21

Indeed, many researchers in the past were unconvinced that the photoelectric effect shows the existence of photons. Historically, the most important argument for the *necessity* of light quanta was given by Henri Poincaré. In 1911 and 1912, aged 57 and only a few months before his death, he published two influential papers proving that the radiation law of black bodies – in which the quantum of action had been discovered by Max Planck – *requires* the existence of photons. He also showed that the amount of radiation emitted by a hot body is *finite* only because of the *quantum* nature of the processes leading to light emission. A description of these processes in terms of classical electrodynamics would

Ref. 22

lead to (almost) *infinite* amounts of radiated energy. Poincaré's two influential papers convinced most physicists that it was worthwhile to study quantum phenomena in more detail. Poincaré did not know about the action limit $S \geqslant \hbar$; yet his argument is based on the observation that light of a given frequency has a minimum intensity, namely a single photon. Such a one-photon beam may be split into two beams, for example by using a half-silvered mirror. However, taken together, those two beams never contain more than a single photon.

Another interesting experiment that requires photons is the observation of 'molecules of photons'. In 1995, Jacobson et al. predicted that the de Broglie wavelength of a *packet* of photons could be observed. According to quantum theory, the packet wavelength is given by the wavelength of a single photon divided by the number of photons in the packet. The team argued that the packet wavelength could be observable if such a packet could be split and recombined without destroying the cohesion within it. In 1999, this effect was indeed observed by de Pádua and his research group in Brazil. They used a careful set-up with a nonlinear crystal to create what they call a *biphoton*, and observed its interference properties, finding a reduction in the effective wavelength by the predicted factor of two. Since then, packages with three and even four entangled photons have been created and observed.

Ref. 23

Ref. 24

Yet another argument for the necessity of photons is the above-mentioned recoil felt by atoms emitting light. The recoil measured in these cases is best explained by the emission of a photon in a particular direction. In contrast, classical electrodynamics predicts the emission of a spherical wave, with no preferred direction.

Page 37

Obviously, the observation of *non-classical light*, also called *squeezed light*, also argues for the existence of photons, as squeezed light proves that photons are indeed an *intrinsic* aspect of light, necessary even when interactions with matter play no role. The same is true for the Hanbury Brown–Twiss effect.

Page 43

Ref. 25

Finally, the spontaneous decay of excited atomic states also requires the existence of photons. This cannot be explained by a continuum description of light.

In summary, the concept of a photon is indeed necessary for a precise description of light; but the details are often subtle, as the properties of photons are unusual and require a change in our habits of thought. To avoid these issues, most textbooks stop discussing photons after coming to the photoelectric effect. This is a pity, as it is only then that things get interesting. Ponder the following. Obviously, all electromagnetic fields are made of photons. At present, photons can be counted for gamma rays, X-rays, ultraviolet light, visible light and infrared light. However, for lower frequencies, such as radio waves, photons have not yet been detected. Can you imagine what would be necessary to count the photons emitted from a radio station?

Challenge 38 ny

This issue leads directly to the most important question of all:

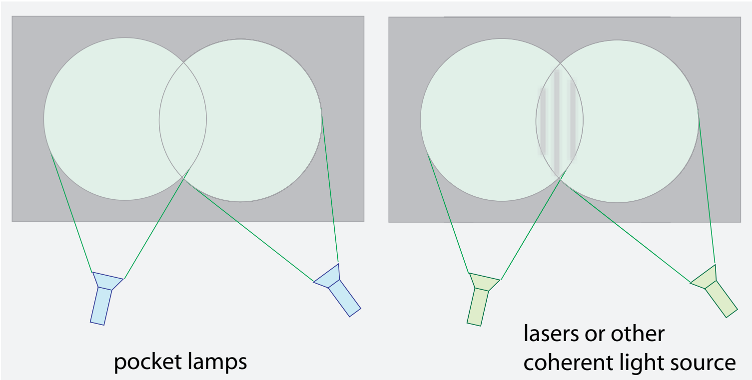


Figure 25 Light crossing light.

How can a wave be made up of particles?

> Fünfzig Jahre intensiven Nachdenkens haben mich der Antwort auf die Frage 'Was sind Lichtquanten?' nicht näher gebracht. Natürlich bildet sich heute jeder Wicht ein, er wisse die Antwort. Doch da täuscht er sich.
>
> Albert Einstein, 1951 *

If a light wave is made of particles, one must be able to explain each and every wave property in terms of photons. The experiments mentioned above already hint that this is possible only because photons are *quantum* particles. Let us take a more detailed look at this.

Light can *cross* other light undisturbed. This observation is not hard to explain with photons; since photons do not interact with each other, and are point-like, they 'never' hit each other. In fact, there *is* an extremely small positive probability for their interaction, as will be found below, but this effect is not observable in everyday life.

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But some problems remain. If two light beams of identical frequency and fixed phase relation cross, we observe alternating bright and dark regions: so-called *interference fringes*.** How do these interference fringes appear? How can it be that photons are not detected in the dark regions? We already know the only possible answer: the brightness at a given place corresponds to the *probability* that a photon will arrive there. The fringes imply that photons behave like little *arrows*. Some further thought leads to the following description:

— The probability of a photon arriving somewhere is given by the *square of an arrow*.
— The final arrow is the *sum* of all the arrows arriving there by all possible paths.
— The arrow is always perpendicular to the direction of motion.
— The arrow's direction stays fixed in space when the photons move.

* 'Fifty years of intense reflection have not brought me nearer to the answer to the question 'What are light quanta?' Of course nowadays every little mind thinks he knows the answer. But he is wrong.' Einstein said this a few years before his death.

Challenge 39 s

** This experiment is only possible if both beams are derived from a single beam by splitting, or if two expensive high-precision lasers are used. (Why?)

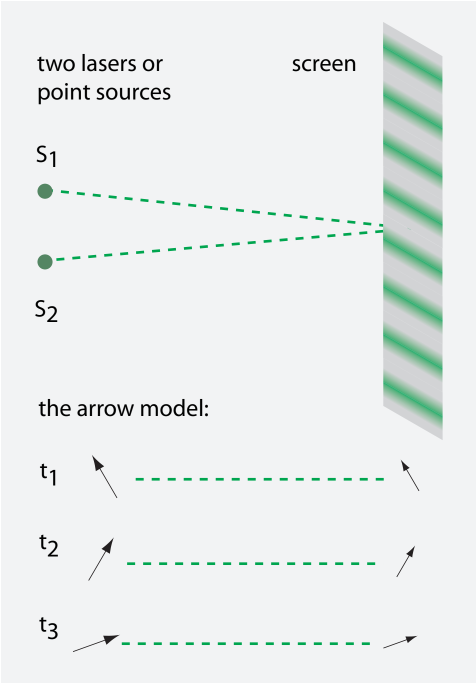

FIGURE 26 Interference and the description of light with arrows (at three instants of time).

— The length of an arrow shrinks with the square of the distance travelled.
— Photons emitted by single-coloured sources are emitted with arrows of constant length pointing in the direction ωt; in other words, such sources spit out photons with a *rotating* mouth.
— Photons emitted by thermal sources, such as pocket lamps, are emitted with arrows of constant length pointing in *random* directions.

With this model* we can explain the stripes seen in laser experiments, such as those of Figure 25 and Figure 26. You can check that in some regions the two arrows travelling through the two slits add up to zero *for all times*. No photons are detected there. In other regions, the arrows always add up to the maximal value. These regions are always bright. Regions in between have intermediate shades. Obviously, in the case of pocket lamps, the brightness also behaves as expected: the averages simply add up, as in the common region shown in the left-hand diagram of Figure 25.

Challenge 40 ny You may wish to calculate the distance between the lines, given the source distance, the colour and the distance to the screen.

Obviously, the photon model implies that an interference pattern like this is built up as the sum of a large number of single-photon hits. Using low-intensity beams, we should therefore be able to see how these little spots slowly build up an interference pattern by accumulating in the bright regions and never hitting the dark regions. This is indeed the case. All experiments have confirmed this description. Page 37

* The model gives a correct description of light except that it neglects polarization. To add it, it is necessary to combine arrows that rotate in both senses around the direction of motion.

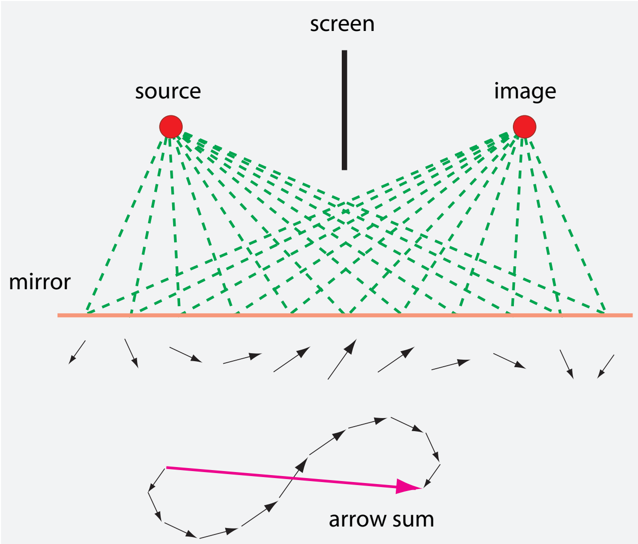

FIGURE 27 Light reflected by a mirror, and the corresponding arrows (at an instant of time).

It is important to note that interference between two light beams is *not* the result of two different photons cancelling each other out or being added together. Such cancellation would contradict conservation of energy and momentum. Interference is an effect applicable to each photon separately, because each photon is spread out over the whole set-up: each photon takes all possible paths. As Paul Dirac said, *each photon interferes only with itself*. Interference only works because photons are quantons, and not classical particles. Ref. 26

Dirac's oft-quoted statement leads to a famous paradox: if a photon can interfere only with itself, how can two laser beams from two different lasers interfere with each other? The answer given by quantum physics is simple but strange: in the region where the beams interfere, it is impossible to say from which source a photon has come. The photons in the crossing region *cannot* be said to come from a specific source. Photons in the interference region are quantons, which indeed interfere only with themselves. In that region, one cannot truly say that light is a flow of photons. Despite regular claims to the contrary, Dirac's statement is correct. It is a strange consequence of the quantum of action.

Waves also show *diffraction*. To understand this phenomenon with photons, let us start with a simple mirror, and study *reflection* first. Photons (like all quantum particles) move from source to detector by *all* possible paths. As Richard Feynman,* who discov-

* Richard ('Dick') Phillips Feynman (b. 1918 New York City, d. 1988), US-American physicist. One of the founders of quantum electrodynamics, he also discovered the 'sum-over-histories' reformulation of quantum theory, made important contributions to the theory of the weak interaction and to quantum gravity, and co-authored a famous textbook, the *Feynman Lectures on Physics*. He is one of those theoretical physicists who made his career mainly by performing complex calculations – but he backtracked with age, most successfully in his teachings and physics books, which are all worth reading. He was deeply dedicated to physics and to enlarging knowledge, and was a collector of surprising physical explanations. He helped building the nuclear bomb, wrote papers in topless bars, avoided to take any professional responsibility, and was famously arrogant and disrespectful of authority. He wrote several popular books on the events

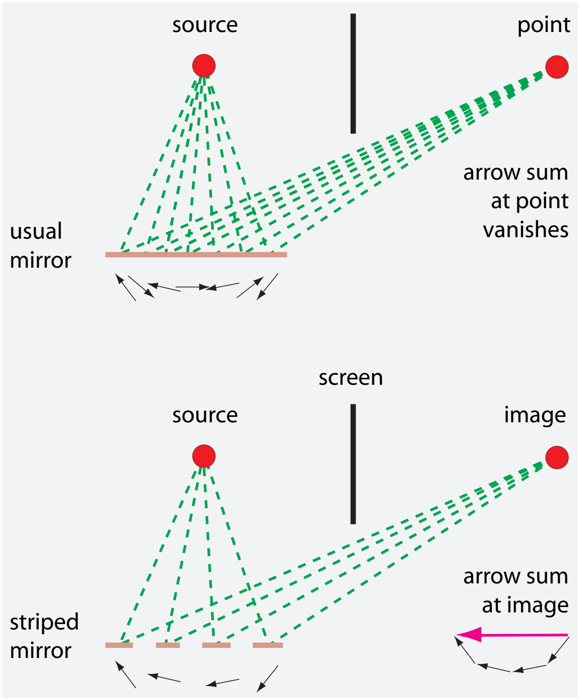

FIGURE 28 The light reflected by a badly-placed mirror and by a grating.

ered this explanation, liked to stress, the term 'all' has to be taken literally. This is not a big deal in the explanation of interference. But in order to understand a mirror, we have to include all possibilities, however crazy they seem, as shown in Figure 27.

As stated above, a light source emits rotating arrows. To determine the probability that light arrives at a certain location within the image, we have to add up all the arrows arriving at the same time at that location. For each path, the arrow orientation at the image is shown – for convenience only – below the corresponding segment of the mirror. The angle and length of the arriving arrow depends on the path. Note that the sum of all the arrows does not vanish: light does indeed arrive at the image. Moreover, the largest contribution comes from the paths near to the middle. If we were to perform the same calculation for another image location, (almost) no light would get there. So the rule that reflection occurs with the incoming angle equal to the outgoing angle is an approximation, following from the arrow model of light.

In fact, a detailed calculation, with more arrows, shows that the approximation is quite precise: the errors are much smaller than the wavelength of the light.

The proof that light does indeed take all these strange paths is given by a more specialized mirror. As show in Figure 28, we can repeat the experiment with a mirror that reflects only along certain stripes. In this case, the stripes have been carefully chosen so that the corresponding path lengths lead to arrows with a bias in one direction, namely to the left. The arrow addition now shows that such a specialized mirror – usually called

of his life. Though he tried to surpass the genius of Wolfgang Pauli throughout his life, he failed in this endeavour. He shared the 1965 Nobel Prize in Physics for his work on quantum electrodynamics.

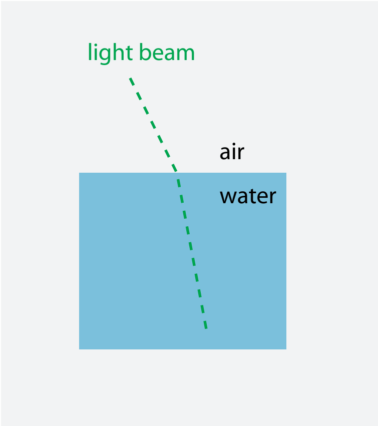

FIGURE 29 If light were made of little stones, they would move faster in water.

a *grating* – allows light to be reflected in unusual directions. Indeed, this behaviour is standard for waves: it is called *diffraction*. In short, the arrow model for photons allows us to describe this wave property of light, provided that photons follow the 'crazy' probability scheme. Do not get upset! As was said above, quantum theory is the theory for crazy people.

You may wish to check that the arrow model, with the approximations it generates by summing over all possible paths, automatically ensures that the quantum of action is indeed the smallest action that can be observed. Challenge 41 ny

All waves have a *signal velocity*. The signal velocity also depends on the medium in which they propagate. As a consequence, waves show *refraction* when they move from one medium into another with different signal velocity. Interestingly, the naive particle picture of photons as little stones would imply that light is faster in materials with high refractive indices: the so-called *dense* materials. Can you confirm this? Challenge 42 ny However, experiments show that light in dense materials moves *slowly*. The wave picture has no difficulty explaining this observation. (Can you confirm this?) Challenge 43 ny Historically, this was one of the arguments *against* the particle theory of light. In contrast, the arrow model of light presented above is able to explain refraction properly. It is not difficult: try it. Challenge 44 ny

Waves also *reflect partially* from materials such as glass. This is one of the most difficult wave properties to explain with photons. But it is one of the few effects that is *not* explained by a classical wave theory of light. However, it *is* explained by the arrow model, as we will find shortly. Partial reflection confirms the first two rules of the arrow model. Page 48 Partial reflection shows that photons indeed behave *randomly*: some are reflected and other are not, without any selection criterion. The distinction is purely statistical. More about this issue shortly.

In waves, the fields *oscillate in time and space*. One way to show how waves can be made of particles is to show how to build up a sine wave using a large number of photons. Ref. 27 A sine wave is a coherent state of light. The way to build them up was explained by Roy Glauber. In fact, to build a pure sine wave, we need a superposition of a beam with one photon, a beam with two photons, a beam with three photons, and so on. Together, they give a perfect sine wave. As expected, its photon number fluctuates to the highest possible degree.

If we repeat the calculation for non-ideal beams, we find that the indeterminacy relation for energy and time is respected: every emitted beam will possess a certain spectral width. Purely monochromatic light does not exist. Similarly, no system that emits a wave *at random* can produce a monochromatic wave. All experiments confirm these results.

Waves can be *polarized*. So far, we have disregarded this property. In the photon picture, polarization is the result of carefully superposing beams of photons spinning clockwise and anticlockwise. Indeed, we know that linear polarization can be seen as a result of superposing circularly-polarized light of both signs, using the proper phase. What seemed a curiosity in classical optics turns out to be a fundamental justification for quantum theory.

Photons are *indistinguishable*. When two photons of the same colour cross, there is no way to say afterwards which of the two is which. The quantum of action makes this impossible. The indistinguishability of photons has an interesting consequence. It is impossible to say which emitted photon corresponds to which arriving photon. In other words, there is no way to follow the path of a photon, as we are used to following the path of a billiard ball. Photons are indeed indistinguishable. In addition, the experiment

Ref. 48 Page 24

by Hanbury Brown and Twiss, implies that photons are bosons. We will discover more

Page 93

details about the specific indistinguishability of bosons later in.

In summary, we find that light waves *can* indeed be built of particles. However, this is only possible with the proviso that photons are not precisely countable, that they are not localizable, that they have no size, no charge and no mass, that they carry an (approximate) phase, that they carry spin, that they are indistinguishable bosons, that they can take any path whatsoever, that one cannot pinpoint their origin, and that their probability of arriving somewhere is determined by the square of the sum of amplitudes for all possible paths. In other words, light can be made of particles only if these particles have very special *quantum* properties. These quantum properties allow them to behave like waves when they are present in large numbers.

Can light move faster than light? – Virtual photons

In a vacuum, light can move faster than c, as well as slower than c. The quantum principle provides the details. As long as this principle is obeyed, the speed of a short light flash can differ – though only by a tiny amount – from the 'official' value. Can you estimate the allowable difference in arrival time for a light flash coming from the dawn of time? Challenge 45 ny

The arrow explanation gives the same result. If we take into account the crazy possibility that photons can move with any speed, we find that all speeds very different from c cancel out. The only variation that remains, translated into distances, is the indeterminacy of about one wavelength in the longitudinal direction, which we mentioned above. Challenge 46 ny

More bizarre consequences of the quantum of action appear when we study *static* electric fields, such as the field around a charged metal sphere. Obviously, such a field must also be made of photons. How do they move? It turns out that static electric fields are made of *virtual* photons. Virtual photons are photons that do not appear as free particles: they only appear for an extremely short time before they disappear again. In the case of a static electric field, they are *longitudinally* polarized, and do not carry energy away. Virtual photons, like other virtual particles, are 'shadows' of particles that obey

$$\Delta x \Delta p \leqslant \hbar/2 . \qquad (13)$$

Rather than obeying the usual indeterminacy relation, they obey the opposite relation, which expresses their very brief appearance. Despite their intrinsically short life, and despite the impossibility of detecting them directly, virtual particles have important effects. Page 163 We will explore virtual particles in detail shortly.

In fact, the vector potential A allows *four* polarizations, corresponding to the four coordinates (t, x, y, z). It turns out that for the photons one usually talks about – the free or *real* photons – the polarizations in the t and z directions cancel out, so that one observes only the x and y polarizations in actual experiments.

For bound or *virtual* photons, the situation is different. All four polarizations are possible. Indeed, the z and t polarizations are the ones that can be said to be the building blocks of *static* electric and magnetic fields.

In other words, static electric and magnetic fields are continuous flows of virtual photons. In contrast to real photons, virtual photons can have mass, can have spin directions not pointing along the path of motion, and can have momentum opposite to their direction of motion. Exchange of virtual photons leads to the *attraction* of bodies of different charge. In fact, virtual photons *necessarily* appear in any description of electromagnetic Vol. V, page 81 interactions. Later on we will discuss their effects further – including the famous attraction of neutral bodies.

In summary, light can indeed move faster than light, though only by an amount allowed by the quantum of action. For everyday situations, i.e., for high values of the action, all quantum effects average out, including light velocities different from c.

There is another point that we should mention here. Not only the position, but also Ref. 29 the energy of a single photon can be undefined. For example, certain materials split one photon of energy $\hbar\omega$ into two photons, whose two energies add up to the original one. Quantum mechanics implies that the energy partitioning is known only when the energy of one of the two photons is measured. Only at that very instant is the energy of the second photon known. Before the measurement, both photons have undefined energies. The process of energy fixing takes place *instantaneously*, even if the second photon is far Page 127 away. We will explain below the background to this and similar strange effects, which *seem* to be faster than light. In fact, despite the appearance, these observations do not Challenge 47 ny involve faster-than-light transmission of energy or information.

Indeterminacy of electric fields

We have seen that the quantum of action implies an indeterminacy for light intensity. Since light is an electromagnetic wave, this indeterminacy implies similar, separate limits for electric and magnetic fields at a given point in space. This conclusion was first drawn Ref. 30 in 1933 by Bohr and Rosenfeld. They started from the effects of the fields on a test particle of mass m and charge q, which are described by:

$$m\boldsymbol{a} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{b}) . \qquad (14)$$

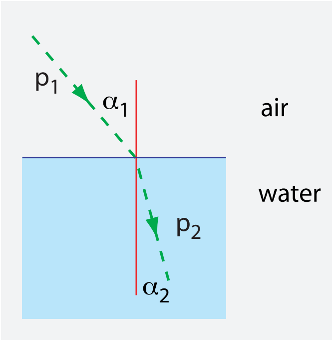

FIGURE 30 Refraction and photons.

Since it is impossible to measure both the momentum and the position of a particle, they deduced an indeterminacy for the electrical field, given by

Challenge 48 ny

$$\Delta E = \frac{\hbar}{q\,\Delta x\, t}\,, \tag{15}$$

where t is the measurement time and Δx is the position indeterminacy. Thus every value of an electric field, and similarly of a magnetic field, possesses an indeterminacy. The state of the electromagnetic field behaves like the state of matter in this respect: both follow an indeterminacy relation.

Curiosities and fun challenges about photons

Can one explain refraction with photons? Newton was not able to do so, but today we can. In refraction by a horizontal surface, as shown in Figure 30, the situation is translationally invariant along the horizontal direction. Therefore, the momentum component along this direction is also conserved: $p_1 \sin\alpha_1 = p_2 \sin\alpha_2$. The photon energy $E = E_1 = E_2$ is obviously conserved. The index of refraction n is defined in terms of momentum and energy as

$$n = \frac{cp}{E}\,. \tag{16}$$

Challenge 49 e

The 'law' of refraction follows:

$$\frac{\sin\alpha_1}{\sin\alpha_2} = n\,. \tag{17}$$

There is an important issue here. In a material, the velocity of a photon $v = \delta E/\delta p$ in a light ray *differs* from the phase velocity $u = E/p$ that enters into the calculation. In summary, inside matter, the concept of photon must be used with extreme care.

* *

FIGURE 31 The blue shades of the sky and the colours of clouds are due to various degrees of Rayleigh, Mie and Tyndall scattering (© Giorgio di Iorio).

If en electromagnetic wave has amplitude A, the photon density d is

$$d = \frac{A^2}{\hbar\omega} . \tag{18}$$

Challenge 50 ny Can you show this?

* *

A typical effect of the quantum 'laws' is the yellow colour of the lamps used for street illumination in most cities. They emit pure yellow light of a single frequency; that is why no other colours can be distinguished in their light. According to classical electrodynamics, harmonics of that light frequency should also be emitted. Experiments show, however, that this is not the case; so classical electrodynamics is thus wrong. Is this argument correct? Challenge 51 ny

* *

What happens to photons that hit an object but are not absorbed or transmitted? Generally speaking, they are scattered. *Scattering* is the name for any process that changes the motion of light (or that of any other wave). The details of the scattering process depend on the object; some scattering processes only change the direction of motion, others also change the frequency. Table 3 gives an overview of processes that scatter light.

All scattering properties depend on the material. Among others, the study of scattering processes explains many colours of transparent materials, as we will see below. Page 144

We note that the bending of light due to gravity, is not called scattering. Why? Challenge 52 e

TABLE 3 Types of light scattering.

Scattering type	Scatterer	Details	Examples
Rayleigh scattering	atoms, molecules	elastic, angle changes as $1/\lambda^4$, scatterers smaller than $\lambda/10$	blue sky, red evening sky, blue cigarette smoke
Mie scattering	transparent objects, droplets	elastic, angle changes as $1/\lambda^{0.5}$ to $1/\lambda^2$, scatterer size around λ	blue sky, red evenings, blue distant mountains
Geometric scattering	edges	elastic, scatterer size larger than λ	better called *diffraction*, used in interference
Tyndall scattering	non-transparent objects	elastic, angle weakly or not wavelength-dependent	smog, white clouds, fog, white cigarette smoke
Smekal–Raman scattering	excited atoms, molecules	inelastic, light gains energy	used in lidar investigations of the atmosphere
Inverse Raman scattering	atoms, molecules	inelastic, light loses energy	used in material research
Thomson scattering	electrons	elastic	used for electron density determination
Compton scattering	electrons	inelastic, X-ray lose energy	proves particle nature of light page 39
Brillouin scattering	acoustic phonons, density variations in solids/fluids	inelastic, frequency shift of a few GHz	used to study phonons and to diagnose optical fibres
X-ray scattering	crystalline solids	elastic, due to interference at crystal planes	used to determine crystal structures; also called *Bragg diffraction*

A summary on light: particle and waves

In summary, light is a stream of photons. A single photon is the smallest possible light intensity of a given colour. Photons, like all quantons, are quite different from everyday life particles. In fact, we can argue that the only (classical) particle aspects of photons are their quantized energy, momentum and spin. In all other respects, photons are *not* like little stones. Photons cannot be localized in light beams. Photons are indistinguishable. Photons are bosons. Photons have no mass, no charge and no size. It is more accurate to say that *photons are calculating devices to precisely describe observations about light.* Ref. 31

The strange properties of photons are the reason why earlier attempts to describe light as a stream of (classical) particles, such as the attempt of Newton, failed miserably, and were rightly ridiculed by other scientists. Indeed, Newton upheld his theory against all experimental evidence – especially with regard to light's wave properties – which is something that a physicist should never do. Only after people had accepted that light is a wave, and then discovered and understood that quantum particles are different from classical particles, was the approach successful.

The quantum of action implies that *all waves are streams of quantons*. In fact, all waves are *correlated* streams of quantons. This is true for light, for any other form of radiation, and for all forms of matter waves.

The indeterminacy relations show that even a single quanton can be regarded as a wave; however, whenever it interacts with the rest of the world, it behaves as a particle. In fact, it is *essential* that all waves be made of quantons: if they were not, then interactions would be non-local, and objects could not be localized at all, contrary to experience. To decide between the wave and particle descriptions, we can use the following criterion. Whenever matter and light interact, it is more appropriate to describe electromagnetic radiation as a wave if the wavelength λ satisfies

$$\lambda \gg \frac{\hbar c}{kT} , \qquad (19)$$

where $k = 1.4 \cdot 10^{-23}$ J/K is Boltzmann's constant. If the wavelength is much *smaller* than the quantity on the right-hand side, the particle description is most appropriate. If the two sides are of the same order of magnitude, both descriptions play a role. Can you explain the criterion?

Challenge 53 e

Chapter 3

MOTION OF MATTER – BEYOND CLASSICAL PHYSICS

> All great things begin as blasphemies.
>
> George Bernard Shaw

The existence of a smallest action has numerous important consequences for the motion of matter. We start with a few experimental results that show that the quantum of action is indeed the smallest measurable action, also in the case of matter. Then we show that the quantum of action implies the existence of a phase and thus of the wave properties of matter, including the same description that we already found for light: matter particles behave like rotating arrows.

Wine glasses, pencils and atoms – no rest

> Otium cum dignitate.*
>
> Cicero, *De oratore*.

If the quantum of action is the smallest observable change in a physical system, then two observations of the same system must always differ. Thus there cannot be perfect rest in nature. Is that true? Experiments show that this is indeed the case.

A simple consequence of the quantum of action is the impossibility of completely filling a glass of wine. If we call a glass at maximum capacity (including surface tension effects, to make the argument precise) 'full', we immediately see that the situation requires the liquid's surface to be completely at rest. But a completely quiet surface would admit two successive observations that differ by less than $\hbar$. We could try to reduce all motions by reducing the temperature of the system. But absolute rest would imply reaching absolute zero temperature. Experiments show that this is impossible. (Indeed, this impossibility, the so-called *third 'law' of thermodynamics*, is equivalent to the existence of a minimum action.) *There is no rest in nature.* In other words, the quantum of action proves the old truth that a glass of wine is always partially empty *and* partially full.

The absence of microscopic rest, predicted by the quantum of action, is confirmed in many experiments. For example, a pencil standing on its tip *cannot* remain vertical, as shown in Figure 32, even if it is isolated from all disturbances, such as vibrations, air molecules and thermal motion. This follows from the indeterminacy relation. In fact, it
Challenge 54 d
is even possible to calculate the time after which a pencil *must* have fallen over.

But the most important consequence of the absence of rest is another. The absence of

* 'Rest with dignity.'

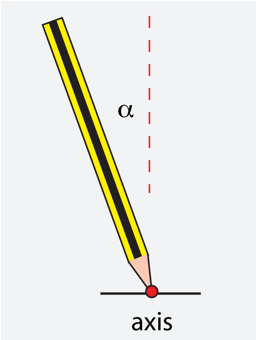

FIGURE 32 A falling pencil.

rest for the electrons inside atoms prevents them from falling into the nuclei, despite their mutual attraction. In short, the existence and the size of atoms, and thus of all matter, is a direct consequence of the absence of microscopic rest. We will explore this in more detail below. We only exist and live because of the quantum of action.

Page 66

No infinite precision

The quantum of action prevents the observation of rest in many ways. In order to check whether an object is at rest, we need to observe its position with high precision. Because of the wave properties of light, we need a high-energy photon: only a high-energy photon has a small wavelength and thus allows a precise position measurement. As a result of this high energy, however, the object is disturbed. Worse, the disturbance itself is not precisely measurable; so there is no way to determine the original position even by taking the disturbance into account. In short, perfect rest cannot be observed even in principle.

Indeed, all experiments in which systems have been observed with high precision confirm that perfect rest does not exist. The absence of rest has been confirmed for electrons, neutrons, protons, ions, atoms, molecules, atomic condensates, crystals, and objects with a mass of up to a tonne, as used in certain gravitational wave detectors. No object is ever at rest.

The same argument also shows that no measurement, of any observable, can ever be performed to infinite precision. This is another of the far-reaching consequences of the quantum of action.

Cool gas

The quantum of action implies that rest is impossible in nature. In fact, even at extremely low temperatures, all particles inside matter are in motion. This fundamental lack of rest is said to be due to the so-called *zero-point fluctuations*. A good example is provided by the recent measurements of Bose–Einstein condensates. They are trapped gases, with a small number of atoms (between ten and a few million), cooled to extremely low temperatures (around 1 nK). These cool gases can be observed with high precision. Using elaborate experimental techniques, Bose–Einstein condensates can be put into states for which $\Delta p \Delta x$ is almost exactly equal to $\hbar/2$ – though never lower than this value. These experiments confirm directly that there is no rest, but a fundamental *fuzziness* in nature.

This leads to an interesting puzzle. In a normal object, the distance between the atoms is much larger than their de Broglie wavelength. (Can you confirm this?) But today it is possible to cool objects to extremely low temperatures. At sufficiently low temperatures, less than 1 nK, the wavelength of the atoms may be larger than their separation. Can you imagine what happens in such cases?

Challenge 55 s
Ref. 32
Challenge 56 s

Flows and the quantization of matter

> Die Bewegung ist die Daseinsform der Materie.
> Friedrich Engels, *Anti-Dühring*.*

Not only does the quantum of action make rest impossible, but it makes any situation that does not change in time impossible. The most important examples of (apparently) stationary situations are *flows*. The quantum of action implies that *no* flow can be stationary. More precisely, a smallest action implies that no flow can be continuous. All flows fluctuate and are made of smallest entities: in nature, all flows are made of quantum particles. We saw above that this is valid for light. Two simple types of flow from our everyday experience directly confirm this consequence from the quantum of action: flows of fluids and flows of electricity.

Fluid flows and quantons

The flow of matter also exhibits smallest units. We mentioned early on in our adventure that a consequence of the particulate structure of liquids is that oil or any other smooth liquid produces *noise* when it flows through even the smoothest of pipes. We mentioned that the noise we hear in our ears in situations of absolute silence – for example, in a snowy and windless landscape in the mountains or in an anechoic chamber – is due to the granularity of blood flow in the veins. Experiments show that all flows of matter produce vibrations. This is a consequence of the quantum of action, and of the resulting granularity of matter.

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Knocking tables and quantized conductivity

If electrical current were a continuous flow, it would be possible to observe action values as small as desired. The simplest counter-example was discovered in 1996, by José Costa-Krämer and his colleagues. They put two metal wires on top of each other on a kitchen table and attached a battery, a current-voltage converter (or simply a resistor) and a storage oscilloscope to them. Then they measured the electrical current while *knocking* on the table. That is all.

Ref. 33, Ref. 34

Knocking the table breaks the contact between the two wires. In the last millisecond before the wires detach, the conductivity and thus the electrical current diminished in regular steps of about 7 μA, as can easily be seen on the oscilloscope (see Figure 33). This simple experiment could have beaten, if it had been performed a few years earlier, a number of other, enormously expensive experiments which discovered this quantization at costs of several million euro each, using complex set-ups at extremely low temperatures.

* 'Motion is matter's way of being.'

Ref. 12

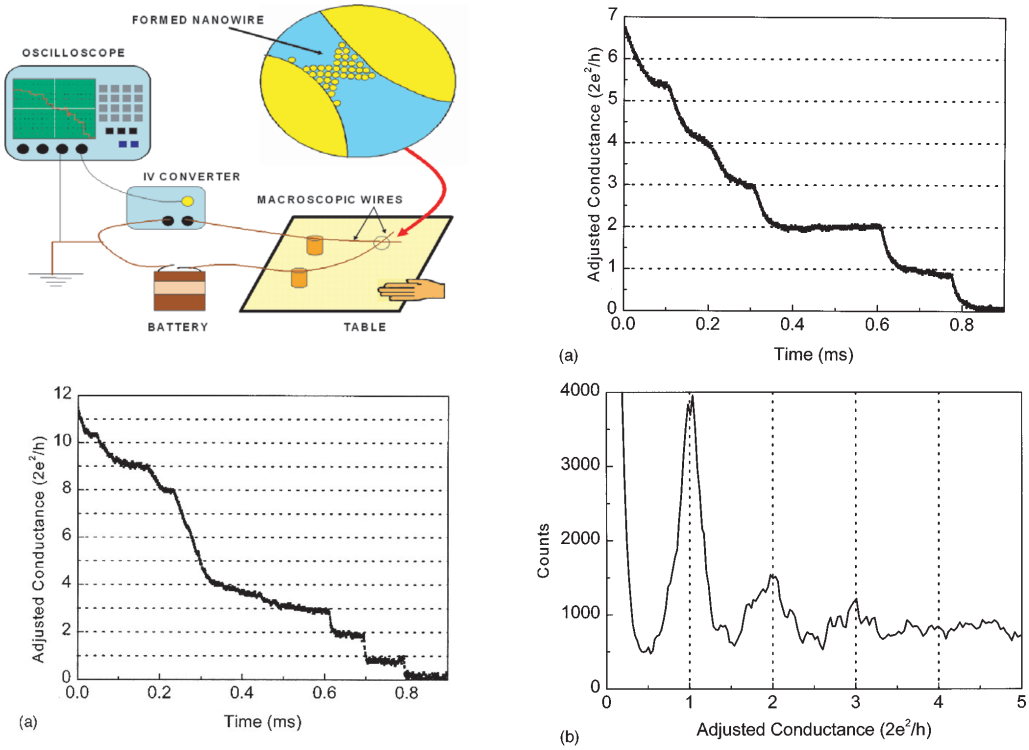

FIGURE 33 Steps in the flow of electricity in metal wire crossings: the set-up, the nanowires at the basis of the effect, and three measurement results (© José Costa-Krämer, AAPT from Ref. 34).

In fact, the quantization of conductivity appears in any electrical contact with a small cross-section. In such situations the quantum of action implies that the conductivity can only be a multiple of $2e^2/\hbar \approx (12\,906\,\Omega)^{-1}$. Can you confirm this result? Note that electrical conductivity can be as small as required; only the *quantized* electrical conductivity has the minimum value of $2e^2/\hbar$.

Challenge 57 ny

Many more elaborate experiments confirm the observation of conductance steps. They force us to conclude that there is a *smallest electric charge* in nature. This smallest charge has the same value as the charge of an electron. Indeed, electrons turn out to be part of every atom, in a construction to be explained shortly. In metals, a large number of electrons can move freely: that is why metals conduct electricity so well and work as mirrors.

In short, matter and electricity flow in smallest units. Depending on the material, the smallest flowing units of matter may be 'molecules', 'atoms', 'ions', or 'electrons'. Electrons, ions, atoms and molecules are *quantum* particles, or *quantons*. The quantum of action implies that matter is made of quantons. Matter quantons share some properties with ordinary stones, but also differ from them in many ways. A stone has position and momentum, mass and acceleration, size, shape, structure, orientation and angular momentum, and colour. Let us explore each of these properties for quantons, and see how they are related to the quantum of action.

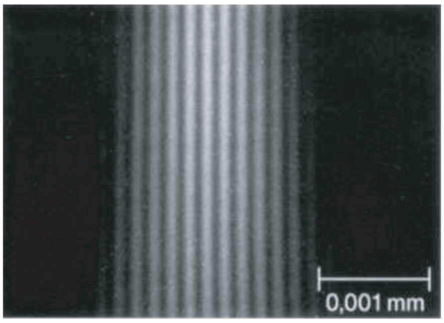

FIGURE 34 Electrons beams diffract and interfere at multiple slits (© Claus Jönsson).

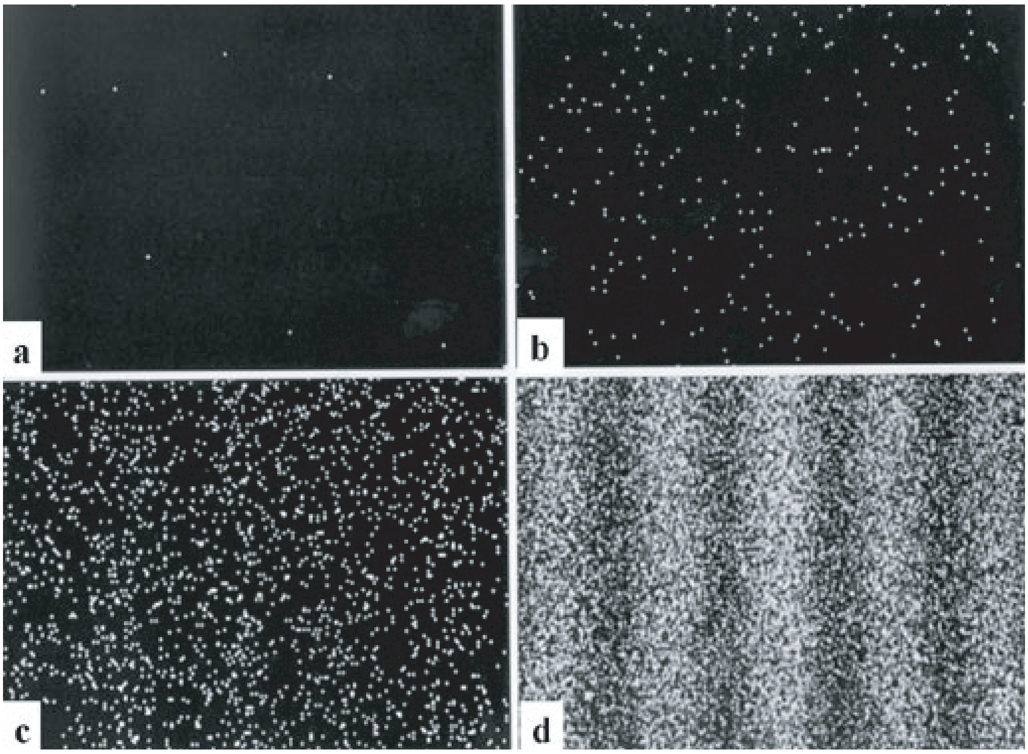

FIGURE 35 Formation over time of the interference pattern of electrons, here in a low-intensity double-slit experiment: (a) 8 electrons, (b) 270 electrons, (c) 2000 electrons, (d) 6000 electrons, after 20 minutes of exposure. The last image corresponds to the situation shown in the previous figure. (© Tonomura Akira/Hitachi).

Matter quantons and their motion – matter waves

Ref. 35 In 1923 and 1924, the French physicist Louis de Broglie pondered the consequences of the quantum of action for matter particles. In the case of light, the quantum of action connects wave behaviour to particle behaviour. He reasoned that the same should apply to matter. It dawned to him that streams of matter particles with the same momentum should behave as waves, just as light quanta do. He thus predicted that like for light, coherent matter flows should have a wavelength and angular frequency given by

$$\lambda = \frac{2\pi\hbar}{p} \quad \text{and} \quad \omega = \frac{E}{\hbar} , \tag{20}$$

where p and E are the momentum and the energy, respectively, of the single particles. Equivalently, we can write the relations as

$$\boldsymbol{p} = \hbar\boldsymbol{k} \quad \text{and} \quad E = \hbar\omega . \tag{21}$$

These relations state that matter also behaves as a wave.

Soon after de Broglie's prediction, experiments began to confirm the statement. It is indeed observed that matter streams can diffract, refract and interfere, with the values predicted by de Broglie. Because of the smallness of the wavelength, careful experiments are needed to detect these effects. But one by one, all experimental confirmations of the wave properties of light were repeated for matter beams. For example, just as light is diffracted when it passes around an edge or through a slit, matter is also diffracted in these situations. This is true even for electrons, the simplest particles of matter, as shown in Figure 34. The experiment with electrons is quite difficult. It was first performed by Claus Jönsson in Tübingen in 1961; in the year 2002 it was voted the most beautiful experiment in all of physics. Many years after Jönsson, the experiment was repeated with a modified electron microscope, as shown in Figure 35.

Ref. 36

Inspired by light interferometers, researchers began to build *matter* interferometers. Today these work with beams of electrons, nucleons, nuclei, atoms, or even large molecules. Just as observations of light interference prove the wave character of light, so the interference patterns observed with matter beams prove the wave character of matter.

Ref. 37

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Like light, matter is made of particles; like light, matter behaves as a wave when large numbers of particles with the same momentum are involved. But although beams of large molecules behave as waves, everyday objects – such as cars on a motorway – do not. There are two main reasons for this. First, for cars on a motorway the relevant wavelength is extremely small. Secondly, the speeds of the cars vary too much: streams of cars with the same speed cannot be made coherent.

If matter behaves like a wave, we can draw a strange conclusion. For any wave, the position and the wavelength cannot both be sharply defined simultaneously: the indeterminacies of the wave number $k = 2\pi/\lambda$ and of the position X obey the relation

$$\Delta k \Delta X \geq \frac{1}{2} . \tag{22}$$

Similarly, for every wave the angular frequency $\omega = 2\pi f$ and the instant T of its peak amplitude cannot both be sharply defined. Their indeterminacies are related by

$$\Delta\omega\Delta T \geq \frac{1}{2} . \tag{23}$$

Using de Broglie's wave properties of matter (21), we get

$$\Delta p \Delta X \geqslant \frac{\hbar}{2} \qquad \text{and} \qquad \Delta E \Delta T \geqslant \frac{\hbar}{2} \,. \tag{24}$$

These famous relations are called *Heisenberg's indeterminacy relations*. They were discovered by the German physicist Werner Heisenberg in 1925. They are valid for all quantum particles, be they matter or radiation. The indeterminacy relations state that there is no way to simultaneously ascribe a precise momentum and position to a quantum system, nor to simultaneously ascribe a precise energy and age. The more accurately one quantity is known, the less accurately the other is.* As a result, matter quantons – rather like stones, but unlike photons – *can* be localized, but always only approximately.

Both indeterminacy relations have been checked experimentally in great detail. All experiments confirm them. In fact, every experiment proving that matter behaves like a wave is a confirmation of the indeterminacy relation – and vice versa.

When two variables are linked by indeterminacy relations, one says that they are *complementary* to each other. Niels Bohr systematically explored all possible such pairs. You can also do that for yourself. Bohr was deeply fascinated by the existence of a complementarity principle, and he later extended it in philosophical directions. In a famous scene, somebody asked him what was the quantity complementary to precision. He answered: 'clarity'.

Challenge 58 s

We remark that the usual, *real*, matter quantons always move more slowly than light. Due to the inherent fuzziness of quantum motion, it should not come to a surprise that there are some exceptions. Indeed, in some extremely special cases, the quantum of action allows the existence of particles that move faster than light – so-called *virtual* particles – which we will meet later on.

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In short, the quantum of action means that matter quantons do not behave like point-like stones, but as waves. In particular, like for waves, the values of position and momentum cannot both be exactly defined for quantons. The values are fuzzy – position and momentum are undetermined. The more precisely one of the two is known, the less precisely the other is known.

Mass and acceleration of quantons

Matter quantons, like stones, have mass. Hits by single electrons, atoms or molecules can be detected. Quantons can also be slowed down or accelerated. We have already explored some of these experiments in the section on electrodynamics. However, quantons differ from pebbles. Using the time–energy indeterminacy relation, you can deduce that

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Challenge 59 s

$$a \leqslant \frac{2mc^3}{\hbar} \,. \tag{25}$$

* A policeman stops the car being driven by Werner Heisenberg. 'Do you know how fast you were driving?' 'No, but I know exactly where I was!'

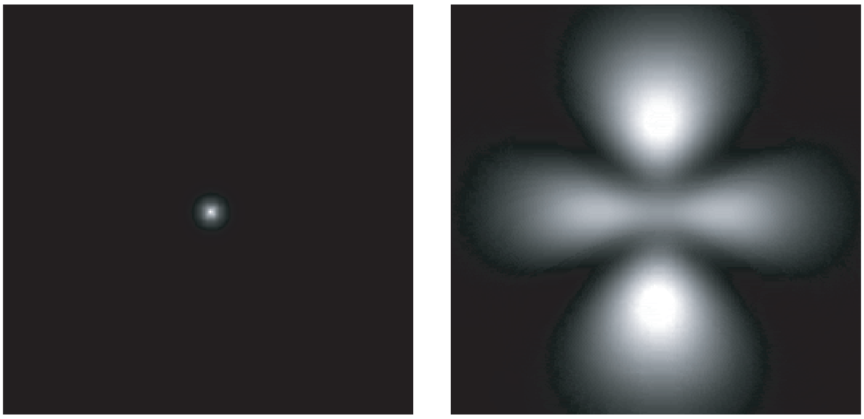

FIGURE 36 Probability clouds: a hydrogen atom in its spherical ground state (left) and in a non-spherical excited state (right) as seen by an observer travelling around it (QuickTime film produced with Dean Dauger's software package 'Atom in a Box', available at daugerresearch.com).

Thus there is a maximum acceleration for quantons.* Indeed, no particle has ever been observed with a higher acceleration than this value. In fact, no particle has ever been observed with an acceleration anywhere *near* this value. Ref. 38

Why are atoms not flat? Why do shapes exist?

The quantum of action determines all sizes in nature. In particular, it determines all *shapes*. Let us explore this topic.

Experiments show that all *composed* quantons have structures of finite sizes and often with complex shapes. All the size and shape of composed quantons is due to the motion of their constituents.

In 1901, Jean Perrin, and independently, in 1904, Nagaoka Hantaro, proposed that atoms are small 'solar systems'. In 1913, Niels Bohr used this idea, combining it with the quantum of action, and found that he could predict a number of properties of the hydrogen atom that had not until then been understood. These calculations, though often found in textbooks, are only of historical interest today. Even Bohr knew that they were not correct, because they assumed that hydrogen atoms were flat. This is observed not to be the case; moreover, it contradicts the quantum of action. The quantum of action implies that the motion of quantum constituents is fuzzy. Therefore, all composed quantons must be made of *clouds* of constituents. Ref. 39 Ref. 40 Challenge 61 e

In short, the quantum of action predicts that *atoms are spherical clouds*. Experiment and theory show that the shape of any atom or molecule is due to the cloud, or probability

* We note that this acceleration limit is different from the acceleration limit due to general relativity:

$$a \leqslant \frac{c^4}{4Gm} \,. \tag{26}$$

In particular, the quantum limit (25) applies to microscopic particles, whereas the general-relativistic limit applies to macroscopic systems. Can you confirm that in each domain the relevant limit is the smaller of the two? Challenge 60 e distribution, of its electrons. The quantum of action thus states that atoms or molecules are not hard balls, as Democritus or Dalton believed, but that they are clouds. Atomic electron clouds are not infinitely hard, but can to a certain degree mix and interpenetrate. Indeed, this mixing leads to molecules, liquids, solids, flowers and people.

Matter is made of clouds. On the other hand, while atoms are spherical, molecules have more complex shapes. A detailed exploration shows that all shapes, from the simplest molecules to the shape of people, are due to the interactions between electrons and nuclei of the constituent atoms. Nowadays, many shapes can be calculated to high precision. Small molecules, like water, have shapes that are fairly rigid, though endowed with a certain degree of elasticity. Large molecules, such as polymers or peptides, have flexible shapes. These shape changes are essential for their effects inside cells and thus for our survival. A large body of biophysical and biochemical research is exploring molecular shape effects.

In summary, the quantum of action thus implies that shapes exist and that they fluctuate. For example, if a long molecule is held fixed at its two ends, it cannot remain at rest in between. Such experiments are easy to perform nowadays, for example with DNA; they again confirm that perfect rest does not exist, and that the quantum of action is at the basis of chemistry and life.

All shapes are due to the quantum of action. Every object with a non-spherical shape is able to rotate. Let us explore what the quantum of action can say about rotation.

Rotation, quantization of angular momentum, and the lack of north poles

> Tristo è quel discepolo che non avanza il suo maestro.
>
> Leonardo da Vinci*

In everyday life, rotation is a frequent type of motion. Wheels are all around us. It turns out that the quantum of action has important consequences for rotational motion. First of all, we note that action and angular momentum have the same physical dimension: both are measured in Js or Nms. It only takes a little thought to show that if matter or radiation has a momentum and wavelength related by the quantum of action, then angular momentum is fixed in multiples of the quantum of action. This famous argument is due to Dicke and Wittke. Ref. 41

Imagine a circular fence, made of N steel bars spaced apart at a distance $a = 2\pi R/N$, as shown in Figure 37. At the centre of the fence, imagine a source of matter or radiation that can emit particles towards the fence in any chosen direction. The linear momentum of such a particle is $p = \hbar k = 2\pi\hbar/\lambda$. At the fence slits, the wave will interfere. Outside the fence, the direction of the motion of the particle is determined by the condition of positive interference. In other words, the angle θ, describing the direction of motion outside the fence, is given by $a \sin\theta = M\lambda$, where M is an integer. Through the deflection due to the interference process, the fence receives a linear momentum $p \sin\theta$, or an angular momentum $L = pR \sin\theta$. Using all these expressions, we find that the angular

* 'Sad is that disciple who does not surpass his master.' This statement is sculpted in large letters in the chemistry aula of the University of Rome *La Sapienza*.

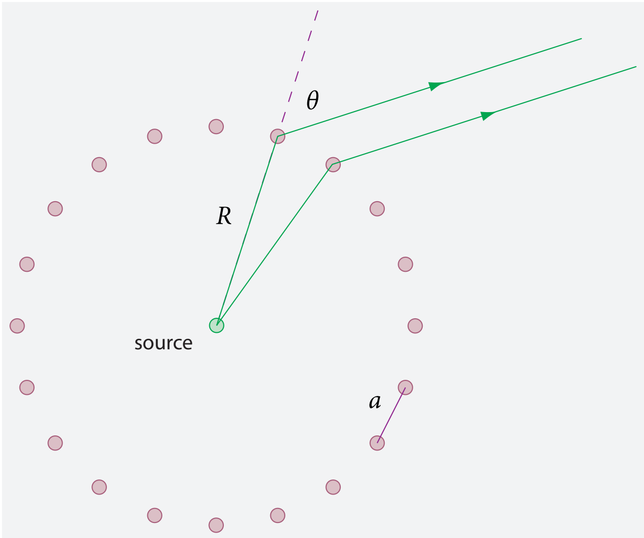

FIGURE 37 The quantization of angular momentum.

momentum transferred to the fence is

$$L = NM\hbar . \qquad (27)$$

In other words, the angular momentum of the fence is an integer multiple of $\hbar$. Fences can only have integer intrinsic angular momenta (in units of $\hbar$). Of course, this is only a hint, not a proof. Nevertheless, the generalization of argument to all bodies is correct. The measured intrinsic angular momentum of bodies is always a multiple of $\hbar$. Quantum theory thus states that every object's angular momentum increases in steps. Angular momentum is quantized.

But rotation has more interesting aspects. Thanks to the quantum of action, just as linear momentum is usually fuzzy, so is angular momentum. There is an indeterminacy
Ref. 42 relation for angular momentum L. The complementary variable is the phase angle φ of
Ref. 43 the rotation. The indeterminacy relation can be expressed in several ways. The simplest
Page 42 – though not the most precise – is

$$\Delta L \, \Delta\varphi \geqslant \frac{\hbar}{2} . \qquad (28)$$

(This is obviously an approximation: the relation is only valid for large angular momenta. It cannot be valid for small values, as $\Delta\varphi$ by definition cannot grow beyond 2π. In particular, angular-momentum eigenstates have $\Delta L = 0$.*) The indeterminacy of angular

* An *exact* formulation of the indeterminacy relation for angular momentum is

$$\Delta L \, \Delta\varphi \geqslant \frac{\hbar}{2} |1 - 2\pi P(\pi)| , \qquad (29)$$

where $P(\pi)$ is the normalized probability that the angular position has the value π. For an angular-momentum eigenstate, one has $\Delta\varphi = \pi/\sqrt{3}$ and $P(\pi) = 1/2\pi$. This expression has been tested and conmomentum appears for all macroscopic bodies, or alternatively, for all cases when the angular phase of the system can be measured.

The quantization and indeterminacy of angular momentum have important consequences. Classically speaking, the poles of the Earth are the places that do not move when observed by a non-rotating observer. Therefore, at those places matter would have a defined position and a defined momentum. However, the quantum of action forbids this. There cannot be a North Pole on Earth. More precisely, the idea of a rotational axis is an approximation, not valid in general.

Rotation of quantons

Even more interesting are the effects of the quantum of action on microscopic particles, such as atoms, molecules or nuclei. We note again that action and angular momentum have the same units. The precision with which angular momentum can be measured depends on the precision of the rotation angle. But if a *microscopic* particle rotates, this rotation might be unobservable: a situation in fundamental contrast with the case of *macroscopic* objects. Experiments indeed confirm that many microscopic particles have unobservable rotation angles. For example, in many (but not all) cases, an atomic nucleus rotated by half a turn cannot be distinguished from the unrotated nucleus.

If a microscopic particle has a *smallest* unobservable rotation angle, the quantum of action implies that the angular momentum of that particle *cannot* be zero. It must always be rotating. Therefore we need to check, for each particle, what its smallest unobservable angle of rotation is. Physicists have checked all particles in nature in experiments, and found smallest unobservable angles (depending on the particle type) of 0, 4π, 2π, $4\pi/3$, π, $4\pi/5$, $2\pi/3$, etc.

Let us take an example. Certain nuclei have a smallest unobservable rotation angle of *half* a turn. This is the case for a prolate nucleus (one that looks like a rugby ball) turning around its short axis. Both the largest observable rotation and the indeterminacy are thus a *quarter* turn. Since the change, or action, produced by a rotation is the number of turns multiplied by the angular momentum, we find that the angular momentum of this nucleus is $2 \cdot \hbar$.

As a general result, we deduce from the minimum angle values that the angular momentum of a microscopic particle can be 0, $\hbar/2$, $\hbar$, $3\hbar/2$, $2\hbar$, $5\hbar/2$, $3\hbar$, etc. In other words, the intrinsic angular momentum of a particle, usually called its *spin*, is an integer multiple of $\hbar/2$. Spin describes how a particle behaves under rotations. (It turns out that all spin-0 particles are composed of other particles, thus respecting the quantum of action as the limit for rotational motion in nature.)

How can a particle rotate? At this point, we do not yet know how to *picture* the rotation. But we can *feel* it – just as we showed that light is made of rotating entities: all matter, including electrons, can be *polarized*. This is shown clearly by the famous Stern–Gerlach experiment.

Ref. 44 firmed by experiments.

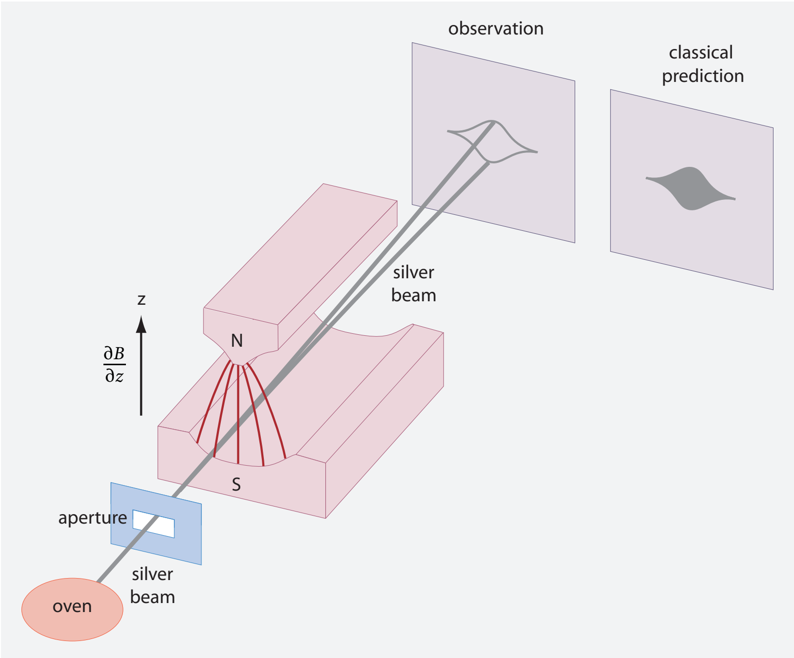

FIGURE 38 The Stern–Gerlach experiment.

Silver, Stern and Gerlach – polarization of quantons

After a year of hard work, in 1922, Otto Stern and Walther Gerlach* completed a beautiful experiment to investigate the polarization of matter quantons. They knew that inhomogeneous magnetic fields act as polarizers for rotating charges. Rotating charges are present in every atom. Therefore they let a beam of silver atoms, extracted from an oven by evaporation, pass an inhomogeneous magnetic field. They found that the beam splits into *two separate* beams, as shown in Figure 38. No atoms leave the magnetic field region in intermediate directions. This is in full contrast to what would be expected from classical physics. Ref. 45

The splitting into two beams is an intrinsic property of silver atoms; today we know that it is due to their spin. Silver atoms have spin $\hbar/2$, and depending on their orientation in space, they are deflected either in the direction of the field inhomogeneity or against it. The splitting of the beam is a pure quantum effect: there are no intermediate options. Indeed, the Stern–Gerlach experiment provides one of the clearest demonstrations that classical physics does not work well in the microscopic domain. In 1922, the result seemed so strange that it was studied in great detail all over the world.

When one of the two beams – say the 'up' beam – is passed through a second set-up, all the atoms end up in the 'up' beam. The other possible exit, the 'down' beam, remains unused in this case. In other words, the up and down beams, in contrast to the original beam, cannot be split. This is not surprising.

* Otto Stern (1888–1969) and Walther Gerlach (1889–1979), both German physicists, worked together at the University of Frankfurt. For his subsequent measurement of the anomalous magnetic moment of the proton, Stern received the Nobel Prize for physics in 1943, after he had to flee Germany.

But if the second set-up is rotated by $\pi/2$ with respect to the first, again two beams – 'right' and 'left' – are formed, and it does not matter whether the incoming beam is directly from the oven or from the 'up' part of the beam. A partially-rotated set-up yields a partial, uneven split. The proportions depend on the angle.

We note directly that if we split the beam from the oven first vertically and then horizontally, we get a different result from splitting the beam in the opposite order. Splitting processes do not commute. (When the order of two operations makes a difference to the net result, physicists call them 'non-commutative'.) Since all measurements are also physical processes, we deduce that, in general, measurements and processes in quantum systems are non-commutative.

Challenge 62 e

Beam splitting is direction-dependent. Matter beams behave almost in the same way as polarized light beams. Indeed, the inhomogeneous magnetic field acts on matter somewhat like a polarizer acts on light. The up and down beams, taken together, define a polarization direction. In fact, the polarization direction can be rotated (with the help of a *homogeneous* magnetic field). Indeed, a rotated beam in a unrotated magnet behaves like an unrotated beam in a rotated magnet.

Curiosities and fun challenges about quantum matter

> It is possible to walk while reading, but not to read while walking.
>
> Serge Pahaut

The quantum of action implies that there are *no fractals* in nature. Everything is made of particles. And particles are clouds. Quantum theory requires that all shapes in nature be 'fuzzy' *clouds*.

* *

Can atoms rotate? Can an atom that falls on the floor roll under the table? Can atoms be put into high-speed rotation? The answer is 'no' to all these questions, because angular momentum is quantized; moreover, atoms are not solid objects, but clouds. The macroscopic case of an object turning more and more slowly until it stops does not exist in the microscopic world. The quantum of action does not allow it.

Ref. 57

* *

Light is refracted when it enters dense matter. Do matter waves behave similarly? Yes, they do. In 1995, David Pritchard showed this for sodium waves entering a gas of helium and xenon.

Ref. 59

* *

Many quantum effects yield curves that show steps. An important example is the molar heat of hydrogen H_2 gas. In creasing the temperature from 20 to 8 000 K, the molar heat is shows two steps, first from $3R/2$ to $5R/2$, and then to $7R/2$. Can you explain why?

* *

Most examples of quantum motion given so far are due to electromagnetic effects. Can you argue that the quantum of action must also apply to nuclear motion?

Challenge 63 ny

First summary on the motion of quantum particles

In summary, the 'digital' beam splitting seen in the Stern–Gerlach experiment and the wave properties of matter force us to rethink our description of motion. In special relativity, the existence of a maximum speed forced us to introduce the concept of space-time, and then to refine our description of motion. In general relativity, the maximum force obliged us to introduce the concepts of horizon and curvature, and then again to refine our description of motion. At this point, the existence of the quantum of action and the wave behaviour of matter force us to take the same two similar steps. We first introduce the concept of a wave function, and then we refine our description of motion.

Chapter 4

THE QUANTUM DESCRIPTION OF MATTER AND ITS MOTION

In everyday life and in classical physics, we say that a system *has* a position, that it *is* oriented in a certain direction, that it *has* an axis of rotation, and that it *is* in a state with specific momentum. In classical physics, we can talk in this way because the *state* – the situation a system 'is' in and the properties a system 'has' – and the *results of measurement* coincide. They coincide because measurements can be imagined to have a negligible effect on the system.

Because of the existence of a smallest action, the interaction necessary to perform a measurement on a system *cannot* be made arbitrarily small. Therefore, the quantum of action makes it *impossible* for us to continue saying that a system *has* momentum, position or an axis of rotation. We are forced to use the idea of the rotating arrow and to introduce the concept of *wave function* or *state function*. Let us see why and how.

The Stern–Gerlach experiment shows that the measured values of spin orientation are *not* intrinsic, but result from the measurement process itself (in this case, the interaction with the inhomogeneous field). This is in contrast to the spin *magnitude*, which is intrinsic and state-independent.

Therefore, the quantum of action forces us to distinguish three entities:

— the state of the system;
— the operation of measurement;
— the result of the measurement.

In contrast to the classical, everyday case, the state of a quantum system (the properties a system 'has') is *not* described by the outcomes of measurements. The simplest illustration of this difference is the system made of a single particle in the Stern–Gerlach experiment. The experiment shows that a spin measurement on a general (oven) particle state sometimes gives 'up' (say +1), and sometimes gives 'down' (say −1). So a general atom, in an oven state, has no intrinsic orientation. Only after the measurement, an atom is either in an 'up' state or in a 'down' state. It is also found that feeding 'up' states into the measurement apparatus gives 'up' states: thus certain special states, called *eigenstates*, do remain unaffected by measurement. Finally, the experiment shows that states can be rotated by applied fields: they have a direction in space.

These details can be formulated in a straightforward way. Since measurements are operations that take a state as input and produce as output an output state and a measurement result, we can say:

— *States* are described by rotating arrows.

— *Measurements* of observables are operations on the state vectors.
— *Measurement results* are *real numbers*; and like in classical physics, they usually depend on the observer.

Thus we have distinguished three quantities that are not distinguished in classical physics: states, measurement of observables and measurement results. Given this distinction, quantum theory follows quite simply, as we shall see.

Quantum physics describes observables as operators, and thus as *transformations* in Hilbert space, because any measurement is an interaction with a system and thus a transformation of its state. The Stern–Gerlach experiment shows this clearly: the interaction with the field transforms some atoms in one way, and other atoms in another way.

Page 201

— Mathematically, states are *complex vectors* in an abstract space. The space of all possible states is a *Hilbert space*.
— Mathematically, measurements are linear transformations, more precisely, they are described by self-adjoint, or *Hermitean*, *operators* (or matrices).
— Mathematically, *changes of viewpoint* are described by *unitary operators* (or matrices) that act on states and on measurement operators.

Quantum-mechanical experiments also show that a measurement of an observable can only give a result that is an *eigenvalue* of this transformation. The resulting states, those exceptional states that are *not* influenced when the corresponding variable is measured, are the *eigenvectors*. Therefore every expert on motion must know what an eigenvalue and an eigenvector is.

For any linear transformation T, those special vectors ψ that are transformed into multiples of themselves,

$$T\psi = \lambda\psi \tag{30}$$

are called *eigenvectors* (or *eigenstates*), and the multiplication factor λ is called the associated *eigenvalue*. Experiments show that the state of the system *after* a measurement is given by the eigenvector corresponding to the measured eigenvalue. In the Stern–Gerlach experiment, the eigenstates are the 'up' and the 'down' states. Eigenstates are those states that do not change when the corresponding variable is measured. Eigenvalues of Hermitean operators are always real, so that consistency is ensured.

In summary, the quantum of action obliges us to distinguish between three concepts that are mixed together in classical physics: the *state* of a system, a *measurement* on the system, and the *measurement result*. The quantum of action forces us to change the vocabulary with which we describe nature, and obliges to use more differentiated concepts. Now follows the main step: the description of *motion* with these concepts. This is what is usually called 'quantum theory'.

Visualizing the wave function: rotating arrows and probability clouds

The state of a quantum particle is described by a wave function. To be able to visualize the wave function, we first imagine that a quantum particle is localized as much as possible. In this case, the wave function for a free quanton can be described simply by a rotating arrow.

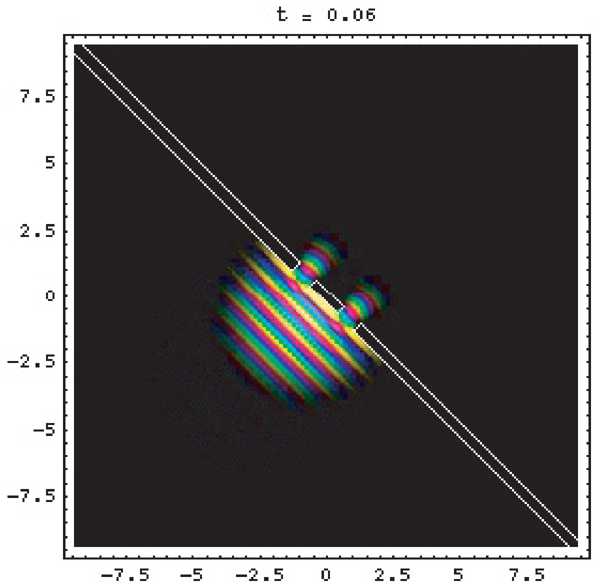

FIGURE 39 The motion of a wave function, the quantum state, through a double slit, showing both the particle and the wave properties of matter. The density of the state is displayed by brightness, and the local phase is encoded in the colour. (QuickTime film © Bernd Thaller).

When a *localized* quanton travels through space, the attached arrow rotates. If the particle is non-relativistic and if spin can be neglected, the rotation takes place in a plane perpendicular to the direction of motion. The end of the arrow then traces a *helix* around the direction of motion. In this case, the state at a given time is described by the angle of the arrow. This angle is the quantum phase. The quantum phase is responsible for the wave properties of matter, as we will see. The wavelength and the frequency of the helix are determined by the momentum and the kinetic energy of the particle.

If the particle is *not localized* – but still non-relativistic and with negligible spin effects – the state, or the wave function, defines a rotating arrow at each point in space. The rotation still takes place in a plane perpendicular to the direction of motion. But now we have a distribution of arrows that trace helices parallel to the direction of motion. At a given point in space and time, the state has a quantum phase and a length of the arrow.

Figure 39 Shows the evolution of a wave function. The direction of the arrow at each point is shown by the colour at the specific point. The wave function $\psi(t, x)$ for non-relativistic particles with negligible spin effects is described by a length and a phase: it is a *complex* number at each point in space. The phase is essential for interference and many other wave effects. What measurable property does the amplitude, the length of the local arrow, describe? The answer was given by the German physicist Max Born:the amplitude of the wave function is a *probability amplitude*. The square of the amplitude, i.e., the quantity $|\psi(t, x)|^2$ gives the probability to find the particle at the place x at time t.

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Note that even though the wave function can be seen as defining an arrow at every point in space, the wave function as a whole can also be described as one, single vector, this time in a Hilbert space. For free particles, the Hilbert space is infinite dimensional! Nevertheless, it is not hard to calculate in such spaces. The scalar product of two wave functions is the spatial integral of the product of the complex conjugate of the first function and the (unconjugated) second function. With this definition, all vector concepts (unit vectors, null vectors, basis vectors, etc.) can be meaningfully applied to wave functions.

In short, we can imagine the state or *wave function* of non-relativistic quantum particles as an arrow at every point in space. The rotation frequency of the arrow is its *kinetic energy*; the wavelength of the arrow motion – the period of the helical curve that the tip of the arrow traces during motion – is the *momentum* of the quantum particle. An arrow at each point in space is a *field*; since the field is concentrated in the region where the particle is located, and since the amplitude of the field is related to the probability to find the particle, the wave field can best be called a *cloud*.

The state or wave function of a quantum particle is a rotating cloud. One aspect of this rotating cloud is unusual: the cloud is made of little arrows. Every point of the cloud is described by a local density and a local orientation. This last property is not present in any cloud of everyday life. Therefore, for many decades it was tacitly assumed that no simple visualization of such a cloud is possible. Only the last years have shown that there is a simple visualization for such clouds; this visualization is presented in the last volume of this series.

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The state evolution – the Schrödinger equation

The description of the state of a non-relativistic quanton with negligible spin effects as rotating cloud *completely* determines how the wave function evolves in time. For such quantum particles the evolution follows from the total energy, the sum of kinetic and potential energy $T + V$, and the properties of matter waves.

The local rate of change of the state arrow ψ is produced by the local total energy, or Hamiltonian, $H = T + V$:

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \; . \qquad (31)$$

This famous equation is *Schrödinger's equation of motion*.* This evolution equation applies to all quantum systems and is one of the high points of modern physics.

Ref. 46
Ref. 47

In fact, Erwin Schrödinger had found his equation in two different ways. In his first paper, he deduced it from a variational principle. In his second paper, he deduced the evolution equation directly, by asking a simple question: how does the state evolve? He knew that the state of a quanton behaves both like a wave and like a particle. A wave is described by a field, which he denoted $\psi(t, x)$. If the state ψ behaves like a wave, then the corresponding wave function' must be an amplitude W multiplied by a phase factor

* Erwin Schrödinger (b. 1887 Vienna, d. 1961 Vienna) was famous for being a *physicien bohémien*, always living in a household with two women. In 1925 he discovered the equation that brought him international fame, and the Nobel Prize for physics in 1933. He was also the first to show that the radiation discovered by Victor Hess in Vienna was indeed coming from the cosmos. He left Germany, and then again Austria, out of dislike for National Socialism, and was a professor in Dublin for many years. There he published his famous and influential book *What is life?*. In it, he came close to predicting the then-unknown nucleic acid DNA from theoretical insight alone.

FIGURE 40 Erwin Schrödinger (1887–1961)

$e^{i\boldsymbol{kx}-\omega t}$. The state can thus be written as

$$\psi(t,x) = W(t,x)e^{i\boldsymbol{kx}-\omega t} . \tag{32}$$

The amplitude W is the length of the local arrow; the phase is the orientation of the local arrow. Equivalently, the amplitude is the local density of the cloud, and the phase is the local orientation of the cloud.

We know that the quantum wave must also behave like a particle of mass m. In particular, the non-relativistic relation between energy and momentum $E = \boldsymbol{p}^2/2m + V(\boldsymbol{x})$ – where $V(\boldsymbol{x})$ is the potential at position $\boldsymbol{x}$ – must be fulfilled for these waves. The two de Broglie relations (21) for matter wavelength and frequency then imply Page 62

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V(\boldsymbol{x})\psi . \tag{33}$$

As mentioned, this equation states how the arrow wave associated to a particle, the wave function ψ, evolves over time. In 1926, this *wave equation* for the complex field ψ became instantly famous when Schrödinger used it, by inserting the potential felt by an electron near a proton, to calculate the energy levels of the hydrogen atom. In a hydrogen atom, light is emitted by the single electron inside that atom; therefore a precise description of the motion of the electron in a hydrogen atom allows us to describe the light frequencies it can emit. (We will perform the calculation and the comparison with experiment below.) Page 154 First of all, the Schrödinger equation explained that only *discrete* colours are emitted by hydrogen; in addition, the frequencies of the emitted light were found to be in agreement with the prediction of the equation to five decimal places. This was an important result, especially if we keep in mind that classical physics cannot even explain the existence of atoms, let alone their light emission! In contrast, quantum physics explains all properties of atoms and their colours to high precision. The discovery of the quantum of action led the description of the motion of matter to a new high point.

In fact, the exact description of matter is found when spin and the *relativistic* energy–momentum relation are taken into account. Page 158 We do this below. No deviations between relativistic calculations and experiments have ever been found. Even today, predictions and measurements of atomic spectra remain the most precise and accurate in the whole study of nature: in the cases that experimental precision allows it, the calculated values agree with experiments to 13 decimal places.

Self-interference of quantons

Waves interfere. We saw above that all experiments confirm that all quantum particles show interference. Figure 39 shows that interference is a direct consequence of the Schrödinger equation. The film shows the solution of the Schrödinger equation for a double slit. The film visualizes how the wave function behaves when a double slit induces diffraction and interference. It turns out that the Schrödinger equation indeed reproduces and explains the observations shown in Figure 34 and Figure 35: interference is due to the evolution of rotating clouds.

Like in all interference phenomena, the local intensity of the interference pattern turns out to be proportional to the square $|W|^2$ of the state amplitude. We also note that even though the wave function is spread out over the whole detection screen just before it hits the screen, it nevertheless yields only a localized spot on the screen. This effect, the so-called *collapse of the wavefunction*, is explored in detail below. Page 127

The speed of quantons

Let us delve a little into the details of the description given by the Schrödinger equation (33). The equation expresses a simple connection: the classical speed of a matter particle is the *group velocity* of the wave function ψ. We know from classical physics that the group velocity is not always well defined: in cases where the group dissolves in several peaks, the concept of group velocity is not of much use. These are the cases in which quantum motion is very different from classical motion, as we will soon discover. But for the other, more well-behaved cases, we find that the wave function moves in the same way as a classical particle does.

The Schrödinger equation makes another point: velocity and position of matter are not independent variables, and cannot be chosen at will. Indeed, the initial condition of a system is given by the initial value of the wave function alone. No derivatives have to be (or can be) specified. In other words, quantum systems are described by a *first-order* evolution equation, in stark contrast to classical systems.

Dispersion of quantons

For free quantum particles, the evolution equation has a simple consequence, shown in Figure 41. Imagine a wave function that is localized around a given starting position. This wave function describes a quantum system at rest. When time passes, this wave function will *spread out* in space. Indeed, the evolution equation is similar, mathematically, to a diffusion equation. In the same way that a drop of ink in water spreads out, also a localized quantum particle will spread out in space. The most probable position stays unchanged, but the probability to find the particle at large distances from the starting position increases over time. For quantum particles, this spreading effect is indeed observed in experiment; it is a consequence of the wave aspect of matter. For macroscopic objects, the spreading effect is not observed, however: cars rarely move away from parking spaces. Indeed, quantum theory predicts that for macroscopic systems, the effect of

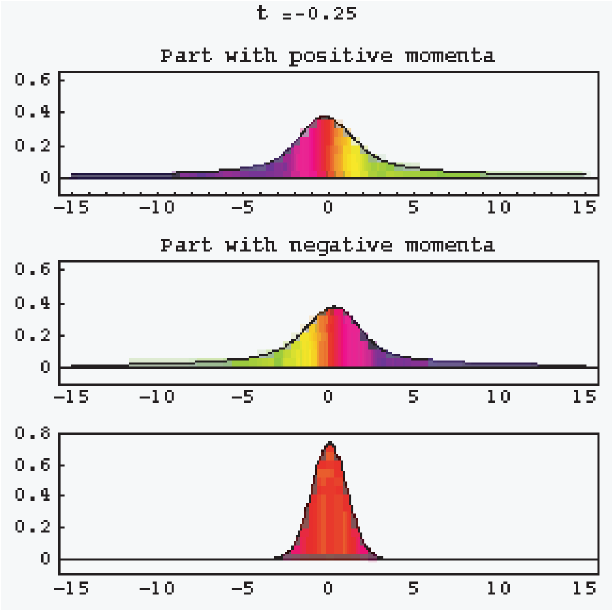

FIGURE 41 The evolution of a wave function (lowest curve) with zero momentum, and the motion of its parts with positive and negative momenta. Local phase is encoded in the colour. (QuickTime film © Bernd Thaller).

Challenge 64 ny spreading is negligibly small. Can you show why?

In summary, the wave aspect of matter leads to the spreading of wave functions. Wave functions show dispersion.

Tunnelling and limits on memory – damping of quantons

'Common sense' says that a slow ball cannot roll over a high hill. More precisely, classical physics says that if the kinetic energy T is smaller than the potential energy V that the ball would have at the top of the hill, then the ball cannot reach the top of the hill. In contrast, according to quantum theory, there is a positive probability of passing the hill for *any* energy of the ball.

In quantum theory, hills and obstacles are described by potential barriers, and objects by wave functions. Any initial wave function will spread *beyond* any potential barrier of finite height and width. The wave function will also be non-vanishing at the location of the barrier. In short, any object can overcome any hill or barrier. This effect is called the *tunnelling effect*. It is in complete contrast to everyday experience – and to classical mechanics.

The tunnelling effect results from a new aspect contained in the quantum description of hills: the statement that all obstacles in nature can be overcome with a *finite* effort. No obstacle is infinitely difficult to surmount. Indeed, only for a potential of infinite height would the wave function vanish and fail to spread to the other side. But such potentials exist only as approximations; in nature potentials are always of finite value.

Challenge 65 ny How large is the tunnelling effect? Calculation shows that the transmission probability

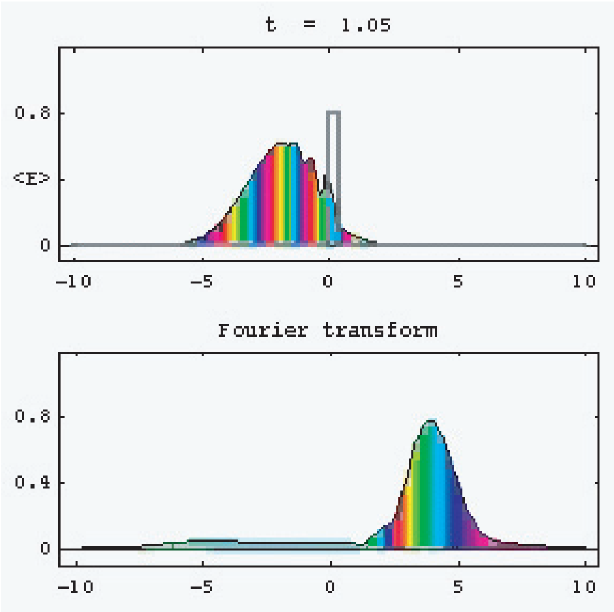

FIGURE 42 The tunnelling of a wave function through a potential hill (the rectangular column): most of the wave function is reflected, and part of the wave function passes to the other side. Local phase is encoded in the colour. (QuickTime film © Bernd Thaller).

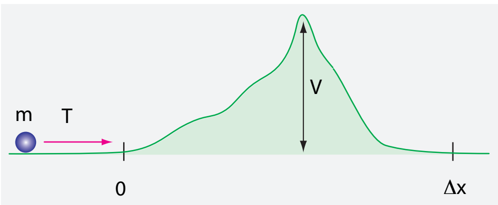

FIGURE 43 Climbing a hill.

P is given approximately by

$$P \approx \frac{16T(V-T)}{V^2}\, \mathrm{e}^{-\frac{2w}{\hbar}\sqrt{2m(V-T)}} \qquad (34)$$

where w is the width of the hill, v its height, and m and T the mass and the kinetic energy of the particle. For a system of large number of particles, the probability is the product of the probabilities for the different particles.

Let us take the case of a car in a garage, and assume that it is made of 10^{28} atoms at room temperature. A typical garage wall has a thickness of 0.1 m and a potential height of $V = 1\,\mathrm{keV} = 160\,\mathrm{aJ}$ for the passage of an atom. We get that the probability of finding the car outside the garage is

$$P \approx \left(10^{-(10^{12})}\right)^{(10^{28})} \approx 10^{-(10^{40})} . \qquad (35)$$

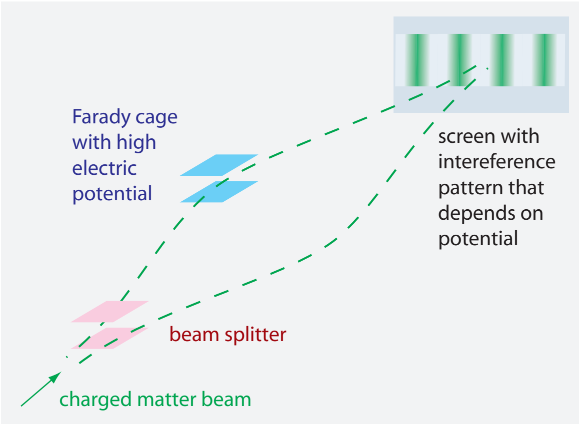

FIGURE 44 A localized electric potential in an interferometer leads to a shift of the interference pattern.

Challenge 66 e

The smallness of this value (just try to write it down, to be convinced) is the reason why it is never taken into account by the police when a car is reported missing. (Actually, the probability is considerably smaller. Can you name at least one effect that has been forgotten in this simple calculation?)

Challenge 67 ny

Obviously, tunnelling can be important only for small systems, made of a few particles, and for thin barriers, with a thickness of the order of $\hbar/\sqrt{2m(V-T)}$. For example, tunnelling of single atoms is observed in solids at high temperature, but is not important in daily life. For electrons, the effect is larger: the barrier width w for an appreciable tunnelling effect is

$$w \approx \frac{0.5\,\mathrm{nm}\,\sqrt{\mathrm{aJ}}}{\sqrt{V-T}} \,. \tag{36}$$

At room temperature, the kinetic energy T is of the order of 6 zJ; increasing the temperature obviously increases the tunnelling. As a result, electrons tunnel quite easily through barriers that are a few atoms in width. Indeed, every TV tube uses tunnelling at high temperature to generate the electron beam producing the picture. The necessary heating is the reason why television tubes take some time to switch on.

The tunnelling of electrons also limits the physical size of computer memories. Memory chips cannot be made arbitrary small. Silicon integrated circuits with one terabyte (TB) of random-access memory (RAM) will probably never exist. Can you imagine why?

Challenge 68 s

In fact, tunnelling limits the working of any type of memory, including that of our brain. Indeed, if we were much hotter than 37°C, we could not remember anything!

Since light is made of particles, it can also tunnel through potential barriers. The best – or highest – potential barriers for light are mirrors; mirrors have barrier heights of the order of one attojoule. Tunnelling implies that light *can* be detected behind any mirror. These so-called *evanescent waves* have indeed been detected; they are used in various high-precision experiments.

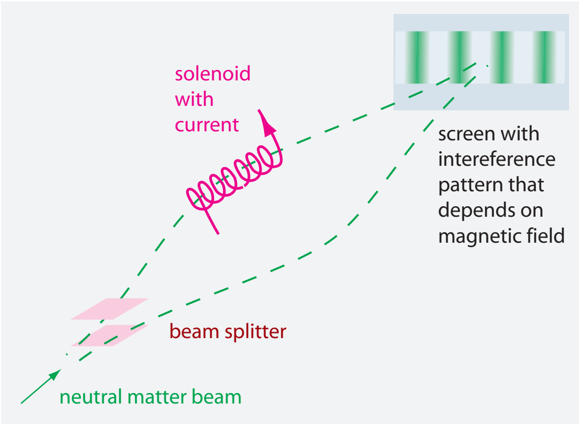

FIGURE 45 Magnetic fields change the phase of a spinning particle.

The quantum phase

The motion of the cloud for a single microscopic particle is described by two quantities: the *amplitude* and the *phase*. Whereas the amplitude is easy to picture – just think of the (square root of the) density of a real cloud – the phase takes more effort. States or wave functions are *clouds with a local phase*: they are clouds of objects that can be rotated. In case of an everyday water cloud, local rotation of droplets has no effect of the cloud. In contrast, in quantum theory, the local rotation of the cloud, thus the change of its phase, does have a measurable effect. Let us explore this point.

Page 48 The phase of matter waves behaves like the phase of photons: it evolves with time, and thus increases along the path of a moving particle. The phase can be pictured by a small rotating arrow. The angular velocity with which the phase rotates is given by the famous relation $\omega = E/\hbar$. In short, *we can picture the wave function of a free particle as a moving cloud that rotates and disperses.*

Above all, the phase is that aspect of the wave function that leads to the wave effects of matter. In particular, the phase of the wave function leads to interference effects. When two partial wave functions are separated and recombined after a relative phase change, the phase change will determine the interference. This is the origin of the electron beam interference observations shown in Figure 34. Without quantum phase, there would be no extinction and no interference.

The phase of a wave function can be influenced in many ways. The simplest way is the use of electric fields. If the wave function of a charged particle is split, and one part is led through a region with an electric field, a phase change will result. The arrangement is shown in Figure 44. A periodic change of the electric potential yields a periodic shift of the interference pattern. This is indeed observed.

Another simple case of phase manipulation is shown in Figure 45: also a magnetic field changes the phase of a spinning particle, and thus influences the interference behaviour.

A famous experiment shows the importance of the phase in an even more surprising way: the *Aharonov–Bohm effect*. The effect is famous because it is counter-intuitive and because it was predicted before it was observed. Look at the set-up shown in Figure 46.
Ref. 48

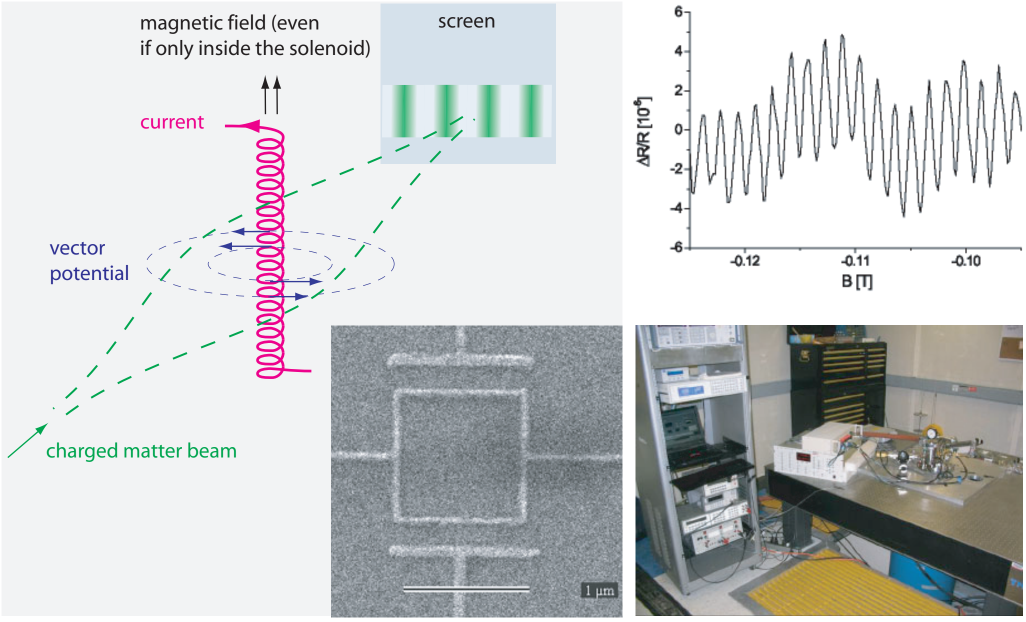

FIGURE 46 The Aharonov–Bohm effect: the influence of the magnetic vector potential on interference (left) and a measurement confirmation (right), using a microscopic sample that transports electrons in thin metal wires (© Doru Cuturela).

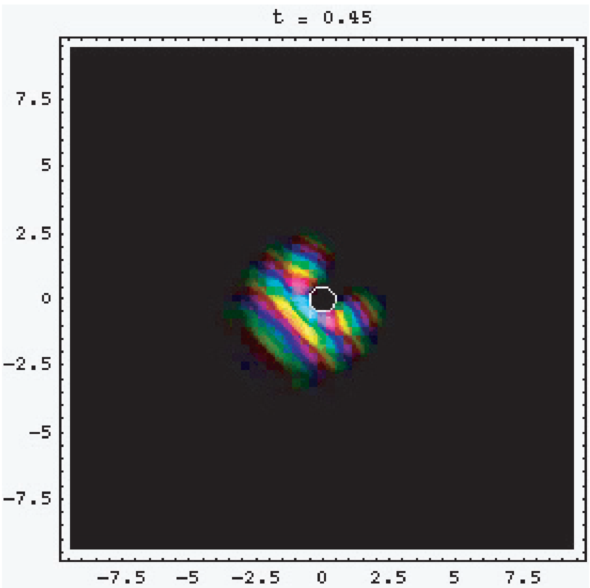

FIGURE 47 The motion of a wave function around a solenoid showing the Aharonov–Bohm effect. The density of the state is displayed by brightness, and the local phase is encoded in the colour. (QuickTime film © Bernd Thaller).

A matter wave of charged particles is split into two by a cylinder – positioned at a right angle to the matter's path – and the matter wave recombines behind it. Inside the cylinder there is a magnetic field; outside, there is none. (A simple way to realize such a cylinder is

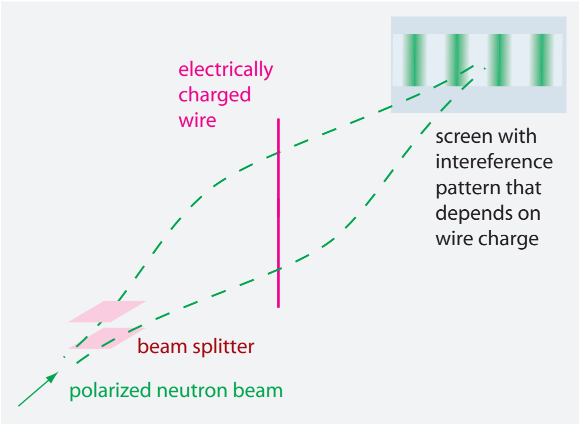

FIGURE 48 The Aharonov–Casher effect: the influence of charge on the phase leads to interference even for interfering neutrons.

a long solenoid.) Quantum physics predicts that an interference pattern will be observed, and that the position of the stripes will depend on the value of the magnetic field. This happens even though the wave never enters the region with the field! The effect has been observed in countless experiments.

The Aharonov–Bohm effect is surprising. The reason for the effect is simple: for a charged particle, the phase of a wave function is determined by the vector potential $\boldsymbol{A}$, not by the magnetic field $\boldsymbol{B}$. The vector potential around a solenoid does not vanish, as Vol. III, page 69 we know from the section on electrodynamics, but circulates around the solenoid. This circulation distinguishes the two sides of the solenoid and leads to a phase shift – one that indeed depends on the magnetic field value – and thus produces interference, even though the particle never interacts with the magnetic field.

A further example for phase manipulation is the so-called *Aharonov–Casher effect*, which even occurs for neutral particles, as long as they have a magnetic moment, such as neutrons. The phase of a *polarized* neutron will be influenced by an electric field, so that the arrangement shown in Figure 48 will show an interference pattern that depends on the applied electric potential.

Another case of phase manipulation will be presented later on: also gravitational fields can be used to rotate wave functions. In fact, even the acceleration due to rotational motion can do so. In fact, it has been possible to measure the rotation of the Earth by Ref. 49 observing the change of neutron beam interference patterns.

Another important class of experiments that manipulate the phase of wave functions are possible with macroscopic quantum phenomena. In superconductivity and in superfluidity, the phase of the wave function is regularly manipulated with magnetic and electric fields. This possibility has many important technical applications. For example, the Josephson effect is used to measure electric potential differences by measuring the frequency of emitted radio waves, and so-called superconducting quantum interference devices, or SQIDs, are used to measure tiny magnetic fields.

We note that all these experiments confirm that the *absolute* phase of a wave function *cannot* be measured. However, *relative phases* – phase differences or phase changes – *can* Challenge 69 e be measured. Can you confirm this?

All the phase shift effects just explained have been observed in numerous experiments.

The phase of a wave function is an essential aspect of it: the phase is the reason for calling it *wave* function in the first place. Like in any wave, the phase evolves over time and it can be influenced by various external influences. Above all, the experiments show that a localized quantum particle – thus when the spread of the wave function can be neglected – is best imagined as a rotating arrow; if the spread cannot be neglected, the wave function is best imagined as a wave of arrows rotating at each point in space.

The least action principle in quantum physics

In nature, motion happens in a way that minimizes change. Indeed, in classical physics,
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the principle of least action states that in nature, the motion of a particle happens along that particular path – out of all possible paths with the same end points – for which the action is minimal. This principle of cosmic laziness was stated mathematically by saying that in nature, the *variation* δS of the action is zero. Change minimization explains all classical evolution equations. We now transfer this idea to the quantum domain.

For quantum systems, we need to redefine both the concept of action and the concept of variation: first of all, we to find a description of action that is based on operators; secondly, we need to define the action variation without paths, as the concept of 'path' does not exist for quantum systems; thirdly, since there is a smallest action in nature, a vanishing variation is not a clearly defined concept, and we must overcome this hurdle. There are two main ways to achieve this: to describe the motion of quantum systems as a superposition of all possible paths, or to describe action with the help of wave functions. Both approaches are equivalent.

In the first approach, the *path integral formulation*, the motion of a quantum particle is described as a democratic superposition of motions along all possible paths. For each path, the evolution of the arrow is determined, and at the end point, the arrows from all paths are added. The action for each path is the number of turns that the arrow performs along the path. The result from this exercise is that the path for which the arrow makes the smallest number of turns is usually (but not always!) the most probable path. A more precise investigation shows that classical, macroscopic systems always follows the path of smallest action, but quantum systems do not.

In the second approach to quantum physics, action is defined with help of wave functions. In classical physics, we defined the action (or change) as the integral of the Lagrangian between the initial and final points in time, and the Lagrangian itself as the
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difference between kinetic and potential energy. In quantum physics, the simplest definition is the *quantum action* defined by Julian Schwinger. Let us call the initial and final states of the system $\psi_{\rm i}$ and $\psi_{\rm f}$. The action S between these two states is defined as

$$S = \langle \psi_{\rm i} \mid \int L \, {\rm d}t \mid \psi_{\rm f} \rangle \;, \qquad (37)$$

where L is the Lagrangian (operator). The angle brackets represent the 'multiplication' of states and operators as defined in quantum theory. (We skip the details of notation and mathematics here.) In simple words, also in quantum theory, action – i.e., the change occurring in a system – is the integral of the Lagrangian. The Lagrangian operator L is defined in the same way as in classical physics: the Lagrangian $L = T - V$ is the difference between the kinetic energy T and the potential energy V operators. The only difference is that, in quantum theory, the momentum and position *variables* of classical physics are replaced by the corresponding *operators* of quantum physics.*

To transfer the concept of action variation δS to the quantum domain, Julian Schwinger introduced the straightforward expression

$$\delta S = \langle \psi_{\mathrm{i}} \,|\, \delta \int L \,\mathrm{d}t |\, \psi_{\mathrm{f}} \rangle \ . \qquad (38)$$

The concept of path is not needed in this expression, as the variation of the action is based on varying wave functions instead of varying particle paths.

The last classical requirement to be transferred to the quantum domain is that, because nature is lazy, the variation of the action must vanish. However, in the quantum domain, the variation of the action cannot be zero, as the smallest observable action is the quantum of action. As Julian Schwinger discovered, there is only one possible way to express the required minimality of action:

$$\delta S = \langle \psi_{\mathrm{i}} \,|\, \delta \int L \,\mathrm{d}t |\, \psi_{\mathrm{f}} \rangle = -i\hbar \, \delta \langle \psi_{\mathrm{i}} | \psi_{\mathrm{f}} \rangle \ . \qquad (39)$$

This so-called *quantum action principle* describes all motion in the quantum domain. Classically, the right-hand side is zero – since $\hbar$ is taken to be zero – and we then recover the minimum-action principle $\delta S = 0$ of classical physics. But in quantum theory, whenever we try to achieve small variations, we encounter the quantum of action and changes of (relative) phase. This is expressed by the right-hand side of the expression.

In simple words, all quantum motion – i.e., the quantum evolution of a state ψ or $|\psi\rangle$ – happens in such a way that the action variation is the same as $-i$ times the quantum of action $\hbar$ times the variation of the scalar product between initial and final states. In other words, in the actual motion, the intermediate states are fixed by the requirement that they must lead from the initial state to the final state with the smallest number of effective turns of the state phase. The factor $-i$ expresses the dependence of the action on the rotation of the wave function.

In summary, the least action principle is also valid in quantum physics.

The motion of quantons with spin

“Everything turns.”

Anonymous

What is the origin of the quantum phase? Classical physics helps to answer the question. Quantons can rotate around an axis: we speak of particle *spin*. But if quantum particles can spin, they should possess angular momentum. And indeed, experiments confirm this deduction.

In particular, electrons have spin. The full details of electron spin were deduced from
Ref. 50 experiments by two Dutch students, George Uhlenbeck and Samuel Goudsmit, in 1925.

* More precisely, there is also a condition governing the ordering of operators in a mixed product, so that the non-commutativity of operators is taken into account. We do not explore this issue here.

They had the guts to publish what Ralph Kronig had also suspected: that electrons rotate around an axis with a projected component of the angular momentum given by $\hbar/2$. In fact, this value – often called spin 1/2 for short – is valid for *all* elementary *matter* particles. (In contrast, all elementary *radiation* particles have spin values of $\hbar$, or spin 1 for short.)

If a spinning particle has angular momentum, it must be possible to rearrange the axis by applying a torque, to observe precession, to transfer the spin in collisions etc. All this is Page 70 indeed observed; in fact, the Stern–Gerlach experiment allows all these observations. The only difference between particle spin and classical angular momentum is that particle Page 67 spin is quantized, as we deduced above.

In other words, the *spin* describes how a particle behaves under rotations. One result of spin is that *charged* quantum particles also act as small dipole magnets. The strength of the dipole magnet is described by the so-called *g-value*. Uhlenbeck and Goudsmit Page 89 deduced a g-value of 2 for the electron in order to explain the optical spectra. This value Ref. 51 was explained by Llewellyn Thomas as a relativistic effect a few months later; today one often speaks of *Thomas precession*.

By 2004, experimental techniques had become so sensitive that the magnetic effect of a single electron spin attached to an impurity (in an otherwise non-magnetic material) could be detected. Researchers now hope to improve these so-called 'magnetic-resonance-force microscopes' until they reach atomic resolution.

In 1927, the Austrian physicist Wolfgang Pauli* discovered how to include spin 1/2 in a quantum-mechanical description: instead of a state function described by a single complex number, a state function with *two* complex components is needed. The reason for this expansion is simple. In general, the little rotating arrow that describes a quantum state does *not* rotate around a *fixed* plane, as is assumed by the Schrödinger equation; the *plane of rotation* has also to be specified at each position in space. This implies that two additional parameters are required at each space point, bringing the total number of parameters to four real numbers, or, equivalently, two complex numbers. Nowadays, Pauli's equation is mainly of conceptual interest, because – like that of Schrödinger – it does not comply with special relativity. However, the idea of including the local rotation plane remains valid. The idea was used by Dirac when he introduced the relativistic description of the electron, and the idea is also used in all other wave equations for particles with spin.

In summary, the description of a quanton with spin implies the use of wave functions that specify *two* complex numbers at each point in space and time.

* Wolfgang Ernst Pauli (b. 1900 Vienna, d. 1958 Zürich), at the age of 21, wrote one of the best texts on special and general relativity. He was the first to calculate the energy levels of hydrogen using quantum theory, discovered the exclusion principle, incorporated spin into quantum theory, elucidated the relation between spin and statistics, proved the CPT theorem, and predicted the neutrino. He was admired for his intelligence, and feared for his biting criticisms, which led to his nickname, 'conscience of physics'. Despite this, he helped many people in their research, such as Heisenberg with quantum theory, without claiming any credit for Ref. 52 himself. He was seen by many, including Einstein, as the greatest and sharpest mind of twentieth-century physics. He was also famous for the 'Pauli effect', i.e., his ability to trigger disasters in laboratories, machines and his surroundings by his mere presence. As we will see shortly, one can argue that Pauli actually received the Nobel Prize for physics in 1945 (officially 'for the discovery of the exclusion principle') for finally settling the question of how many angels can dance on the tip of a pin.

Relativistic wave equations

In 1899, Max Planck had discovered the quantum of action. In 1905, Albert Einstein published the theory of special relativity, which was based on the idea that the speed of light c is independent of the speed of the observer. The first question Planck asked himself was whether the value of the quantum of action would be independent of the speed of the observer. It was his interest in this question that led him to invite Einstein to Berlin. With this invitation, he made the patent-office clerk famous in the world of physics.

Experiments show that the quantum of action is indeed independent of the speed of the observer. All observers find the same minimum value. To include special relativity into quantum theory, we therefore only need to find the correct quantum Hamiltonian H.

Given that the classical Hamiltonian of a free particle is given by

$$H = \beta\sqrt{c^4m^2 + c^2\boldsymbol{p}^2} \quad \text{with} \quad \boldsymbol{p} = \gamma m\boldsymbol{v} \,, \tag{40}$$

Ref. 53
one might ask: what is the corresponding Hamilton operator? The simplest answer was given, only in 1950, by L.L. Foldy and S.A. Wouthuysen. The operator is almost the same one:

$$H = \beta\sqrt{c^4m^2 + c^2\boldsymbol{p}^2} \quad \text{with} \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{41}$$

The signs appearing in the matrix operator β distinguish between particles and antiparticles. The numbers $+1$ and -1 appear twice, to take care of the two possible spin directions for each case.

With this relativistic Hamiltonian operator for spin 1/2 particles – and with all others – the wave function is described by *four* complex numbers, *two* for particles and *two* for antiparticles. That each type of particles requires two complex components follows from the requirement to specify, at each point in space, the length of the arrow, its phase, and its plane of rotation. Long ago we also found that relativity automatically introduces antimatter, and we will explore the issue in detail below. Both matter and antimatter are part of any relativistic description of quantum effects. The wave function for a particle has vanishing antiparticle components, and vice versa. In total, the wavefunction for relativistic spin 1/2 systems has thus four complex components.

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The Hamilton operator yields the velocity operator $\boldsymbol{v}$ through the same relation that is valid in classical physics:

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \beta\frac{\boldsymbol{p}}{\sqrt{c^4m^2 + c^2\boldsymbol{p}^2}} . \tag{42}$$

This velocity operator shows a continuum of eigenvalues, from minus to plus the speed of light. The velocity $\boldsymbol{v}$ is a constant of motion, as are the momentum $\boldsymbol{p}$ and the energy

$$E = \sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \ . \tag{43}$$

Also the orbital angular momentum $\boldsymbol{L}$ is defined as in classical physics, through

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} \ . \tag{44}$$

Ref. 54 The orbital angular momentum $\boldsymbol{L}$ and the spin $\boldsymbol{\sigma}$ are separate constants of motion. A particle (or antiparticle) with positive (or negative) angular momentum component has a wave function with only one non-vanishing component; the other three components vanish.

But alas, the representation of relativistic motion given by Foldy and Wouthuysen is not the simplest when it comes to take electromagnetic interactions into account. The simple identity between the classical and quantum-mechanical descriptions is lost when electromagnetism is included. We will solve this problem below, when we explore Dirac's evolution equation for relativistic wave functions. Page 159

Composite vs. elementary quantons

When is an object composite, and not elementary? Quantum theory gives several pragmatic answers. The first one is somewhat strange: an object is composite when its gyromagnetic ratio is different from the one predicted by QED. Ref. 55 The *gyromagnetic ratio* γ is defined as the ratio between the magnetic moment $\boldsymbol{M}$ and the angular momentum $\boldsymbol{L}$:

$$\boldsymbol{M} = \gamma \boldsymbol{L} \ . \tag{45}$$

The gyromagnetic ratio γ is measured in units of $s^{-1}T^{-1}$, i.e., C/kg, and determines the energy levels of magnetic spinning particles in magnetic fields; it will reappear later in the context of magnetic resonance imaging. Vol. V, page 118 All candidates for elementary particles have spin 1/2. The gyromagnetic ratio for spin-1/2 particles of magnetic moment M and mass m can be written as

$$\gamma = \frac{M}{\hbar/2} = g \frac{e}{2m} \ . \tag{46}$$

The criterion for being elementary can thus be reduced to a condition on the value of the dimensionless number g, the so-called *g-factor*. (The expression $e\hbar/2m$ is often called the *magneton* of the particle. Confusingly, the dimensionless factor $g/2$ is often called the gyromagnetic ratio as well.) If the g-factor *differs* from the value predicted by QED for point particles – about 2.0 – the object is *composite*. For example, a $^4He^+$ helium ion has spin 1/2 and a g value of $14.7 \cdot 10^3$. Challenge 70 ny Indeed, the radius of the helium ion is $3 \cdot 10^{-11}$ m, obviously a finite value, and the ion is a composite entity. For the proton, one measures a g-factor of about 5.6. Indeed, experiments yield a finite proton radius of about 0.9 fm and show that it contains several constituents.

The neutron, which has a magnetic moment despite being electrically neutral, must therefore be composite. Indeed, its radius is approximately the same as that of the proton. Similarly, molecules, mountains, stars and people must be composite. According to this first criterion, the only elementary particles are *leptons* (i.e., electrons, muons, tauons and neutrinos), *quarks*, and *intermediate bosons* (i.e., photons, W-bosons, Z-bosons and gluons). More details on these particles will be revealed in the chapters on the nucleus. Vol. V, page 118

Another simple criterion for compositeness has just been mentioned: *any object with a measurable size is composite*. This criterion yields the same list of elementary particles as the first. Indeed, the two criteria are related. The simplest model for composite structures predicts that the g-factor obeys Ref. 56

$$g - 2 = \frac{R}{\lambda_C} \tag{47}$$

where R is the radius and $\lambda_C = h/mc$ is the Compton wavelength of the system. This expression is surprisingly precise for helium-4 ions, helium-3, tritium ions and protons, as you may wish to check. The tables in Appendix B make the same point. Challenge 71 e

A third criterion for compositeness is more general: *any object larger than its Compton length is composite*. The argument is simple. An object is composite if one can detect *internal* motion, i.e., motion of some components. Now the action of any part with mass m_{part} moving inside a composed system of size r obeys

$$S_{part} < 2\pi\, r\, m_{part}\, c < \pi\, r\, m\, c \tag{48}$$

where m is the mass of the *composite* object. On the other hand, following the principle of quantum theory, this action, to be observable, must be larger than $\hbar/2$. Inserting this condition, we find that for any composite object*

$$r > \frac{\hbar}{2\pi\, m\, c} \,. \tag{49}$$

The right-hand side differs only by a factor $4\pi^2$ from the so-called *Compton (wave)length*

$$\lambda = \frac{h}{m\, c} \tag{50}$$

of an object. Thus any object *larger* than its own Compton wavelength is composite; and any object *smaller* than the right-hand side of expression (49) is elementary. Again, only leptons, quarks and intermediate bosons pass the test. All other objects are composite. This third criterion produces the same list as the previous ones. Can you explain why? Challenge 73 ny

A fourth criterion is regularly cited by Steven Weinberg: a particle is elementary if it appears in the Lagrangian of the standard model of particle physics. Can you show that this criterion follows from the previous ones? Challenge 74 ny

Interestingly, we are not yet finished with this topic. Even stranger statements about compositeness will appear when gravity is taken into account. Just be patient: it is worth it. Vol. VI, page 269

Challenge 72 ny * Can you find the missing factor of 2? And is the assumption that the components must always be lighter than the composite a valid one?

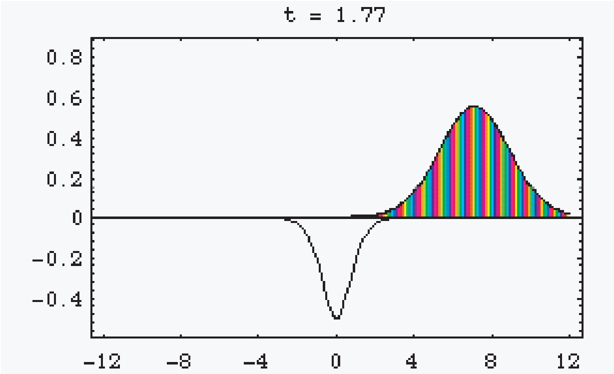


FIGURE 49 A special potential well that does *not* disturb a wave function. Colour indicates phase. (QuickTime film © Bernd Thaller).

Curiosities and fun challenges about quantum motion of matter

Take the sharpest knife edge or needle tip you can think of: the quantum of action implies that they are all fuzzy clouds.

* *

Do hydrogen atoms exist? Most types of atom have been imaged with microscopes, photographed under illumination, levitated one by one, and even moved with needles, one by one, as the picture on page 272 shows. Researchers have even moved single atoms Ref. 58 by using laser beams to push them. However, not a single one of these experiments has measured or imaged hydrogen atoms. Is that a reason to doubt the existence of hydrogen Challenge 75 s atoms? Taking this not-so-serious discussion seriously can be a lot of fun.

* *

Two observables can commute for two different reasons: either they are very *similar* – such as the coordinates x and x^2 – or they are very *different* – such as the coordinate x Challenge 76 ny and the momentum p_y. Can you give an explanation for this?

* *

Space and time translations commute. Why then do the momentum operator and the Challenge 77 ny Hamiltonian not commute in general?

* *

There are some potentials that have no influence on a wave function. Figure 49 shows an example. This potential has reflection coefficient zero for all energies; the scattered wave has no reflected part. The well has the shape of a soliton of the Korteweg–de Vries equation, which is related to the Schrödinger equation.

* *

For a bound system in a non-relativistic state with no angular momentum, one has the

Ref. 60 relation

$$\langle r^2 \rangle \langle T \rangle \geqslant \frac{9\hbar^2}{8m} , \qquad (51)$$

where m is the reduced mass and T the kinetic energy of the components, and r is the size of the system. Can you deduce this result, and check it for hydrogen? Challenge 78 s

* *

One often reads that the universe might have been born from a quantum fluctuation. Can you explain why this statement make no sense? Challenge 79 ny

A summary on motion of quantons

In summary, the motion of quantons can be described in two ways:

— Seen from far away, at low magnification, moving quantum particles and their wave functions behave as advancing, rotating and precessing arrows. The details of the rotation and precession of the arrow depend on the energy and momentum of the particle. The squared length of the arrow is the probability to observe a particle. If a particle can get from a starting point to a final point in several ways, arrows add up.

— At large magnification, quantum particles move like advancing, rotating and precessing clouds. The cloud shape is described by the wave function. The local cloud orientation, the local phase, follows a wobbling motion. The square of the probability amplitude, the density of the cloud, is the probability for finding the particle at a given spot.

Rotating arrows resulting from cloud averages combine particle and wave properties. A full rotation of the arrow corresponds to the quantum of action. These central feature imply that a non-relativistic particle whose spin can be neglected follows the Schrödinger equation, and that a relativistic electron follows the Dirac equation. They describe all of chemistry and biology.

To continue with the greatest efficiency on our path across quantum physics, we explore three important topics: the issue of indistinguishability of several particles, the topic of spin, and the issue of the interpretation of the probabilities.

Chapter 5

PERMUTATION OF PARTICLES – ARE PARTICLES LIKE GLOVES?

Why are we able to distinguish twins from each other? Why can we distinguish what looks alike, such as a copy from an original? Most of us are convinced that whenever we compare an original with a copy, we can find a difference. This conviction turns out to be correct. A short exploration shows that this conviction is in contrast with classical physics. The possibility to distinguish originals from copies is a pure quantum effect.

Indeed, quantum theory has a lot to say about copies and their differences. Think about any method that allows you to distinguish objects: you will find that it runs into trouble for point-like particles. Therefore in the quantum domain something must change about our ability to distinguish particles and objects. Let us explore the issue.

Challenge 80 s

Some usually forgotten properties of objects are highlighted by studying a pretty combinatorial puzzle: the *glove problem*. It asks:

> How many surgical gloves (for the right hand) are necessary if m doctors need to operate w patients in a hygienic way, so that nobody gets in contact with the body fluids of anybody else?

The same problem also appears in other settings. For example, it also applies to condoms, men and women – and is then called the *condom problem* – or to computers, interfaces and computer viruses. To be clear, the optimal number of gloves is *not* the product mw. In fact, the problem has three subcases.

Ref. 61

Challenge 81 s

— The simple case $m = w = 2$ already provides the most important ideas needed. Are you able to find the optimal solution and procedure?
— In the case $w = 1$ and m odd or the case $m = 1$ and w odd, the solution is $(m + 1)/2$ gloves. This is the optimal solution, as you can easily check yourself.
— A solution with a simple procedure for all other cases is given by $\lceil 2w/3 + m/2 \rceil$ gloves, where $\lceil x \rceil$ means the smallest integer greater than or equal to x. For example, for two doctors and three patients this gives only three gloves. (However, this formula does not always give the optimal solution; better values exist in certain subcases.)

Challenge 82 e
Ref. 62
Challenge 83 e

Two basic properties of gloves determine the solution to the puzzle. First, gloves have two sides, an interior and an exterior one. Secondly, gloves can be distinguished from each other. Do these two properties also apply to particles? We will discuss the issue of double-sidedness in the last part of the mountain ascent. In fact, the question whether particles can be turned inside out will be of importance for their description and their

Vol. VI, page 103

motion. (In fact, particles do behave like gloves in the sense that one can distinguish right-handed from left-handed ones.) Vol. V, page 177 In the present chapter we concentrate on the second issue, namely whether objects and particles can always be distinguished. We will find that *elementary* particles do not behave like gloves but in an even more surprising manner.

In everyday life, distinction of objects can be achieved in two ways. We are able to distinguish objects – or people – from each other because they differ in their *intrinsic properties*, such as their mass, colour, size or shape. In addition, we are also able to distinguish objects if they have the *same* intrinsic properties. Any game of billiard suggests that by following the path of each ball, we can distinguish it from the others. In short, objects with identical properties can also be distinguished using their *state*.

The state of a billiard ball is given by its position and momentum. In the case of billiard balls, the state allows distinction because the measurement error for the position of the ball is much smaller than the size of the ball itself. However, in the microscopic domain this is not the case. Let us take single atoms. Atoms of the same type have the same intrinsic properties. To distinguish them in collisions, we would need to keep track of their motion. But we have no chance to achieve this. Already in the nineteenth century it was shown experimentally that even nature itself is not able to do it! This profound result was discovered studying systems which incorporate a large number of colliding atoms of the same type: *gases*.

Vol. I, page 297 The calculation of the entropy S of a simple gas, made of N simple particles of mass m moving in a volume V, gives

$$\frac{S}{kN} = \ln\left[\frac{V}{\Lambda^3}\right] + \frac{3}{2} + \frac{\ln\alpha}{N} \tag{52}$$

where k is the Boltzmann constant, ln the natural logarithm, T the temperature, and $\Lambda = \sqrt{2\pi\hbar^2/mkT}$ is the thermal wavelength (approximately the de Broglie wavelength of the particles making up the gas). In this formula, the pure number α is equal to 1 if the particles are distinguishable like billiard balls, and equal to $1/N!$ if they are not distinguishable at all. Measuring the entropy thus allows us to determine α and therefore whether particles are distinguishable. It turns out that only the second case describes nature. Challenge 84 e This can easily be checked without even performing the measurement: only in the second case, $\alpha = 1/N!$ does the entropy of two volumes of identical gas *add* up.* The result, often called *Gibbs' paradox*,** thus proves that the microscopic components of matter are *indistinguishable*: Ref. 63 in a system of microscopic particles, there is no way to say which particle is which. Indistinguishability is an experimental property of nature.***

The properties of matter would be completely different without indistinguishability.

* Indeed, the entropy values observed by experiment, for a monoatomic gas, are given by the so-called Sackur–Tetrode formula Challenge 85 d

$$\frac{S}{kN} = \ln\left[\frac{V}{N\Lambda^3}\right] + \frac{5}{2} \tag{53}$$

which follows when $\alpha = 1/N!$ is inserted above. It was deduced independently by the German physicist Otto Sackur (1880–1914) and the Dutch physicist Hugo Tetrode (1895–1931). Note that the essential parameter is the ratio between V/N, the volume per particle, and Λ^3, the de Broglie volume of a particle.

** Josiah Willard Gibbs (1839–1903), US-American physicist who was, with Maxwell and Planck, one of the three founders of statistical mechanics and thermodynamics; he introduced the concept of *ensemble* and

FIGURE 50 Willard Gibbs (1839–1903)

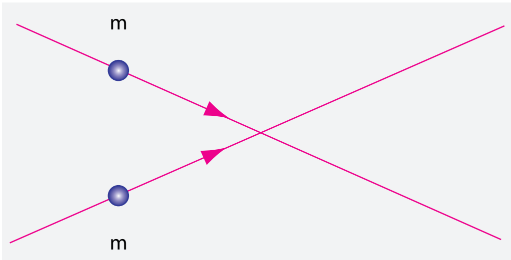


FIGURE 51 Identical objects with crossing paths.

For example, we will discover that without it, knifes and swords would not cut. In addition, the soil would not carry us; we would fall right through it. To illuminate the issue in more detail, we explore the following question.

Why does indistinguishability appear in nature?

Take two microscopic particles with the same mass, the same composition and the same shape, such as two atoms. Imagine that their paths cross, and that they approach each other to small distances at the crossing, as shown in Figure 51. In a gas, both a collision of atoms or a near miss are examples. Now, experiments show that at small distances it is impossible to say whether the two particles have switched roles or not. This is the main reason that makes it *impossible* in a gas to follow particles moving around and to determine which particle is which. This impossibility is a direct consequence of the quantum of action.

For a path that brings two approaching particles very close to each other, a role switch requires only a small amount of change, i.e., only a small (physical) action. However, we know that there is a smallest observable action in nature. Keeping track of each particle at small distances would require action values *smaller* than the minimal action observed in nature. The existence of a smallest action thus makes it impossible to keep track of microscopic particles when they come too near to each other. Any description of several

the term *thermodynamic phase*.
*** When radioactivity was discovered, people thought that it contradicted the indistinguishability of atoms, as decay seems to single out certain atoms compared to others. But quantum theory then showed that this is not the case and that atoms do remain indistinguishable.

particles must thus take into account that after a close encounter, it is impossible to say which particle is which.

If we remember that quantum theory describes particles as clouds, the indistinguishability appears more natural. Whenever two clouds meet and depart again, it is impossible to say which one is which.

If two particles are kept distant enough, one does have an effective distinguishability; indistinguishability thus appears only when the particles come close. In short, indistinguishability is a consequence of the existence of a minimal action in nature. This result leads straight away to the next question:

Can particles be counted?

In everyday life, objects can be counted because they can be distinguished. Since quantum particles cannot always be distinguished, we need some care in determining how to count them. The first step in counting particles is the definition of what is meant by a situation without any particle at all. This seems an easy thing to do, but later on we will encounter situations where already this step runs into difficulties. In any case, the first step is thus the *specification of the vacuum*. Any counting method requires that the situation without particles is clearly separated from situations with particles.

The second step is the specification of an observable useful for determining particle number. The easiest way is to chose one of those quantum numbers which add up under composition, such as electric charge.* Counting is then performed by measuring the total charge and dividing by the unit charge.

This method has several advantages. First of all, it is not important whether particles are distinguishable or not; counting always works. Secondly, virtual particles are not counted. This is a welcome state of affairs, as we will see, because for virtual particles, i.e., particles for which $E^2 \neq p^2c^2 + m^2c^4$, there is *no way* to define a particle number anyway.

Vol. V, page 85

The side effect of the counting method is that antiparticles count negatively! Also this consequence is a result of the quantum of action. We saw above that the quantum of action implies that even in vacuum, particle–antiparticle pairs are observed at sufficiently high energies. As a result, an antiparticle must count as minus one particle. In other words, any way of counting particles can produce an error due to this effect. In everyday life this limitation plays no role, as there is no antimatter around us. The issue does play a role at higher energies, however. It turns out that there is no general way to count the exact number of particles and antiparticles separately; only the sum can be defined. In short, quantum theory shows that particle counting is never perfect.

In summary, nature does provide a way to count particles even if they cannot be distinguished, though only for everyday, low energy conditions; due to the quantum of action, antiparticles count negatively. Antiparticles thus provide a limit to the counting of particles at high energies, when the mass–energy relation becomes important.

* In everyday life, the weight or mass is commonly used as observable. However, it cannot be used in the quantum domain, except for simple cases. Can you give at least two reasons, one from special relativity and one from general relativity?

Challenge 86 s

What is permutation symmetry?

Since particles are countable but indistinguishable, there exists a symmetry of nature for systems composed of several identical particles. *Permutation symmetry*, also called *exchange symmetry*, is the property of nature that observations are unchanged under exchange of identical particles. Together with space-time symmetry, gauge symmetry and the not yet encountered renormalization symmetry, permutation symmetry forms one of the four pillars of quantum theory. Permutation symmetry is a property of *composed* systems, i.e., of systems made of many (identical) subsystems. Only for such systems does indistinguishability play a role.

In other words, 'indistinguishable' is not the same as 'identical'. Two particles are not the *same*; they are more like *copies* of each other. On the other hand, everyday life experience shows us that two copies can always be distinguished under close inspection, so that the term is not fully appropriate either. In the microscopic domain, particles are countable and completely indistinguishable.* Particles are perfect copies of each other.

We will discover shortly that permutation is partial rotation. Permutation symmetry thus is a symmetry under partial rotations. Can you find out why? Challenge 87 e

Indistinguishability and symmetry

The indistinguishability of particles leads to important conclusions about the description of their state of motion. This happens because it is impossible to formulate a description of motion that includes indistinguishability right from the start. (Are you able to confirm this?) Challenge 88 s We need to describe a n-particle state with a state $\Psi_{1...i...j...n}$ which assumes that distinction is possible, as expressed by the ordered indices in the notation, and we introduce the indistinguishability afterwards.

Indistinguishability means that the exchange of any two particles results in the same physical system.** Now, two quantum states have the same physical properties if they differ at most by a phase factor; indistinguishability thus requires

$$\Psi_{1...i...j...n} = e^{i\alpha}\, \Psi_{1...j...i...n} \tag{54}$$

for some unknown angle α. Applying this expression twice, by exchanging the same couple of indices again, allows us to conclude that $e^{2i\alpha} = 1$. This implies that

$$\Psi_{1...i...j...n} = \pm\, \Psi_{1...j...i...n} \,, \tag{55}$$

in other words, a wave function is either *symmetric* or *antisymmetric* under exchange of indices. (One can also say that the eigenvalue for the exchange operator is either +1 or −1.) *Quantum theory thus predicts that particles can be indistinguishable in one of two distinct ways.**** Particles corresponding to symmetric wave functions – those which

* The word 'indistinguishable' is so long that many physicists sloppily speak of 'identical' particles nevertheless. Take care.

** We therefore have the same situation that we encountered already several times: *an overspecification of the mathematical description*, here the explicit ordering of the indices, *implies a symmetry of this description*, which in our case is a symmetry under exchange of indices, i.e., under exchange of particles.

*** This conclusion applies to three-dimensional space only. In two dimensions there are more possibilities.

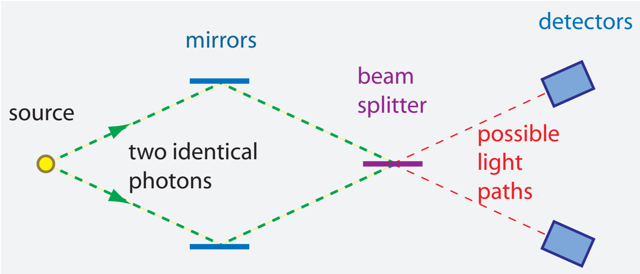


FIGURE 52 Two-photon emission and interference: both photons are always found arriving together, at the same detector.

transform with a '+' in equation (55) – are called *bosons*, those corresponding to antisymmetric wave functions – those which transform with a '−' in equation (55) – are called *fermions*.*

Experiments show that the exchange behaviour depends on the *type* of particle. Photons are found to be bosons. On the other hand, electrons, protons and neutrons are found to be fermions. Also about half of the atoms are found to behave as bosons (at moderate energies), the other half are fermions. In fact, a composite of an *even* number of fermions (at moderate energies) – or of any number of bosons (at any energy) – turns out to be a boson; a composite of an *odd* number of fermions is (always) a fermion.

For example, almost all of the known molecules are bosons (electronically speaking). Fermionic molecules are rather special and even have a special name in chemistry; they are called *radicals* and are known for their eagerness to react and to form normal bosonic molecules. Inside the human body, too many radicals can have adverse effects on health; it is well known that vitamin C is important because it is effective in reducing the number of radicals.

Challenge 89 s To which class of particles do mountains, trees, people and all other macroscopic objects belong?

The behaviour of photons

A simple experiment allows observing the behaviour of photons. Take a source that emits two photons of identical frequency and polarization at the same time, as shown in Figure 52. In the laboratory, such a source can be realized with a down-converter, a material that converts a photon of frequency 2ω into two photons of frequency ω. Both photons, after having travelled exactly the same distance, are made to enter the two sides of a beam splitter (for example, a half-silvered mirror). At the two exits of the beam splitter are two detectors. Experiments show that both photons are always detected together Ref. 65

* The term 'fermion' is derived from the name of the Italian physicist and Nobel Prize winner Enrico Fermi (b. 1901 Rome, d. 1954 Chicago) famous for his all-encompassing genius in theoretical and experimental physics. He mainly worked on nuclear and elementary particle physics, on spin and on statistics. For his experimental work he was called 'quantum engineer'. He is also famous for his lectures, which are still published in his own hand-writing, and his brilliant approach to physical problems. Nevertheless, his highly deserved Nobel Prize was one of the few cases in which the prize was given for a discovery which turned out to be incorrect.

'Bosons' are named after the Indian physicist Satyenra Nath Bose (b. 1894 Calcutta, d. 1974 Calcutta) Ref. 64 who first described the statistical properties of photons. The work was later expanded by Albert Einstein.

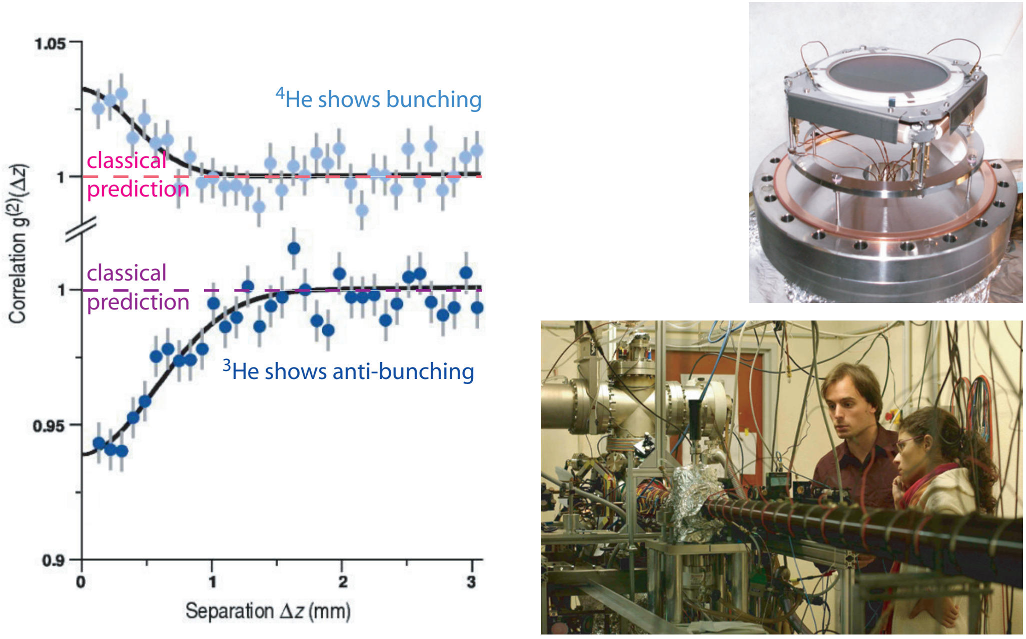


FIGURE 53 Bunching and antibunching of ^{3}He and ^{4}He helium atoms: the measurement result, the detector and the experiment (from atomoptic.iota.u-psud.fr/research/helium/helium.html, photo © Denis Boiron, Jerome Chatin).

on the *same* side, and never separately on opposite sides. This result shows that photons are bosons. Fermions behave in exactly the opposite way; two fermions are always detected separately on *opposite* sides, never together on the same side.

Bunching and antibunching

A beautiful way to test the exchange character of a particle is the Hanbury Brown–Twiss Page 44 experiment described earlier on. First of all, the experiment shows that quantum particles behave differently than classical particles. In addition, compared to classical particles, fermions show antibunching – because of Pauli's exclusion principle – and bosons show bunching. Hanbury Brown and Twiss performed the experiment with photons. In 2005, Ref. 66 a French–Dutch research collaboration performed the experiment with atoms. By using an extremely cold helium gas at 500 nK and a clever detector principle, they were able to measure the correlation curves typical for the effect. The result, shown in Figure 53 shows that, as predicted by quantum theory, ^{3}He is a fermion and ^{4}He is a boson.

The energy dependence of permutation symmetry

If experiments force us to conclude that nobody, not even nature, can distinguish any two particles of the same type, we deduce that they do not form two separate entities, but some sort of unity. Our naive, classical sense of particle as a separate entity from the rest of the world is thus an incorrect description of the phenomenon of 'particle'. Indeed, no experiment can track particles with identical intrinsic properties in such a way that they can be distinguished with certainty. This impossibility has been checked experimentally

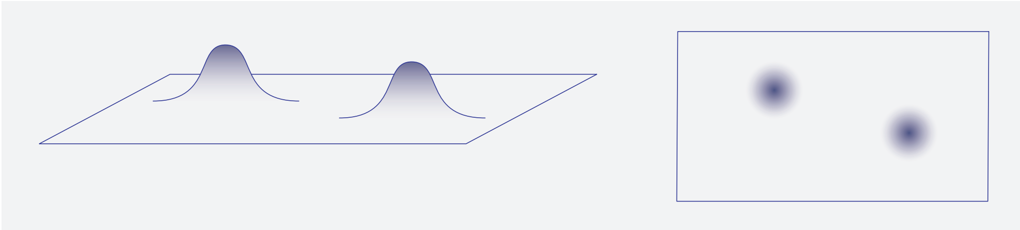


FIGURE 54 Picturing particles as localized excitations (left) or clouds (right).

with all elementary particles, with nuclei, with atoms and with numerous molecules.

How does this fit with everyday life, i.e., with classical physics? Photons do not worry us much here. Let us focus the discussion on matter particles. We know to be able to distinguish electrons by pointing to the wire in which they flow, and we can distinguish our fridge from that of our neighbour. While the quantum of action makes distinction impossible, everyday life allows it. The simplest explanation is to imagine a microscopic particle, especially an elementary one, as a bulge, i.e., as a localized excitation of the vacuum, or as a tiny cloud. Figure 54 shows two such bulges and two clouds representing particles. It is evident that if particles are too near to each other, it makes no sense to distinguish them; we cannot say any more which is which.

The bulge image shows that either for large distances or for high potential walls separating them, distinction of identical particles does become possible. In such situations, measurements allowing to track them independently do exist. In other words, we can specify a limit energy at which permutation symmetry of objects or particles separated by a distance d becomes important. It is given by

$$E = \frac{c\,\hbar}{d} \,. \tag{56}$$

Challenge 90 ny Are you able to confirm the expression? For example, at everyday temperatures we *can* distinguish atoms inside a solid from each other, since the energy so calculated is much higher than the thermal energy of atoms. To have fun, you might want to determine at Challenge 91 e what energy two truly identical human twins become indistinguishable. Estimating at what energies the statistical character of trees or fridges will become apparent is then straightforward.

The bulge image of particles thus purveys the idea that distinguishability exists for objects in everyday life but not for particles in the microscopic domain. To sum up, in daily life we are able to distinguish objects and thus people for two reasons: because they are made of *many* parts, and because we live in a *low energy* environment.

The energy issue immediately adds a new aspect to the discussion. How can we describe fermions and bosons in the presence of virtual particles and of antiparticles?

Indistinguishability in quantum field theory

Quantum field theory, as we will see shortly, simply puts the cloudy bulge idea of Figure 54 into mathematical language. A situation with no bulge is called *vacuum state*. Quantum field theory describes all particles of a given type as *excitations* of a single fundamental field. Particles are indistinguishable because each particle is an excitation of the same basic substrate and each excitation has the same properties. A situation with one particle is then described by a vacuum state acted upon by a *creation operator*. Adding a second particle is described by adding a second creation operator, and subtracting a particle by adding a *annihilation operator*; the latter turns out to be the adjoint of the former.

Quantum field theory studies how creation and annihilation operators must behave to describe observations.* It arrives at the following conclusions:

— Fields with half-integer spin are fermions and imply (local) anticommutation.
— Fields with integer spin are bosons and imply (local) commutation.
— For all fields at space-like separations, the commutator, respectively anticommutator, vanishes.
— Antiparticles of fermions are fermions, and antiparticles of bosons are bosons.
— Virtual particles behave under exchange like their real counterparts.

These connections are at the basis of quantum field theory. They describe how particles are identical. But why are they? Why are all electrons identical? Quantum field theory describes electrons as identical excitations of the vacuum, and as such as identical by construction. Of course, this answer is only partially satisfying. We will find a better one only in the final part of our mountain ascent.

How accurately is permutation symmetry verified?

A simple but effective experiment testing the fermion behaviour of electrons was carried out by Ramberg and Snow. They sent an electric current of 30 A through a copper wire for one month and looked for X-ray emission. They did not find any. They concluded that electrons are always in an antisymmetric state, with a symmetric component of less than

Ref. 67

$$2 \cdot 10^{-26} \tag{59}$$

of the total state. In short, electrons are always in an antisymmetric state: they are fermions.

The reasoning behind this elegant experiment is the following. If electrons would not always be fermions, every now and then an electron could fall into the lowest energy level of a copper atom, leading to X-ray emission. The lack of such X-rays implies that electrons are fermions to a very high accuracy. X-rays could be emitted only if they were bosons, at least part of the time. Indeed, two electrons, being fermions, cannot be in the

* Whenever the relation

$$[b, b^\dagger] = bb^\dagger - b^\dagger b = 1 \tag{57}$$

holds between the creation operator $b^\dagger$ and the annihilation operator b, the operators describe a *boson*. The dagger can thus be seen as describing the operation of adjoining; a double dagger is equivalent to no dagger. If the operators for particle creation and annihilation anticommute

$$\{d, d^\dagger\} = dd^\dagger + d^\dagger d = 1 \tag{58}$$

they describe a *fermion*. The so defined bracket is called the *anticommutator bracket*.

same quantum state: this restriction is called the Pauli exclusion principle. It applies to all fermions and is our next topic.

Copies, clones and gloves

Can classical systems be indistinguishable? They can: large molecules are examples – provided they are made of exactly the same isotopes. Can *large* classical systems, made of a mole or more particles be indistinguishable? This simple question effectively asks whether a perfect copy, or (physical) *clone* of a system is possible.

It could be argued that any factory for mass-produced goods, such as one producing shirt buttons or paper clips, shows that copies are possible. But the appearance is deceiving. On a microscope there is usually some difference. Is this always the case? In 1982, the Dutch physicist Dennis Dieks and independently, the US-American physicists Wootters and Zurek, published simple proofs that quantum systems cannot be copied. This is the famous *no-cloning theorem*.

Ref. 68

A copying machine is a machine that takes an original, reads out its properties and produces a copy, leaving the original unchanged. This seems definition seems straightforward. However, we know that if we extract information from an original, we have to interact with it. As a result, the system will change at least by the quantum of action. We thus expect that due to quantum theory, copies and originals can never be identical.*

Quantum theory proves this in detail. A copying machine is described by an operator that maps the state of an original system to the state of the copy. In other words, a copying machine is linear. This linearity leads to a problem. Simply stated, if a copying machine were able to copy originals either in state $|A\rangle$ or in state $|B\rangle$, it could not decide what to do if the state of the original were $|A\rangle + |B\rangle$. On the one hand, the copy should be $|A\rangle + |B\rangle$; on the other hand, the linearity of the copier forbids this. Indeed, a copier is a device described by an operator U that changes the starting state $|s\rangle_c$ of the copy in the following way:

— If the original is in state $|A\rangle$, a copier acts as

$$U|A\rangle|s\rangle_c = |A\rangle|A\rangle_c \,. \tag{60}$$

— If the original is in state $|B\rangle$, a copier acts as

$$U|B\rangle|s\rangle_c = |B\rangle|B\rangle_c \,. \tag{61}$$

As a result of these two requirements, an original in the state $|A + B\rangle$ is treated by the copier as

$$U|A + B\rangle|s\rangle_c = |A\rangle|A\rangle_c + |B\rangle|B\rangle_c \,. \tag{62}$$

* This seems to provide a solution against banknote forgeries. In fact, Steve Wiesner proposed to use quantum theory already in 1970; he imagined to use polarizations of stored single photons as bits of serial numbers. Can you explain why this cannot work?

Challenge 92 ny

This is in contrast to what we want, which would be

$$U_{\text{wanted}}|A+B\rangle|s\rangle_{c} = (|A\rangle + |B\rangle)(|A\rangle_{c} + |B\rangle_{c}) \; . \tag{63}$$

In other words, a copy machine cannot copy a state completely.* This is the no-cloning theorem.

The impossibility of copying is implicit in quantum theory. If we were able to clone systems, we could to measure a variable of a system and a second variable on its copy. We would be thus able to beat the indeterminacy relation. This is impossible. Copies are and always must be imperfect.

Other researchers then explored how near to perfection a copy can be, especially in the Ref. 69 case of classical systems. To make a long story short, these investigations show that also the copying or cloning of macroscopic systems is impossible. In simple words, copying machines do not exist. Copies can always be distinguished from originals if observations are made with sufficient care. In particular, this is the case for biological clones; biological clones are identical twins born following separate pregnancies. They differ in their finger prints, iris scans, physical and emotional memories, brain structures, and in many other Challenge 93 s aspects. (Can you specify a few more?) In short, biological clones, like identical twins, are not copies of each other.

The lack of quantum mechanical copying machines is disappointing. Such machines, or teleportation devices, could be fed with two different inputs, such as a lion and a goat, and produce a superposition: a chimaera. Quantum theory shows that all these imaginary beings cannot be realized.

In summary, everyday life objects such as photocopies, billiard balls or twins are always distinguishable. There are two reasons: first, quantum effects play no role in everyday life, so that there is no danger of unobservable exchange; secondly, perfect clones of classical systems do not exist anyway, so that there always are tiny differences between any two objects, even if they look identical at first sight. Gloves, being classical systems, can thus always be distinguished.

* The no-cloning theorem puts severe limitations on quantum computers, as computations often need copies of intermediate results. It also shows that faster-than-light communication is impossible in EPR experiments. In compensation, quantum cryptography becomes possible – at least in the laboratory. Indeed, the no-cloning theorem shows that nobody can copy a quantum message without being noticed. The specific ways to use this result in cryptography are the 1984 Bennett–Brassard protocol and the 1991 Ekert protocol.

Chapter 6

ROTATIONS AND STATISTICS – VISUALIZING SPIN

Spin is the observation that matter beams can be polarized: rays can be rotated. Spin thus describes how particles behave under rotations. Particles are thus not simple spheres shrunk to points. We also saw that spin describes a fundamental difference between quantum particles and gloves: spin specifies the indistinguishability of quantum particles and quantum systems in general. We now explore this connection in more detail.

Quantum particles and symmetry

Ref. 70 The general background for the appearance of spin was clarified by Eugene Wigner in 1939.* He started by recapitulating that any quantum particle, if elementary, must behave like an *irreducible representation* of the set of all viewpoint changes. This set of viewpoint changes forms the symmetry group of flat space-time, the so-called *inhomogeneous Lorentz group*. Why?

Vol. I, page 215 We have seen in the chapter on symmetry that the symmetry of any composite system leads to certain requirements for its components. If the components do not follow these requirements, they cannot build a symmetric composite. We know from everyday life and precision experiments that all physical systems are symmetric under translation in time and space, under rotation in space, under boosts, and – in many cases – under mirror reflection, matter–antimatter exchange and motion reversal.

We know these symmetries known from everyday life; for example, the usefulness of what we call 'experience' in everyday life is simply a consequence of time translation symmetry. The set of all these common symmetries, more precisely, of all these symmetry transformations, is called the *inhomogeneous Lorentz group*.

These *symmetries*, i.e., these changes of viewpoints, lead to certain requirements for the components of physical systems, i.e., for the elementary quantum particles. In mathematical language, the requirement is expressed by saying that *elementary particles* must be irreducible representations of the symmetry group.

Following Wigner, every textbook on quantum theory carries out this reasoning systematically. One obtains a list of all possible irreducible representations, in other words, a list of all possible ways that elementary particles can behave. ** Cataloguing the possibil-

* Eugene Wigner (b. 1902 Budapest, d. 1995 Princeton), Hungarian–US-American theoretical physicist, received the Nobel Prize for physics in 1963. He wrote over 500 papers, many about various aspects of symmetry in nature. He was also famous for being the most polite physicist in the world.

** To be of physical relevance for quantum theory, representations have to be *unitary*. The full list of irreities, one finds first of all that every elementary particle is described by *four-momentum* – no news so far – and by an internal angular momentum, the *spin*. Four-momentum results from the translation symmetry of nature, and spin from its rotation symmetry. The momentum value describes how a particle behaves under translation, i.e., under position and time shift of viewpoints. The spin value describes how an object behaves under rotations in three dimensions, i.e., under orientation change of viewpoints.* As is well known, the magnitude of four-momentum is an invariant property, given by the mass, whereas its orientation in space-time is free. Similarly, the magnitude of spin is an invariant property, and its orientation has various possibilities with respect to the direction of motion. In particular, the spin of massive quantum particles behaves differently from that of massless quantum particles.

For *massive* quantum particles, the inhomogeneous Lorentz group implies that the invariant magnitude of spin is $\sqrt{J(J+1)}\,\hbar$, often written, by oversimplification, as J. It is thus customary to say and write 'spin J' instead of the cumbersome 'spin $\sqrt{J(J+1)}\,\hbar$'. Since the value of the quantum number J specifies the magnitude of the angular momentum, it gives the representation under rotations of a given particle type. The exploration shows that the spin quantum number J can be any multiple of $1/2$, i.e., it can take the values $0, 1/2, 1, 3/2, 2, 5/2$, etc. Experiments show that electrons, protons and neutrons have spin 1/2, the W and Z particles spin 1 and helium atoms spin 0. In addition, the representation of spin J is $2J+1$ dimensional, meaning that the spatial orientation of the spin has $2J+1$ possible values. For electrons, with $J = 1/2$, there are thus two possibilities; they are usually called 'up' and 'down'. Spin thus only takes *discrete* values. This is in contrast with linear momentum, whose representations are infinite dimensional and whose possible values form a *continuous* range.

Also *massless* quantum particles are characterized by the value of their spin. It can take the same values as in the massive case. For example, photons and gluons have spin 1. For massless particles, the representations are one-dimensional, so that massless particles are completely described by their *helicity*, defined as the projection of the spin onto the direction of motion. Massless particles can have positive or negative helicity, often also called right-handed and left-handed polarization. There is no other freedom for the orientation of spin in the massless case.

To complete the list of particle properties, the remaining symmetries must be included. Since *motion inversion*, *spatial parity* and *charge inversion* are parities, each elementary particle has to be described by three additional numbers, called T, P and C, each of which can only take the values $+1$ or -1. Being parities, these numbers must be *multiplied* to yield the value for a composed system.

In short, symmetry investigations lead to the classification of all particles by their mass, their momentum, their spin and their P, C and T parities.

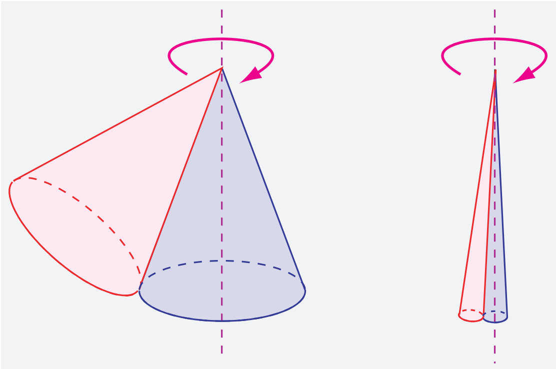


FIGURE 55 An argument showing why rotations by 4π are equivalent to no rotation at all.

Types of quantum particles

The spin values observed for all quantum particles in nature is given in Table 4. The parities and all known intrinsic properties of the elementary particles are given in Table 5. Spin and parities together are called *quantum numbers*. The other intrinsic properties are related to interactions, such as electric charge or isospin. We will explore them in the next volume. But let us return to spin.

TABLE 5 Elementary particle properties.

Particle	Mass m [a]	Lifetime τ or energy width, [b] main decay modes	Isospin I, spin J, [c] parity P, charge parity C	Charge, isospin, strangeness, [c] charm, beauty, topness: $QISCBT$	Lepton & baryon [e] numbers LB
Elementary radiation (bosons)					
photon γ	0 ($<10^{-53}$ kg)	stable	$I(J^{PC}) = 0, 1(1^{--})$	000000	0, 0
$W^{\pm}$	80.398(25) GeV/c^2	2.124(41) GeV 67.60(27)% hadrons, 32.12(36)% $l^+\nu$	$J = 1$	±100000	0, 0
Z	91.1876(21) GeV/c^2	$2.65(2) \cdot 10^{-25}$ s or 2.4952(23) GeV/c^2 69.91(6)% hadrons, 10.0974(69)% l^+l^-	$J = 1$	000000	0, 0

ducible unitary representations of viewpoint changes thus provides the range of possibilities for any particle that wants to be *elementary*.

* The group of physical rotations is also called SO(3), since mathematically it is described by the group of Special Orthogonal 3 by 3 matrices.

TABLE 5 (Continued) Elementary particle properties.

Particle	Mass m [a]	Lifetime τ or energy width, [b] main decay modes	Isospin I, spin J, [c] parity P, charge parity C	Charge, isospin, strangeness, [c] charm, beauty, topness: $QISCBT$	Lepton & baryon [e] numbers $L\,B$
gluon	0	stable	$I(J^P) = 0(1^-)$	000000	0, 0
Elementary matter (fermions): leptons					
electron e	$9.109\,382\,15(45)\cdot 10^{-31}\,\mathrm{kg} = 81.871\,0438(41)\,\mathrm{pJ}/c^2$	$> 13\cdot 10^{30}\,\mathrm{s}$	$J = \frac{1}{2}$	$-100\,000$	1, 0
	$= 0.510\,998\,910(13)\,\mathrm{MeV}/c^2 = 0.000\,548\,579\,909\,43(23)\,\mathrm{u}$				
	gyromagnetic ratio $\mu_e/\mu_B = -1.001\,159\,652\,1811(7)$				
	electric dipole moment $d = (0.7 \pm 0.7)\cdot 10^{-29}\,e\,\mathrm{m}$ [f]				
muon μ	$0.188\,353\,130(11)\,\mathrm{yg}$	$2.197\,03(4)\,\mathrm{\mu s}$ 99 % $e^-\bar{\nu}_e\nu_\mu$	$J = \frac{1}{2}$	-100000	1, 0
	$= 105.658\,3668(38)\,\mathrm{MeV}/c^2 = 0.113\,428\,9256(29)\,\mathrm{u}$				
	gyromagnetic ratio $\mu_\mu/(e\hbar/2m_\mu) = -1.001\,165\,9208(6)$				
	electric dipole moment $d = (3.7 \pm 3.4)\cdot 10^{-21}\,e\,\mathrm{m}$				
tau τ	$1.776\,84(17)\,\mathrm{GeV}/c^2$	$290.6(1.0)\,\mathrm{fs}$	$J = \frac{1}{2}$	-100000	1, 0
el. neutrino ν_e	$< 2\,\mathrm{eV}/c^2$		$J = \frac{1}{2}$		1, 0
muon neutrino ν_μ	$< 2\,\mathrm{eV}/c^2$		$J = \frac{1}{2}$		1, 0
tau neutrino ν_τ	$< 2\,\mathrm{eV}/c^2$		$J = \frac{1}{2}$		1, 0
Elementary matter (fermions): quarks [g]					
up u	1.5 to 3.3 MeV/c^2	see proton	$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$	$+\frac{2}{3}+\frac{1}{2}0000$	$0, \frac{1}{3}$
down d	3.5 to 6 MeV/c^2	see proton	$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$	$-\frac{1}{3}-\frac{1}{2}0000$	$0, \frac{1}{3}$
strange s	70 to 130 MeV/c^2		$I(J^P) = 0(\frac{1}{2}^+)$	$-\frac{1}{3}0{-1}000$	$0, \frac{1}{3}$
charm c	$1.27(11)\,\mathrm{GeV}/c^2$		$I(J^P) = 0(\frac{1}{2}^+)$	$+\frac{2}{3}00{+1}00$	$0, \frac{1}{3}$
bottom b	$4.20(17)\,\mathrm{GeV}/c^2$	$\tau = 1.33(11)\,\mathrm{ps}$	$I(J^P) = 0(\frac{1}{2}^+)$	$-\frac{1}{3}000{-1}0$	$0, \frac{1}{3}$
top t	$171.2(2.1)\,\mathrm{GeV}/c^2$		$I(J^P) = 0(\frac{1}{2}^+)$	$+\frac{2}{3}0000{+1}$	$0, \frac{1}{3}$
Hypothetical elementary matter (boson)					
Higgs [h] H	$> 114\,\mathrm{GeV}/c^2$		$J = 0$		

Notes:

a. See also the table of SI prefixes on page 175. About the eV/c^2 mass unit, see page 179.

b. The *energy width* Γ of a particle is related to its lifetime τ by the indeterminacy relation $\Gamma\tau = \hbar$. There

TABLE 4 Particle spin as representation of the rotation group.

Spin [$\hbar$]	System unchanged after rotation by	Massive examples elementary	Massive examples composite	Massless examples elementary
0	any angle	none [a,b]	mesons, nuclei, atoms	none [b]
1/2	2 turns	e, μ, τ, q, ν_e, ν_μ, ν_τ	nuclei, atoms, molecules	none, as neutrinos have a tiny mass
1	1 turn	W, Z	mesons, nuclei, atoms, molecules, toasters	g, γ
3/2	2/3 turn	none [b]	baryons, nuclei, atoms	none [b]
2	1/2 turn	none	nuclei	'graviton' [c]
5/2	2/5 turn	none	nuclei	none
3	1/3 turn	none	nuclei [d]	none
etc.[d]	etc.[d]	etc.[d]	etc.[d]	etc.[d]

a. Whether the Higgs boson exists, and whether it is elementary is still unknown.
b. Supersymmetry predicts particles in these and other boxes.
c. The graviton has not yet been observed.
d. Nuclei exist with spins values up to at least 101/2 and 51 (in units of $\hbar$). Ref. 71

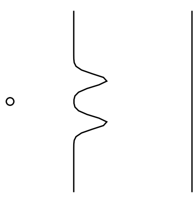
is a difference between the *half-life* $t_{1/2}$ and the lifetime τ of a particle: they are related by $t_{1/2} = \tau \ln 2$, where $\ln 2 \approx 0.693\,147\,18$; the half-life is thus shorter than the lifetime. The unified *atomic mass unit* u is defined as 1/12 of the mass of a carbon 12 atom at rest and in its ground state. One has $1\,\mathrm{u} = \frac{1}{12} m(^{12}\mathrm{C}) = 1.660\,5402(10)\,\mathrm{yg}$.

c. To keep the table short, the header does not explicitly mention *colour*, the charge of the strong interactions. This has to be added to the list of basic object properties. Quantum numbers containing the word 'parity' are multiplicative; all others are additive. Time parity T (not to be confused with topness T), better called motion inversion parity, is equal to CP. The isospin I (or I_Z) is defined only for up and down quarks and their composites, such as the proton and the neutron. In the literature one also sees references to the so-called G-parity, defined as $G = (-1)^{IC}$.

The header also does not mention the *weak charge* of the particles. The details on weak charge g, or, more precisely, on the *weak isospin*, a quantum number assigned to all left-handed fermions (and right-handed anti-fermions), but to no right-handed fermion (and no left-handed antifermion), are given in the section on the weak interactions. Vol. V, page 177

d. 'Beauty' is now commonly called *bottomness*; similarly, 'truth' is now commonly called *topness*. The signs of the quantum numbers S, I, C, B, T can be defined in different ways. In the standard assignment shown here, the sign of each of the non-vanishing quantum numbers is given by the sign of the charge of the corresponding quark.

e. If supersymmetry exists, R-parity must be added to this column. R-parity is a multiplicative quantum number related to the lepton number L, the baryon number B and the spin J through the definition $R = (-1)^{3B+L+2J}$. All particles from the standard model are R-even in this case, whereas their superpartners would be R-odd.



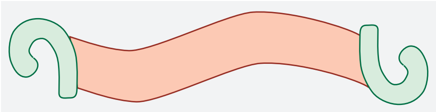


FIGURE 56 Two belt buckles connected by a belt, visualizing two spin 1/2 particles.

Ref. 73, Ref. 72 *f*. The electron radius is less than 10^{-22} m. It is possible to store single electrons in traps for many months.
g. See page 169 for the precise definition and meaning of the quark masses.
h. Currently a hypothetical particle.

A central result of quantum theory is that spin 1/2 is a possibility in nature, even though this value does not appear in everyday life. For a system to have spin 1/2 means that for such a system only a rotation of 720 degrees is equivalent to one of 0 degrees, while one of 360 degrees is not. No such systems exist in everyday life, but they do exist in microscopic systems: electrons, silver atoms and molecular radicals have spin 1/2. A full list of spins of particles is given in Table 4.

Vol. I, page 45 The mathematician Hermann Weyl used a simple image explaining the connection between spin 1/2 and invariance under rotation by 4π. Take two cones, touching each other at their tips as well as along a line. Hold one cone and roll the other around it, as shown in Figure 55. When the rolling cone, after a full turn around the other cone, has come back to the original position, it has rotated by some angle. If the cones are wide, the rotation angle is small. If the cones are very thin, like needles, the moving cone has rotated by (almost) 720 degrees. A rotation of 720 degrees is thus similar to one by 0 degrees. If we imagine the cone angle to vary continuously, this visualization also shows that a 720 degree rotation can be continuously deformed into a 0 degree rotation, whereas a 360 degree rotation cannot.

To sum up, the list of possible representations thus shows that rotations *require* the existence of spin. But why then do experiments show that all fermions have half-integer spin and that all bosons have integer spin? Why do electrons obey the Pauli exclusion principle? At first, it is not clear what the spin has to do with the statistical properties of a particle.

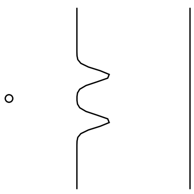
In fact, there are several ways to show that rotations and statistics are connected. The first proof, due to Wolfgang Pauli, used the details of quantum field theory and was so complicated that its essential ingredients were hidden. It took several decades to convince everybody that a simple observation about belts was the central part of the proof. Ref. 74

The belt trick and its extension

The well-known *belt trick* – also called *scissor trick* or *plate trick* – was often used by Dirac to explain the features of spin 1/2. Starting from Figure 54, which models particles as indistinguishable excitations, it is not difficult to imagine a sort of sheet connecting them, similar to a belt connecting two belt buckles, as shown in Figure 56. The buckles represent the particles. If one belt buckle is rotated by 2π along any axis, a twist is inserted into the belt. Now rotate the same buckle by another 2π, bringing the total to 4π. It turns out that the ensuing double twist can easily be undone without moving or rotating the buckles. The animation of Figure 57 shows the details. You may want to do this yourself, using a real belt or a strip of paper, in order to believe it. In short, belt buckles return to

Ref. 75

Challenge 94 e



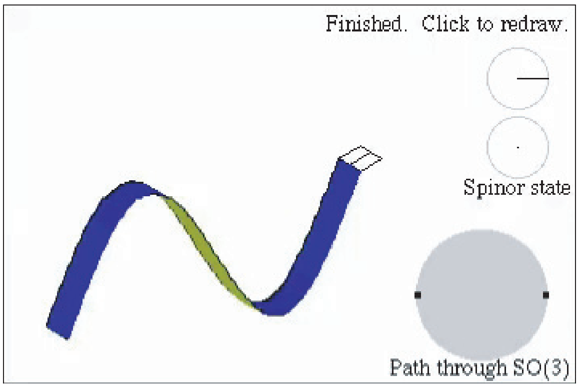


FIGURE 57 The belt trick: a double rotation of the belt buckle is equivalent to no rotation (QuickTime film © Greg Egan).

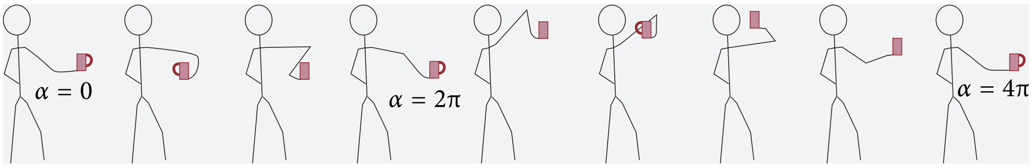


FIGURE 58 The human arm as spin 1/2 model.

their original state only after rotations by 4π, and not after rotations by 2π.

Now look again at Figure 56. If you take the two buckles and simply *swap* positions, a twist is introduced into the belt. Now swap them again: this will undo the twist. In short, two belt buckles return to their original state only after a double exchange, and not after a single exchange.

In other words, if we take each buckle to represent a particle and a twist to mean a factor -1, the belt exactly describes the phase behaviour of spin $1/2$ wave functions, both under rotation and under exchange. In particular, we see that rotation and exchange behaviour are related.

The human body has such a belt built in: the *arm*. Just take your hand, put an object on it for clarity such as a cup, and turn the hand and object by 2π by twisting the arm. After a second rotation the whole system will be untangled again. This is sometimes called the *plate trick*. The trick is even more impressive when *many* arms are used. You can put your two hands (if you chose the correct starting position) under the cup or you can take a friend or two who each keep a hand attached to the cup. The feat can still be performed: the whole system untangles after two full turns.

Challenge 95 e

Challenge 96 e

This leads us to the most complete way to show the connection between rotation and exchange. Just glue any number of threads, belts or tubes, say half a metre long, to an asymmetric object. (With many such tails, is not appropriate any more to call it a belt buckle.) Like the arm of a human being, each band is supposed to go to infinity and be attached there. If the object, which represents the particle, is rotated by 2π, twists appear in its tails. If the object is rotated by an additional turn, to a total of 4π, as shown in Figure 59, all twists and tangles can be made to disappear, without moving or turning the object. You really have to experience this in order to believe it. And the trick

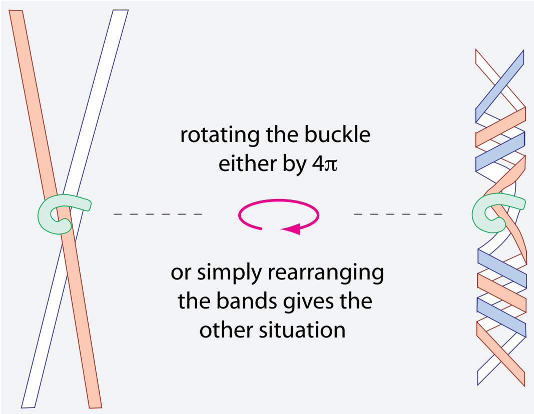


FIGURE 59 The extended belt trick, modelling the rotation behaviour of a spin 1/2 particle: independently of the number of bands or tubes or strings attached, the two situations can be transformed into each other, either by rotating the central object by 4π or by keeping the central object fixed and moving the bands around it.

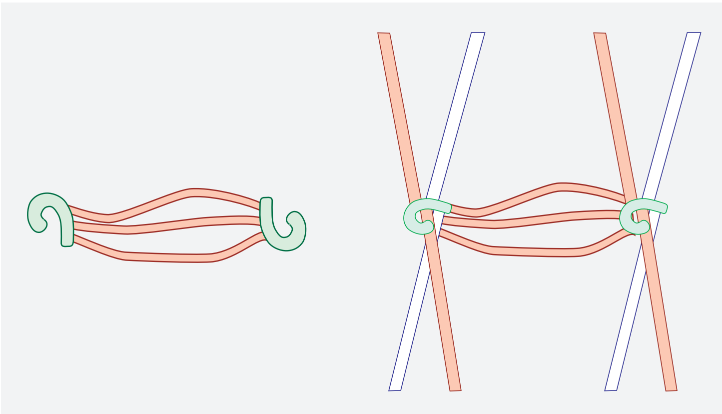


FIGURE 60 Extended belt models for two spin 1/2 particles.

really works with *any* number of bands glued to the object. The website www.evl.uic.edu/hypercomplex/html/dirac.html provides a beautiful animation showing this. Again we find that an object attached to belts behaves under rotations like a spin 1/2 particle.

Similarly, the belt trick can be extended to many bands also for the topic of exchange. Take two buckles that are connected with *many* bands or threads, like in Figure 60. An exchange of the two buckles produces quite a tangle, even if one takes paths that go 'between' the bands; nevertheless, in both cases a second exchange leads back to the original situation.

But this is not all. Take *two* particles with any number of tails, as shown on the right side of Figure 60. You can also add belts going from one to the other particle. If you exchange the positions of two such spin 1/2 particles, always keeping the ends at infinity fixed, a tangled mess is created. But incredibly, if you exchange the two objects a second time, everything untangles neatly, independently of the number of attached strings. You might want to test yourself that the behaviour is also valid if additional particles are involved, as long as you always exchange the same two particles twice. Unfortunately, no animation or video showing this is yet available on the internet. In any case, we conclude that objects attached to belts behave like a spin 1/2 particle also under exchange.

Challenge 97 e

All these observations together form the spin–statistics theorem for spin 1/2 particles: *spin and exchange behaviour are related*. Indeed, these almost 'experimental' arguments can be put into exact mathematical language by studying the behaviour of the configuration space of particles. These investigations result in the following statements: Ref. 76

▷ *Objects of spin 1/2 are fermions.**

▷ *Exchange and rotation of spin 1/2 particles are similar processes.*

In short, objects that behave like spin 1/2 particles under rotations also behave like spin 1/2 particles under exchange. And vice versa. The exchange behaviour of particles determines their statistical properties; the rotation behaviour determines their spin. By extending the belt trick to several buckles, each with several belts, we thus visualized the spin–statistics theorem for fermions.

Note that all these arguments require three dimensions, because there are no tangles (or knots) in fewer or more dimensions.** And indeed, spin exists only in three spatial dimensions.

Here is a challenge. A spin 1/2 object can be modelled with one belt attached to it. If you want to model the spin behaviour with attached one-dimensional *strings* instead of *bands*, what is the minimum number of strings required? Challenge 99 s

Angels, Pauli's exclusion principle and the hardness of matter

Why are we able to knock on a door? Why can stones not fly through tree trunks? How does the mountain we are walking on carry us? Why can't we walk across walls? In classical physics, we avoided this issue, by taking solidity as a defining property of matter. But doing so, we cheated: we have seen that matter consists mainly of low density electron clouds. Thus we have to understand the issue without any sneaky way out. The answer is famous: interpenetration of bodies is made impossible by *Pauli's exclusion principle* among the electrons inside atoms. The exclusion principle states:

▷ *Two fermions cannot occupy the same quantum state.*

All experiments known confirm the statement.

Why do electrons and other fermions obey the Pauli exclusion principle? The answer can be given with a beautifully simple argument. We know that exchanging two fermions produces a minus sign in the total wave function. Imagine these two fermions being, as a classical physicist would say, located at the same spot, or as a quantum physicist would say, in the same state. If that could be possible, an exchange would change nothing in the system. But an exchange of fermions must produce a minus sign for the total state. Both Ref. 77

* A mathematical observable behaving like a spin 1/2 particle is neither a vector nor a tensor, as you may want to check. An additional concept is necessary; such an observable is called a *spinor*. We will introduce it in detail later on. Challenge 98 e Page 159

** Of course, knots and tangles do exist in higher dimensions. Instead of considering knotted one-dimensional lines, one can consider knotted planes or knotted higher-dimensional hyperplanes. For example, deformable planes can be knotted in four dimensions and deformable 3-spaces in five dimensions. However, the effective dimensions that produce the knot are always three.

possibilities – no change at all as well as a minus sign – cannot be realized at the same time. There is only one way out: two fermions must avoid to ever be in the same state. This is Pauli's exclusion principle.

The exclusion principle is the reason that two pieces of matter in everyday life cannot penetrate each other, but have to repel each other. For example, bells only work because of the exclusion principle. A bell would not work if the colliding pieces that produce the sound would interpenetrate. But in any example of two interpenetrating pieces electrons from two atoms would have to be at the same spot: they would have to be in the same states. This is forbidden. Pauli's exclusion principle forbids interpenetration of matter.

Why don't we fall through the floor, even though gravity pulls us down, but remain standing on its surface? Again, the reason is Pauli's exclusion principle. Why does the floor not fall? It does not fall, because the matter of the Earth cannot be compressed further. Why? Pauli's exclusion principle does not allow atoms to be compressed. In other words, the exclusion principle implies that matter cannot be compressed indefinitely, as at a certain stage an effective Pauli pressure appears, so that a compression limit ensues. For this reason for example, planets or neutron stars do not collapse under their own gravity.

The exclusion principle is the reason that atoms are *extended* electron clouds. In fact, the exclusion principle forces the electrons to form shells, and when one shell is filled, a next one is started. The size of the atom is the size of the last shell. Without the exclusion principle, atoms would be point-like. The same argument applies to the protons (and the neutrons) in nuclei.

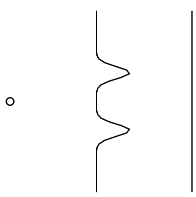
The exclusion principle also answers the question about how many angels can dance on the top of a pin. (Note that angels, if at all, must be made of fermions, as you might
Challenge 100 s
want to deduce from the information known about them.) Both theory and experiment
Ref. 78
confirm the answer already given by Thomas Aquinas in the Middle Ages: Only one! The fermion exclusion principle could also be called 'angel exclusion principle'. To stay in the topic, the principle also shows that *ghosts* cannot be objects, as ghosts are supposed to be able to traverse walls.

Whatever the interpretation, the exclusion principle keeps things in shape; without it, there would be no three-dimensional objects. Only the exclusion principle keeps the cloudy atoms of nature from merging, holding them apart. This repulsion keeps the size of planets to a finite value, and that of neutron stars. All shapes of solids and fluids are a direct consequence of the exclusion principle. In other words, when we knock on a table or on a door, we prove experimentally that both objects are made of fermions.

Since permutation properties and spin properties of fermions are so well described by the belt model, we could be led to the conclusion that these properties might really be consequence of such belt-like connections between particles and the outside world. Maybe for some reason we only observe the belt buckles, not the belts themselves. In the final part of this walk we will discover whether this idea is correct.

So far, we have only considered spin 1/2 particles. We will not talk much about systems with odd spin of higher value, such as 3/2 or 5/2. Such systems can all be seen as
Challenge 101 ny
being composed of spin 1/2 entities. Can you confirm this?

We did not talk about lower spins than 1/2 either. A famous theorem states that a spin value between 0 and 1/2 is impossible, because the largest angle that can be measured



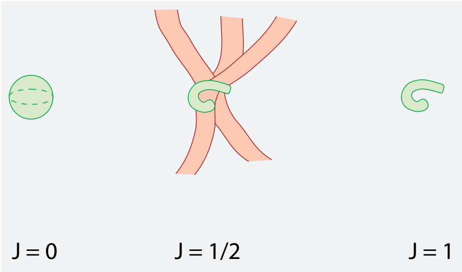


FIGURE 61 Some visualizations of spin representations.

in three dimensions is 4π. There is no way to measure a larger angle;* the quantum of action makes this impossible. Thus there cannot be any spin value between 0 and 1/2 in nature.

Spin, statistics and composition

Under rotations, integer spin particles behave differently from half-integer particles. Integer spin particles do not show the strange sign changes under rotations by 2π. In the belt imagery, integer spin particles need no attached strings. In particular, a spin 0 particle obviously corresponds to a sphere. Models for other spin values are shown in Figure 61. Exploring their properties in the same way as above, we arrive at the full *spin–statistics theorem*:

▷ *Exchange and rotation of objects are similar processes.*

▷ *Objects of half-integer spin are fermions. They obey the Pauli exclusion principle.*

▷ *Objects of integer spin are bosons.*

Challenge 102 ny You might prove by yourself that this suffices to show the following rule:

▷ *Composites of bosons, as well as composites of an even number of fermions (at low energy), are bosons; composites of an uneven number of fermions are fermions.***

These connections express basic characteristics of the three-dimensional world in which we live.

Is spin a rotation about an axis?

The spin of a particle behaves experimentally like an intrinsic angular momentum, adds up like angular momentum, is conserved as part of angular momentum, is described

* This statement, like all statements about spin 1/2, is tied to the three-dimensionality of space. In two dimensions, other largest angles and other 'spin' values are possible.

Challenge 103 ny ** This rule implies that spin 1 and higher can also be achieved *with* tails; can you find such a representation?

Note that composite fermions can be bosons only up to that energy at which the composition breaks down. Otherwise, by packing fermions into bosons, we could have fermions in the same state.

like angular momentum and has a name synonymous with angular momentum. Despite all this, for many decades a strange myth was spread in many physics courses and textbooks around the world, namely that spin $1/2$ is *not* a rotation about an axis. The myth maintains that any rotating object must have integer spin. Since half integer spin is not possible in classical physics, it is argued that such spin is not due to rotation. It is time to finish with this example of incorrect thinking.

Electrons do have spin $1/2$ and are charged. Electrons and all other charged particles with spin $1/2$ do have a magnetic moment.* A magnetic moment is expected for any rotating charge. In other words, spin $1/2$ does behave like rotation. However, assuming that a particle consists of a continuous charge distribution in rotational motion gives the wrong value for the magnetic moment. In the early days of the twentieth century, when physicists were still thinking in classical terms, they concluded that spin $1/2$ particles thus cannot be rotating. This myth has survived through many textbooks. The correct deduction, however, is that the assumption of continuous charge distribution is wrong. Indeed, charge is quantized; nobody today expects that elementary charge is continuously spread over space, as that would contradict its quantization.

Let us recall what rotation is. Both the belt trick for spin 1/2 as well as the integer spin case remind us: a *rotation* of one body around another is a fraction or a multiple of an exchange. What we call a rotating body in everyday life is a body continuously exchanging the positions of its parts. Rotation and exchange are the same.

Now, we just found that spin is exchange behaviour. Since rotation is exchange and spin is exchange, it follows that spin *is* rotation. Since we deduced, like Wigner, spin from rotation invariance, this consequence is not a surprise.

The belt model of a spin 1/2 particle tells us that such a particle can rotate continuously
Page 110 without any hindrance. In short, we are allowed to maintain that spin is rotation about an axis, without any contradiction to observations, even for spin 1/2. The belt model helps us to keep two things in mind: we must assume that in the belt model only the buckle
Ref. 79 can be observed and does interact, not the belt(s), and we must assume that elementary charge is not continuously distributed in space.**

Why is fencing with laser beams impossible?

When a sword is approaching dangerously, we can stop it with a second sword. Many old films use such scenes. When a laser beam is approaching, it is impossible to fend it off with a second beam, despite all science fiction films showing so. Banging two laser beams against each other is impossible. The above explanation of the spin–statistics theorem shows why.

The electrons in the swords are fermions and obey the Pauli exclusion principle. Fermions make matter impenetrable. On the other hand, the photons in laser beams are bosons. Two bosons *can* be in the same state; bosons allow interpenetration. Matter

* This magnetic moment can easily be measured in an experiment; however, not one of the Stern–Gerlach
Challenge 104 ny type. Why not?

** Obviously, the detailed structure of the electron still remains *unclear* at this point. Any angular momentum S is given classically by $S = \Theta\omega$; however, neither the moment of inertia Θ, connected to the rotation radius and electron mass, nor the angular velocity ω are known at this point. We have to wait quite a while, until the final part of our adventure, to find out more.

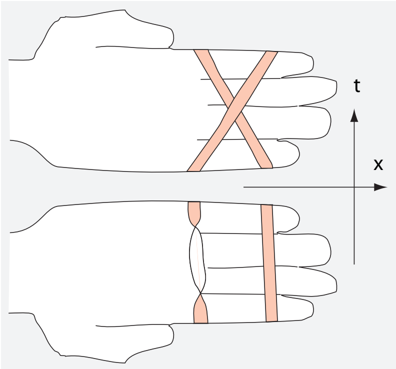


FIGURE 62 Equivalence of exchange and rotation in space-time.

is impenetrable because at the fundamental level it is composed of fermions. Radiation is composed of bosons; light beams can cross each other. The distinction between fermions and bosons thus explains why objects can be touched while images cannot. In the first Vol. I, page 86 part of our mountain ascent we started by noting this difference; now we know its origin.

Rotation requires antiparticles

The connection between rotation and antiparticles may be the most astonishing conclusion from the experiments showing the existence of spin. So far, we have seen that rotation requires the existence of spin, that spin appears when relativity is introduced into Vol. II, page 65 quantum theory, and that relativity requires antimatter. Taking these three statements together, the conclusion of the title is not surprising any more: rotation requires antiparticles. Interestingly, there is a simple argument making the same point with the belt model, if it is extended from space alone to full *space-time*.

To learn how to think in space-time, let us take a particle spin 1, i.e., a particle looking like a detached belt buckle in three dimensions. When moving in a 2+1 dimensional Challenge 105 ny space-time, it is described by a ribbon. Playing around with ribbons in space-time, instead of belts in space, provides many interesting conclusions. For example, Figure 62 shows that wrapping a rubber ribbon around the fingers can show that a rotation of a body by 2π in presence of a second one is the same as exchanging the positions of the two bodies.* Both sides of the hand transform the same initial condition, at one edge of the hand, to the same final condition at the other edge. We have thus successfully extended a known result from space to space-time: rotation and exchange are equivalent.

If you think that Figure 62 is not a satisfying explanation, you are right. A more satisfying explanation must include a smooth sequence of steps realizing the equivalence between rotation and exchange. This is shown in Figure 63. We assume that each particle is described by a segment; in the figure, the two segments lie horizontally. The leftmost diagram shows two particles: one at rest and one being rotated by 2π. The deformation of the ribbons shows that this process is equivalent to the exchange in position of two particles, which is shown in the rightmost diagram.

* Obviously, the full argument would need to check the full spin 1/2 model of Figure 59 in four-dimensional Challenge 106 ny space-time. But doing this is not an easy task; there is no good visualization yet.

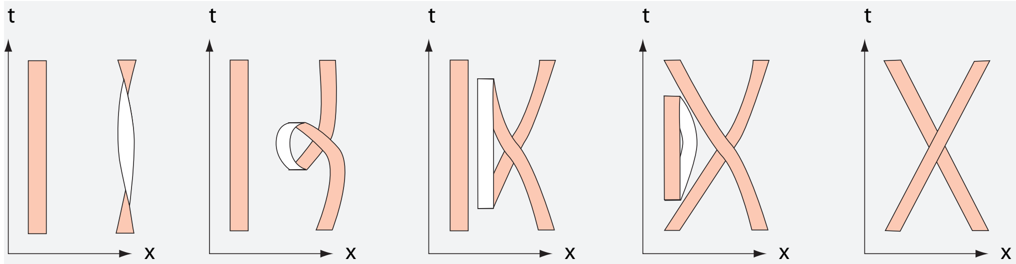


FIGURE 63 Belts in space-time: rotation and antiparticles.

But the main point is made by the intermediate diagrams. One notes that the sequence that shows the equivalence between rotation and exchange requires the use of a loop. But such a loop is the appearance of a particle–antiparticle pair. In other words, without antiparticles, the equivalence of rotation and exchange would not hold. Rotation in space-time requires the existence of antiparticles.

A summary on spin and indistinguishability

The quantum of action implies that physical systems are made of two types of indistinguishable particles: *bosons* and *fermions*. The two possible exchange behaviours are related to their spin value, because exchange is related to rotation.

Experiments show that radiation is made of elementary particles that behave as bosons. They have integer spin. Two or more bosons, such as two photons, *can share* the same state. This sharing makes laser light possible.

Experiments show that matter is made of elementary particles that behave as fermions. They have half-integer spin. They obey Pauli's exclusion principle: two fermions *cannot be* in the same state. The exclusion principle between electrons explains the structure and (partly) the size of atoms, as well as the chemical behaviour of atoms, as we will find out later on. Together with the electrostatic repulsion of electrons, the exclusion principle explains the incompressibility of matter and its lack of impenetrability.

Despite the indistinguishability of particles, the quantum of action also implies that exact copies of macroscopic objects cannot be produced, because quantum states for macroscopic objects cannot be copied.

Finally, the connection between spin and rotation implies that antiparticles exist. It also implies that spin is intrinsically a three-dimensional phenomenon.

Challenge 107 e Can you summarize why matter is hard, but radiation is not?

Limits and open questions of quantum statistics

The topic of statistics is an important research field in theoretical and experimental physics. In particular, researchers have searched and still are searching for generalizations of the possible exchange behaviours of particles.

In two spatial dimensions, the result of an exchange of the wave function is not described by a sign, but by a continuous phase. Two-dimensional objects behaving in this way, called *anyons* because they can have 'any' spin, have experimental importance, since in many experiments in solid state physics the set-up is effectively two-dimensional. The fractional quantum Hall effect, perhaps the most interesting discovery of modern experimental physics, has pushed anyons onto the stage of modern research. Vol. V, page 70

Other theorists generalized the concept of fermions in other ways, introducing parafermions, parabosons, plektons and other hypothetical concepts. Oscar Greenberg has spent most of his professional life on this issue. His conclusion is that in 3 + 1 space-time dimensions, only fermions and bosons exist. (Can you show that this implies that the ghosts appearing in Scottish tales do not exist?) Ref. 80 Challenge 108 s

From a different viewpoint, the belt model of spin 1/2 invites to study the behaviour of braids, open links and knots. (In mathematics, braids and open links are made of strands extending to infinity.) This fascinating part of mathematical physics has become important with in modern unified theories, which all state that particles, especially at high energies, are not point-like, but extended entities.

Still another generalization of statistical behaviour at high energies is the concept of quantum group. In all of these cases, the quest is to understand what happens to permutation symmetry in a unified theory of nature. A glimpse of the difficulties appears already above: how can Figures 54, 59 and 63 be reconciled and combined? We will settle this issue in the final part of our mountain ascent. Vol. VI, page 155

Chapter 7

SUPERPOSITIONS AND PROBABILITIES – QUANTUM THEORY WITHOUT IDEOLOGY

> “The fact that an adequate philosophical presentation has been so long delayed is no doubt caused by the fact that Niels Bohr brainwashed a whole generation of theorists into thinking that the job was done fifty years ago.”
>
> Murray Gell-Mann

Ref. 81

Why is this famous physical issue arousing such strong emotions? In particular, who is brainwashed, Gell-Mann, the discoverer of the quarks, or most of the world's physicists working on quantum theory who follow Niels Bohr's opinion? In the twentieth century, quantum mechanics has thrown many in disarray. Indeed, it radically changed the two most basic concepts of classical physics: state and system. The *state* is not described any more by the specific values taken by position and momentum, but by the specific wave function 'taken' by the position and momentum operators.* In addition, in classical physics a *system* was described as a set of permanent aspects of nature; permanence was defined as negligible interaction with the environment. Quantum mechanics shows that this definition has to be modified as well.

The description of nature with quantum theory is unfamiliar for two reasons: it allows superpositions and it leads to probabilities. Let us clarify these issues. A clarification is essential if we want to avoid getting lost on our way to the top of Motion Mountain, as happened to quite a number of people since quantum theory appeared, including Gell-Mann.

Why are people either dead or alive?

The evolution equation of quantum mechanics is *linear* in the wave function; linearity implies the existence of superpositions. Therefore we can imagine and try to construct systems where the state ψ is a superposition of two radically distinct situations, such as those of a dead and of a living cat. This famous fictional animal is called *Schrödinger's cat* after the originator of the example. Is it possible to produce it? And how would it evolve in time? We can ask the same two questions in other situations. For example, can we produce a superposition of a state where a car is inside a closed garage with a state

* It is equivalent, but maybe conceptually clearer, to say that the state is described by a complete set of commuting operators. In fact, the discussion is somewhat simplified in the Heisenberg picture. However, here we study the issue in the Schrödinger picture, using wave functions.

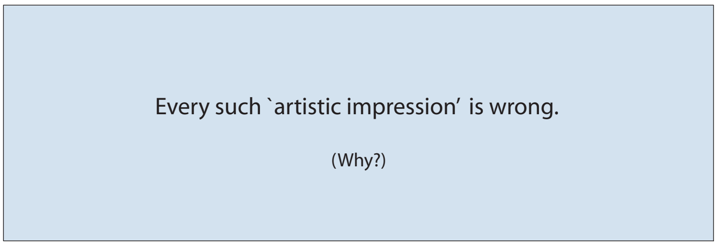


FIGURE 64 An artist's impression of a macroscopic superposition is impossible.

where the car is outside? What happens then?

Such strange situations are not usually observed in everyday life. The reason for this rareness is an important aspect of what is often called the 'interpretation' of quantum mechanics. In fact, such strange situations *are* possible, and the superposition of macroscopically distinct states has actually been observed in a few cases, though not for cats, people or cars. To get an idea of the constraints, let us specify the situation in more detail.*

The object of discussion are linear superpositions of the type $\psi = a\psi_a + b\psi_b$, where ψ_a and ψ_b are macroscopically distinct states of the system under discussion, and where a and b are some complex coefficients. States are called *macroscopically distinct* when each state corresponds to a different macroscopic situation, i.e., when the two states can be distinguished using the concepts or measurement methods of classical physics. In particular, this means that the physical action necessary to transform one state into the other must be much larger than $\hbar$. For example, two different positions of any body composed of a large number of molecules are macroscopically distinct.

A 'strange' situation is thus a superposition of macroscopically distinct states. Let us work out the essence of macroscopic superpositions more clearly. Given two macroscopically distinct states ψ_a and ψ_b, a superposition of the type $\psi = a\psi_a + b\psi_b$ is called a *pure state*. Since the states ψ_a and ψ_b can interfere, one also talks about a *(phase) coherent superposition*. In the case of a superposition of macroscopically distinct states, the scalar product $\psi_a^\dagger \psi_b$ is obviously vanishing. In case of a coherent superposition, the coefficient product $a^* b$ is different from zero. This fact can also be expressed with the help of the *density matrix* ρ of the system, defined as $\rho = \psi \otimes \psi^\dagger$. In the present case it is given by

$$\begin{aligned}\rho_{\text{pure}} = \psi \otimes \psi^\dagger &= |a|^2 \psi_a \otimes \psi_a^\dagger + |b|^2 \psi_b \otimes \psi_b^\dagger + a\, b^* \psi_a \otimes \psi_b^\dagger + a^*\, b\, \psi_b \otimes \psi_a^\dagger \\ &= (\psi_a, \psi_b) \begin{pmatrix} |a|^2 & a\, b^* \\ a^*\, b & |b|^2 \end{pmatrix} \begin{pmatrix} \psi_a^\dagger \\ \psi_b^\dagger \end{pmatrix} . \end{aligned} \qquad (64)$$

We can then say that whenever the system is in a pure state, its density matrix, or *density functional*, contains off-diagonal terms of the same order of magnitude as the diagonal

* Most what can be said about this topic has been said by two people: John von Neumann, who in the nineteen-thirties stressed the differences between evolution and decoherence, and by Hans Dieter Zeh, who in the nineteen-seventies stressed the importance of baths and the environment in the decoherence process. Ref. 82 Ref. 83

ones.* Such a density matrix corresponds to the above-mentioned strange situations that we do not observe in daily life.

We now have a look at the opposite situation, a density matrix for macroscopic distinct states with *vanishing* off-diagonal elements. For two state, the example

$$\begin{aligned} \rho &= |a|^2 \psi_a \otimes \psi_a^\dagger + |b|^2 \psi_b \otimes \psi_b^\dagger \\ &= (\psi_a, \psi_b) \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \begin{pmatrix} \psi_a^\dagger \\ \psi_b^\dagger \end{pmatrix} \end{aligned} \tag{66}$$

describes a system which possesses *no* phase coherence at all. (Here, $\otimes$ denotes the non-commutative dyadic product or tensor product which produces a tensor or matrix starting from two vectors.) Such a diagonal density matrix cannot be that of a pure state; it describes a system which is in the state ψ_a with probability $|a|^2$ and which is in the state ψ_b with probability $|b|^2$. Such a system is said to be in a *mixed state*, because its state is *not known*, or equivalently, to be in a *(phase) incoherent superposition*, because interference effects cannot be observed in such a situation. A system described by a mixed state is always *either* in the state ψ_a *or* in the state ψ_b. In other words, a diagonal density matrix for macroscopically distinct states is not in contrast, but in agreement with everyday experience. In the picture of density matrices, the non-diagonal elements contain the difference between normal, i.e., incoherent, and unusual, i.e., coherent, superpositions.

The experimental situation is clear: for macroscopically distinct states, only diagonal density matrices are observed in everyday life. Any system in a coherent macroscopic superposition somehow loses its off-diagonal matrix elements. How does this process of *decoherence*** take place? The density matrix itself shows the way.

Ref. 84 In thermodynamics, the density matrix for a large system is used for the definition of
Challenge 109 ny its entropy and of all its other thermodynamic quantities. These studies show that

$$S = -k \ \mathrm{tr}(\rho \ln \rho) \tag{67}$$

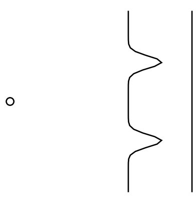
where tr denotes the *trace*, i.e., the sum of all diagonal elements. We also remind ourselves that a system with a large and constant entropy is called a *bath*. In simple physical terms, a bath is a system to which we can ascribe a temperature. More precisely, a *(physical) bath*, or *(thermodynamic) reservoir*, is any large system for which the concept of *equilibrium* can be defined. Experiments show that in practice, this is equivalent to the condition that a bath consists of many interacting subsystems. For this reason, all macroscopic quantities describing the state of a bath show small, irregular *fluctuations*, a fact that will be of central importance shortly.

* Using the density matrix, we can rewrite the evolution equation of a quantum system:

$$\dot{\psi} = -iH\psi \quad \text{becomes} \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] \ . \tag{65}$$

Both are completely equivalent. (The new expression is sometimes also called the *von Neumann equation*.) We won't actually do any calculations here. The expressions are given so that you recognize them when you encounter them elsewhere.

** In certain settings, decoherence is called *disentanglement*, as we will see below.



An everyday bath is also a thermodynamic bath in the physical sense: indeed, a thermodynamic bath is similar to an extremely large warm water bath, one for which the temperature does not change even if one adds some cold or warm water to it. Examples of physical baths are an intense magnetic field, a large amount of gas, or a large solid. (The meanings of 'intense' and 'large' of course depend on the system under study.) The physical concept of bath (or *reservoir*) is thus an abstraction and a generalization of the everyday concept of bath.

It is easy to see from the definition (67) of entropy that the loss of off-diagonal elements corresponds to an increase in entropy. And it is known that any increase in entropy of a reversible system, such as the quantum mechanical system in question, is due to an interaction with a bath.

Challenge 110 s

Where is the bath interacting with the system? It obviously must be outside the system one is talking about, i.e., in its *environment*. Indeed, we know experimentally that any environment is large and characterized by a temperature. Some examples are listed in Table 6. Any environment therefore contains a bath. We can even go further: for every experimental situation, there is a bath *interacting* with the system. Indeed, every system which can be observed is not isolated, as it obviously interacts at least with the observer; and every observer by definition contains a bath, as we will show in more detail shortly. Usually however, the most important baths we have to take into consideration are the atmosphere around a system, the radiation or electromagnetic fields interacting with the system, or, if the system itself is large enough to have a temperature, those degrees of freedom of the system which are not involved in the superposition under investigation.

Since every system is in contact with baths, every density matrix of a macroscopic superposition will lose its diagonal elements eventually. At first sight, this direction of thought is not convincing. The interactions of a system with its environment can be made extremely small by using clever experimental set-ups; that would imply that the time for decoherence can be made extremely large. Thus we need to check how much time a superposition of states needs to decohere. It turns out that there are two standard ways to estimate the *decoherence time:* either by modelling the bath as large number of colliding particles, or by modelling it as a continuous field.

If the bath is described as a set of particles randomly hitting the microscopic system, it is best characterized by the effective wavelength λ_{eff} of the particles and by the average interval t_{hit} between two hits. A straightforward calculation shows that the decoherence time t_d is in any case smaller than this time interval, so that

Challenge 111 ny

$$t_d \leqslant t_{\text{hit}} = \frac{1}{\varphi\sigma} , \tag{68}$$

where φ is the flux of particles and σ the cross-section for the hit.* Typical values are given in Table 6. We easily note that for macroscopic objects, decoherence times are extremely

* The decoherence time is derived by studying the evolution of the density matrix $\rho(x, x')$ of objects localized at two points x and x'. One finds that the off-diagonal elements follow $\rho(x, x', t) = \rho(x, x', 0)\mathrm{e}^{-\Lambda t(x-x')^2}$, where the localization rate Λ is given by

$$\Lambda = k^2 \varphi\sigma_{\text{eff}} \tag{69}$$

where k is the wave number, φ the flux and σ_{eff} the cross-section of the collisions, i.e., usually the size of the macroscopic object.

Ref. 85

TABLE 6 Common and less common baths with their main properties.

Bath type	Temperature T	Wavelength λ_{eff}	Particle flux φ	Cross section (atom) σ	Hit time $1/\sigma\varphi$ for atom[a]	ball[a]
matter baths						
solid, liquid	300 K	10 pm	$10^{31}\,/\mathrm{m^2s}$	$10^{-19}\,\mathrm{m^2}$	10^{-12} s	10^{-25} s
air	300 K	10 pm	$10^{28}\,/\mathrm{m^2s}$	$10^{-19}\,\mathrm{m^2}$	10^{-9} s	10^{-22} s
laboratory vacuum	50 mK	10 μm	$10^{18}\,/\mathrm{m^2s}$	$10^{-19}\,\mathrm{m^2}$	10 s	10^{-12} s
photon baths						
sunlight	5800 K	900 nm	$10^{23}\,/\mathrm{m^2s}$		10^{-4} s	10^{-17} s
‘darkness’	300 K	20 μm	$10^{21}\,/\mathrm{m^2s}$		10^{-2} s	10^{-15} s
cosmic microwaves	2.7 K	2 mm	$10^{17}\,/\mathrm{m^2s}$		10^{2} s	10^{-11} s
terrestrial radio waves						
Casimir effect					very large	
Unruh radiation of Earth	40 zK				very large	
nuclear radiation baths						
radioactivity		10 fm	$1\,/\mathrm{m^2s}$	$10^{-25}\,\mathrm{m^2}$	10^{25} s	10^{12} s
cosmic radiation	>1000 K	10 fm	$10^{-2}\,/\mathrm{m^2s}$	$10^{-25}\,\mathrm{m^2}$	10^{27} s	10^{14} s
solar neutrinos	≈ 10 MK	10 fm	$10^{11}\,/\mathrm{m^2s}$	$10^{-47}\,\mathrm{m^2}$	10^{36} s	10^{15} s
cosmic neutrinos	2.0 K	3 mm	$10^{17}\,/\mathrm{m^2s}$	$10^{-62}\,\mathrm{m^2}$	10^{45} s	10^{24} s
gravitational baths						
gravitational radiation	$5 \cdot 10^{31}$ K	10^{-35} m			very large	

a. Values are rough estimates. The macroscopic ball is assumed to have a 1 mm size.

short. (We also note that nuclear and gravitational effects lead to large decoherence times and can thus all be neglected.) Scattering leads to fast decoherence. However, for atoms or smaller systems, the situation is different, as expected.

A second method to estimate the decoherence time is also common. Any interaction of a system with a bath is described by a relaxation time t_r. The term *relaxation* designates any process which leads to the return to the equilibrium state. The terms *damping* and *friction* are also used. In the present case, the relaxation time describes the return to equilibrium of the combination bath and system. Relaxation is an example of an irreversible evolution. A process is called *irreversible* if the reversed process, in which every component moves in opposite direction, is of very low probability.* For example, it is usual that

One also finds the surprising result that a system hit by a particle of energy E_{hit} collapses the density matrix roughly down to the de Broglie (or thermal de Broglie) wavelength of the hitting particle. Both results together give the formula above. Ref. 86

* Beware of other definitions which try to make something deeper out of the concept of irreversibility, such as claims that ‘irreversible’ means that the reversed process is *not at all* possible. Many so-called ‘contradictions’ between the irreversibility of processes and the reversibility of evolution equations are due to this

a glass of wine poured into a bowl of water colours the whole water; it is very rarely observed that the wine and the water separate again, since the probability of all water and wine molecules to change directions together at the same time is rather low, a state of affairs making the happiness of wine producers and the despair of wine consumers.

Now let us simplify the description of the bath. We approximate it by a single, unspecified, scalar field which interacts with the quantum system. Due to the continuity of space, such a field has an infinity of degrees of freedom. They are taken to model the many degrees of freedom of the bath. The field is assumed to be in an initial state where its degrees of freedom are excited in a way described by a temperature T. The interaction of the system with the bath, which is at the origin of the relaxation process, can be described by the repeated transfer of small amounts of energy E_{hit} until the relaxation process is completed.

The objects of interest in this discussion, like the mentioned cat, person or car, are described by a mass m. Their main characteristic is the maximum energy E_r which can be transferred from the system to the environment. This energy describes the interactions between system and environment. The superpositions of macroscopic states we are interested in are solutions of the Hamiltonian evolution of these systems.

The initial coherence of the superposition, so disturbingly in contrast with our everyday experience, disappears exponentially within a *decoherence time* t_d given by*

Ref. 87

$$t_d = t_r \frac{E_{\text{hit}}}{E_r} \frac{\mathrm{e}^{E_{\text{hit}}/kT} - 1}{\mathrm{e}^{E_{\text{hit}}/kT} + 1} \qquad (72)$$

where k is the *Boltzmann constant* and like above, E_r is the maximum energy which can be transferred from the system to the environment. Note that one always has $t_d \leqslant t_r$. After the decoherence time t_d is elapsed, the system has evolved from the coherent to the incoherent superposition of states, or, in other words, the density matrix has lost its off-diagonal terms. One also says that the phase coherence of this system has been destroyed. Thus, after a time t_d, the system is found either in the state ψ_a or in the state ψ_b, respectively with the probability $|a|^2$ or $|b|^2$, and not any more in a coherent superposition which is so much in contradiction with our daily experience. Which final state is selected depends on the precise state of the bath, whose details were eliminated from the calculation by taking an *average* over the states of its microscopic constituents.

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mistaken interpretation of the term 'irreversible'.

* This result is derived as in the above case. A system interacting with a bath always has an evolution given by the general form

Ref. 88

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] - \frac{1}{2t_o}\sum_j [V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger] \,, \qquad (70)$$

where ρ is the density matrix, H the Hamiltonian, V the interaction, and t_o the characteristic time of the interaction. Are you able to see why? Solving this equation, one finds for the elements far from the diagonal $\rho(t) = \rho_0 \mathrm{e}^{-t/t_0}$. In other words, they disappear with a characteristic time t_o. In most situations one has a relation of the form

Challenge 112 ny

$$t_0 = t_r \frac{E_{\text{hit}}}{E_r} = t_{\text{hit}} \qquad (71)$$

or some variations of it, as in the example above.

The important result is that for all macroscopic objects, the decoherence time t_d is extremely small. In order to see this more clearly, we can study a special simplified case. A macroscopic object of mass m, like the mentioned cat or car, is assumed to be at the same time in two locations separated by a distance l, i.e., in a superposition of the two corresponding states. We further assume that the superposition is due to the object moving as a quantum mechanical oscillator with frequency ω between the two locations; this is the simplest possible system that shows superpositions of an object located in two different positions. The energy of the object is then given by $E_r = m\omega^2 l^2$, and the smallest transfer energy $E_{\text{hit}} = \hbar\omega$ is the difference between the oscillator levels. In a macroscopic situation, this last energy is much smaller than kT, so that from the preceding expression

Ref. 89 we get

$$t_d = t_r \frac{E_{\text{hit}}^2}{2E_r kT} = t_r \frac{\hbar^2}{2mkTl^2} = t_r \frac{\lambda_T^2}{l^2} \tag{73}$$

in which the frequency ω has disappeared. The quantity $\lambda_T = \hbar/\sqrt{2mkT}$ is called the *thermal de Broglie wavelength* of a particle.

It is straightforward to see that for practically all macroscopic objects the typical decoherence time t_d is extremely short. For example, setting $m = 1$ g, $l = 1$ mm and $T = 300$ K we get $t_d/t_r = 1.3 \cdot 10^{-39}$. Even if the interaction between the system and the environment would be so weak that the system would have as relaxation time the age of the universe, which is about $4 \cdot 10^{17}$ s, the time t_d would still be shorter than $5 \cdot 10^{-22}$ s, which is over a million times faster than the oscillation time of a beam of light (about 2 fs for green light). For Schrödinger's cat, the decoherence time would be even shorter. These times are so short that we cannot even hope to *prepare* the initial coherent superposition, let alone to observe its decay or to measure its lifetime.

For microscopic systems however, the situation is different. For example, for an electron in a solid cooled to liquid helium temperature we have $m = 9.1 \cdot 10^{-31}$ kg, and typically $l = 1$ nm and $T = 4$ K; we then get $t_d \approx t_r$ and therefore the system can stay in a coherent superposition until it is relaxed, which confirms that for this case coherent effects can indeed be observed if the system is kept isolated. A typical example is the behaviour of electrons in superconducting materials. We will mention a few more below. Ref. 90

In 1996 the first actual measurement of decoherence times was published by the Paris team around Serge Haroche. It confirmed the relation between the decoherence time and the relaxation time, thus showing that the two processes have to be distinguished at microscopic scale. In the meantime, other experiments confirmed the decoherence process with its evolution equation, both for small and large values of t_d/t_r. A particularly beautiful experiment has been performed in 2004, where the disappearance of two-slit interference for C_{70} molecules was observed when a bath interacts with them. Ref. 91 Ref. 92 Ref. 93

Summary on decoherence, life and death

Our exploration showed that *decoherence results from coupling to a bath in the environment.* Decoherence is a statistical, thermodynamic effect.

The estimates of decoherence times in everyday life told us that both the preparation and the survival of superpositions of macroscopically different states is made impossible by the interaction with any bath found in the environment. This is the case even if the usual measure of this interaction, given by the friction of the motion of the system, is very small. Even if a macroscopic system is subject to an extremely low friction, leading to a very long relaxation time, its decoherence time is still vanishingly short. Only carefully designed and expensive laboratory systems can reach substantial decoherence times.

Our everyday environment is full of baths. Therefore, *coherent superpositions of macroscopically distinct states never appear in everyday life.* Cars cannot be in and out of a garage at the same time. In short, we cannot be dead and alive at the same time.

What is a system? What is an object?

In classical physics, a system is a part of nature which can be isolated from its environment. However, quantum mechanics tells us that isolated systems do not exist, since interactions cannot be made vanishingly small. The results above allow us to define the concept of system with more accuracy. A *system* is any part of nature which interacts *incoherently* with its environment. In other words, an *object* is a part of nature interacting with its environment only through baths.

In particular, a system is called *microscopic* or *quantum mechanical* and can described by a wave function ψ whenever

— it is almost isolated, with $t_{\mathrm{evol}} = \hbar/\Delta E < t_{\mathrm{r}}$, and

Ref. 94 — it is in *incoherent* interaction with its environment.

In short, a microscopic or quantum mechanical system that is described by a wave function interacts incoherently and weakly with its environment. (For such a system, the energy indeterminacy ΔE is larger than the relaxation energy.) In contrast, a bath is never isolated in the sense just given, because the evolution time of a bath is always much larger than its relaxation time. Since all macroscopic bodies are in contact with baths – or even contain one – they cannot be described by a wave function. In particular, it is impossible to describe any measuring apparatus with the help of a wave function.

We thus conclude that a *macroscopic system* is a system with a decoherence time much shorter than any other evolution time of its constituents. Obviously, macroscopic systems also interact incoherently with their environment. Thus cats, cars and television news speakers are all macroscopic systems.

One possibility is left over by the two definitions: what happens in the situation in which the interactions with the environment are *coherent*? We will encounter some examples shortly. Following the definition, they are neither microscopic and macroscopic systems; they are not described by a wave function, and strictly speaking, they are *not* systems. In these situations, when the interaction is coherent, one speaks of *entanglement* or of entangle 'systems'; such a particle or set of particles is said to be *entangled* with its environment.

Entangled, coherently interacting systems can be divided, but must be disentangled when doing so. The act of division leads to detached entities; detached entities interact incoherently. Quantum theory shows that nature is not made of detached entities, but that it is made of *detachable* entities. In quantum theory, the criterion of detachment is the incoherence of interaction. Coherent superpositions imply the surprising consequence that there are systems which, even though they look being made of detached parts, are not. Entanglement poses a limit to detachment. All surprising properties of quantum

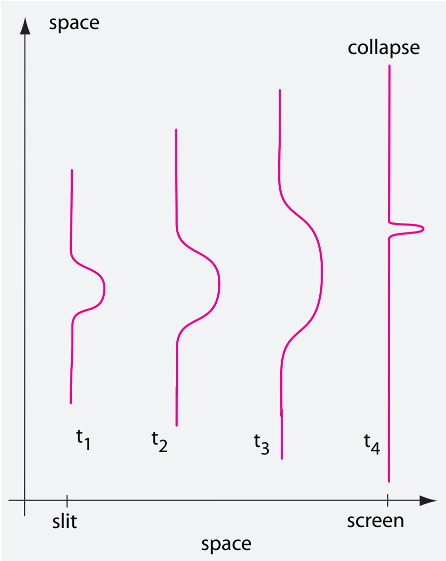


FIGURE 65 Quantum mechanical motion: an electron wave function (actually its module squared) from the moment it passes a slit until it hits a screen.

mechanics, such as Schrödinger's cat, are consequences of the classical prejudice that a system made of two or more parts can obviously be detached into two subsystems without disturbance. But coherent superpositions, or entangled systems, do not allow detachment without disturbance. Whenever we assume to be able to detach entangled systems, we get strange or incorrect conclusions, such as apparent faster-than-light propagation, or, as one says today, non-local behaviour. Let us have a look at a few typical examples.

Is quantum theory non-local? – A bit about the Einstein–Podolsky–Rosen paradox

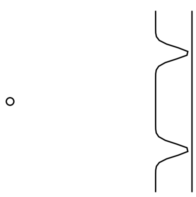
> “[Mr. Duffy] lived a little distance away from his body ...”
>
> James Joyce, *A Painful Case*

We now explore non-locality in quantum mechanics.* To do this, we explore wave function collapse in some detail.

We imagine an electron hitting a screen after passing a slit. Following the description just deduced, the process proceeds schematically as depicted in Figure 65. A film of the same process can be seen in the lower left corners on these pages, starting at page 73. The process has a surprising aspect: due to the short decoherence time, during this (and any other) wave function collapse the maximum of the wave function changes position faster than light. Is this reasonable?

Challenge 113 s
Vol. III, page 107
A situation is called *acausal* or *non-local* if energy is transported faster than light. Using Figure 65 you can determine the energy velocity involved, using the results on signal propagation. The result is a value smaller than c. A wave function maximum moving faster than light does *not* imply energy moving faster than light.

Vol. II, page 257
* This continues a topic that we know already: we have explored a different type of non-locality, in general relativity, earlier on.



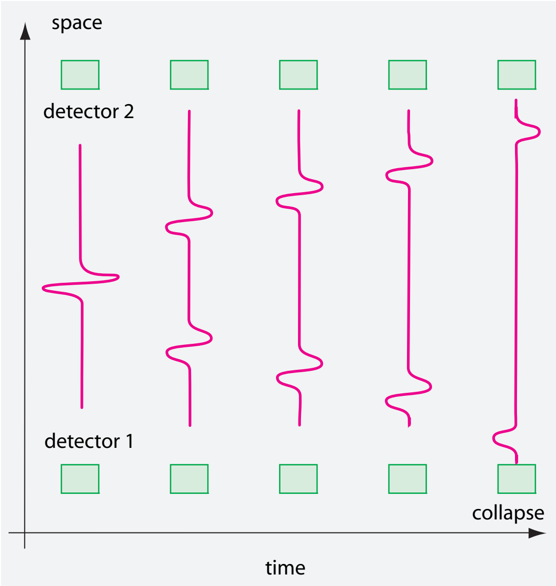


FIGURE 66 Bohm's Gedanken experiment.

In other words, quantum theory has speeds greater than light, but no energy speeds Ref. 95 greater than light. In classical electrodynamics, the same happens with the scalar and the vector potentials if the Coulomb gauge is used. We have also encountered speeds faster Vol. II, page 53 than that of light in the motion of shadows and in many other observations. Any physicist now has two choices: he can be straight, and say that there is no non-locality in nature; or he can be less straight, and claim there is. In the latter case, he has to claim that even classical physics is non-local. However, this never happens. On the other hand, there is a danger in this more provoking usage: a small percentage of those who say that the world is non-local after a while start to believe that there really are faster-than-light effects in nature. These people become prisoners of their muddled thinking; on the other hands, muddled thinking helps to get more easily into newspapers. In short, even though the definition of non-locality is not unanimous, here we stick to the stricter one, and define non-locality as energy transport faster than light.

An often cited Gedanken experiment that shows the pitfalls of non-locality was proposed by Bohm* in the discussion around the so-called Einstein–Podolsky–Rosen Ref. 96, Ref. 97 paradox. In the famous EPR paper the three authors try to find a contradiction between quantum mechanics and common sense. Bohm translated their rather confused paper into a clear Gedanken experiment. When two particles in a spin 0 state move apart, measuring one particle's spin orientation implies an *immediate* collapse also of the other particle's spin, namely in the exactly opposite direction. This happens instantaneously over the whole separation distance; no speed limit is obeyed. In other words, entanglement seems to lead to faster-than-light communication.

However, in Bohm's experiment, no energy is transported faster than light. No non-locality is present, despite numerous claims of the contrary by certain authors. The two

* David Joseph Bohm (1917–1992) American–British physicist. He codiscovered the Aharonov–Bohm effect; he spent a large part of his later life investigating the connections between quantum physics and philosophy.

entangled electrons belong to one system: assuming that they are separate only because the wave function has two distant maxima is a conceptual mistake. In fact, no signal can be transmitted with this method; the decoherence is a case of prediction which looks like a signal without being one. Bohm's experiment, like any other EPR-like experiment, does not allow communication faster than light. We already discussed such cases in the section on electrodynamics. Vol. III, page 111

Bohm's experiment has actually been performed. The first and most famous realization was realized in 1982 by Alain Aspect; he used photons instead of electrons. Like all latter tests, it has fully confirmed quantum mechanics. Ref. 98

In fact, experiments such as the one by Aspect confirm that it is impossible to treat either of the two particles as a system by itself; it is impossible to ascribe any physical property, such as a spin orientation, to either of them alone. (The Heisenberg picture would express this restriction even more clearly.)

The mentioned two examples of apparent non-locality can be dismissed with the remark that since obviously no energy flux faster than light is involved, no problems with causality appear. Therefore the following example is more interesting. Take two identical atoms, one in an excited state, one in the ground state, and call l the distance that separates them. Common sense tells that if the first atom returns to its ground state emitting a photon, the second atom can be excited only after a time $t = l/c$ has been elapsed, i.e., after the photon has travelled to the second atom.

Surprisingly, this conclusion is wrong. The atom in its ground state has a non-zero probability to be excited at the same moment in which the first is de-excited. This has been shown most simply by Gerhard Hegerfeldt. The result has even been confirmed experimentally. Ref. 99

More careful studies show that the result depends on the type of superposition of the two atoms at the beginning: coherent or incoherent. For incoherent superpositions, the intuitive result is correct; the counter-intuitive result appears only for coherent superpositions. Again, a careful discussion shows that no real non-locality of energy is involved.

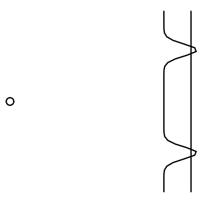
In summary, faster-than-light speeds in wave function collapse do not contradict the limit on energy speed of special relativity. Collapse speeds are phase velocities. In nature, phase velocities are unlimited; unlimited phase velocities never imply energy transport faster than light.

Curiosities and fun challenges about superpositions

Can a photograph show an object at two different places at the same time? Challenge 114 s

∗ ∗

In a few cases, the superposition of different macroscopic states can actually be observed by lowering the temperature to sufficiently small values and by carefully choosing suitably small masses or distances. Two well-known examples of coherent superpositions are those observed in gravitational wave detectors and in Josephson junctions. In the first case, one observes a mass as heavy as 1000 kg in a superposition of states located at different points in space: the distance between them is of the order of 10^{-17} m. In the second case, in superconducting rings, superpositions of a state in which a macroscopic current of the order of 1 pA flows in clockwise direction with one where it flows Ref. 89



Ref. 104 in counter-clockwise direction have been produced.

* *

Ref. 100 Superpositions of magnetization in up and down direction at the same time have also be observed for several materials.

* *

Some people wrongly state that an atom that is in a superposition of states centred at different positions has been photographed. (This lie is even used by some sects to attract Challenge 115 s believers.) Why is this not true?

* *

Since the 1990s, the sport of finding and playing with new systems in coherent macro- Ref. 101 scopic superpositions has taken off across the world. The challenges lie in the clean experiments necessary. Experiments with single atoms in superpositions of states are among Ref. 102 the most popular ones.

* *

Ref. 103 In 1997, coherent atom waves were extracted from a cloud of sodium atoms.

* *

Macroscopic objects usually are in incoherent states. This is the same situation as for light. The world is full of 'macroscopic', i.e., incoherent light: daylight, and all light from lamps, from fire and from glow-worms is incoherent. Only very special and carefully constructed sources, such as lasers or small point sources, emit coherent light. Only these sources allow studying interference effects. In fact, the terms 'coherent' and 'incoherent' originated in optics, since for light the difference between the two, namely the capacity to interfere, had been observed centuries before the case of matter.

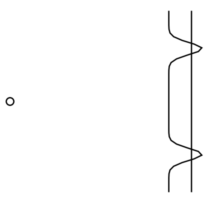
Coherence and incoherence of light and of matter manifest themselves differently, because matter can stay at rest but light cannot and because matter is made of fermions, Page 115 but light is made of bosons. Coherence can be observed easily in systems composed of bosons, such as light, sound in solids, or electron pairs in superconductors. Coherence is less easily observed in systems of fermions, such as systems of atoms with their electron clouds. However, in both cases a decoherence time can be defined. In both cases coherence in many particle systems is best observed if all particles are in the same state (superconductivity, laser light) and in both cases the transition from coherent to incoherent is due to the interaction with a bath. A beam is thus incoherent if its particles arrive randomly in time and in frequency. In everyday life, the rarity of observation of coherent matter superpositions has the same origin as the rarity of observation of coherent light.

* *

We will discuss the relation between the environment and the *decay* of unstable systems Vol. V, page 36 later on. The phenomenon is completely described by the concepts given here.

* *

Challenge 116 ny Can you find a method to measure the *degree* of entanglement? Can you do so for a



system made of many particles?

∗ ∗

The study of entanglement leads to a simple conclusion: *teleportation contradicts correlation*. Can you confirm the statement? Challenge 117 ny

∗ ∗

Are ghost images in TV sets, often due to spurious reflections, examples of interference? Challenge 118 s

∗ ∗

Challenge 119 d What happens when two monochromatic electrons overlap?

∗ ∗

Some people say that quantum theory could be used for quantum computing, by using coherent superpositions of wave functions. Ref. 105 Can you give a general reason that makes this aim very difficult – even though not impossible – even without knowing how such a quantum computer might work, or what the so-called *qubits* might be? Challenge 120 s

What is all the fuss about measurements in quantum theory?

Measurements in quantum mechanics are disturbing. They lead to statements in which *probabilities* appear. For example, we speak about the probability of finding an electron at a certain distance from the nucleus of an atom. Statements like this belong to the general type 'when the observable A is measured, the probability to find the outcome a is p.' In the following we will show that the probabilities in such statements are inevitable for any measurement, because, as we will show, any measurement and any observation is a special case of decoherence or disentanglement process. (Historically, the process of measurement was studied *before* the more general process of decoherence. That explains in part why the topic is so confused in many peoples' minds.)

Vol. III, page 184 What is a measurement? As already mentioned earlier on, a measurement is any interaction which produces a record or a memory. (Any effect of everyday life is a record; but this is not true in general. Can you give some examples of effects that are records and some effects which are not?) Challenge 121 s Measurements can be performed by machines; when they are performed by people, they are called *observations*. In quantum theory, the process of measurement is not as straightforward as in classical physics. This is seen most strikingly when a quantum system, such as a single electron, is first made to pass a diffraction slit, or better – in order to make its wave aspect become apparent – a double slit and then is made to hit a photographic plate, in order to make also its particle aspect appear. Experiment shows that the blackened dot, the spot where the electron has hit the screen, cannot be determined in advance. (The same is true for photons or any other particle.) However, for large numbers of electrons, the spatial distribution of the black dots, the so-called *diffraction pattern*, can be calculated in advance with high precision.

The outcome of experiments on microscopic systems thus forces us to use probabilities for the description of microsystems. We find that the probability distribution $p(\boldsymbol{x})$ of the spots on the photographic plate can be calculated from the wave function ψ of

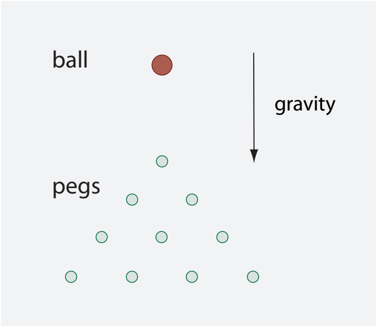


FIGURE 67 A system showing probabilistic behaviour.

the electron at the screen surface and is given by $p(\boldsymbol{x}) = |\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|^2$. This is in fact a special case of the general *first property of quantum measurements*: the measurement of an observable A for a system in a state ψ gives as result one of the eigenvalues a_n, and the probability P_n to get the result a_n is given by

$$P_n = |\varphi_n^\dagger \psi|^2 , \tag{74}$$

where φ_n is the eigenfunction of the operator A corresponding to the eigenvalue a_n.* This experimental result requires an explanation.

Experiments also show a *second property of quantum measurements:* after the measurement, the observed quantum system is in the state φ_n corresponding to the measured eigenvalue a_n. One also says that during the measurement, the wave function has *collapsed* from ψ to φ_n. By the way, these experimental results can also be generalized to the more general cases with degenerate and continuous eigenvalues.

Ref. 106

At first sight, the sort of probabilities encountered in quantum theory are different from the probabilities we encounter in everyday life. Take roulette, dice, pachinko machines or the direction in which a pencil on its tip falls: all have been measured experimentally to be random (assuming no cheating by the designer or operators) to a high degree of accuracy. These everyday systems do not puzzle us. We unconsciously assume that the random outcome is due to the small, but uncontrollable variations of the starting conditions or the environment every time the experiment is repeated.**

But microscopic systems seem to be different. The two properties of quantum measurements just mentioned express what physicists observe in every experiment, even if the initial conditions are taken to be *exactly* the same every time. But why then is the

* All linear transformations transform some special vectors, called *eigenvectors* (from the German word *eigen* meaning 'self') into multiples of themselves. In other words, if T is a transformation, e a vector, and

$$T(e) = \lambda e \tag{75}$$

where λ is a scalar, then the vector e is called an *eigenvector* of T, and λ is associated *eigenvalue*. The set of all eigenvalues of a transformation T is called the *spectrum* of T.

** To get a feeling for the limitations of these unconscious assumptions, you may want to read the already mentioned story of those physicists who built a machine that could predict the outcome of a roulette ball from the initial velocity imparted by the croupier.

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position for a single electron, or most other observables of quantum systems, not predictable? In other words, what happens during the collapse of the wave function? How long does it take? In the beginning of quantum theory, there was the perception that the observed unpredictability is due to the lack of information about the state of the particle. This lead many to search for so-called 'hidden variables'. All these attempts were doomed to fail, however. It took some time for the scientific community to realize that the unpredictability is *not* due to the lack of information about the state of the particle, which is indeed described *completely* by the state vector ψ.

In order to uncover the origin of probabilities, let us recall the nature of a measurement, or better, of a general observation. *Any observation is the production of a record.* The record can be a visual or auditive memory in our brain, or a written record on paper, or a tape recording, or any such type of object. As explained in the previous volume, an object is a record if it cannot have arisen or disappeared by chance. To avoid the influence of chance, all records have to be protected as much as possible from the *external* world; e.g. one typically puts archives in earthquake safe buildings with fire protection, keeps documents in a safe, avoids brain injury as much as possible, etc.

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On top of this, records have to be protected from their *internal* fluctuations. These internal fluctuations are due to the many components any recording device is made of. If the fluctuations were too large, they would make it impossible to distinguish between the possible contents of a memory. Now, fluctuations decrease with increasing size of a system, typically with the square root of the size. For example, if a hand writing is too small, it is difficult to read if the paper gets brittle; if the magnetic tracks on tapes are too small, they demagnetize and lose the stored information. In other words, a record is rendered stable against internal fluctuations by making it of sufficient size. Every record thus consists of many components and shows small fluctuations.

The importance of size can be expressed in another way: every system with memory, i.e., every system capable of producing a record, contains a *bath*. In summary, the statement that any observation is the production of a record can be expressed more precisely as: *Any observation of a system is the result of an interaction between that system and a bath in the recording apparatus.**

In addition, any observation measuring a physical quantity uses an interaction *depending* on that same quantity. With these seemingly trivial remarks, we can describe in more detail the process of observation, or, as it is usually called in the quantum theory, the measurement process.

Any measurement apparatus, or *detector*, is characterized by two main aspects, shown in Figure 68: the interaction it has with the microscopic system, and the bath it contains to produce the record. Any description of the measurement process thus is the description of the evolution of the microscopic system *and* the detector; therefore one needs the Hamiltonian for the particle, the interaction Hamiltonian, and the bath properties (such as the relaxation time t_r). The interaction specifies what is measured and the bath realizes the memory.

Ref. 107

We know that only classical thermodynamic systems can be irreversible; quantum systems are not. We therefore conclude: a measurement system *must* be described classically: otherwise it would have no memory and would not be a measurement system: it

* Since baths imply friction, we can also say: memory needs friction.

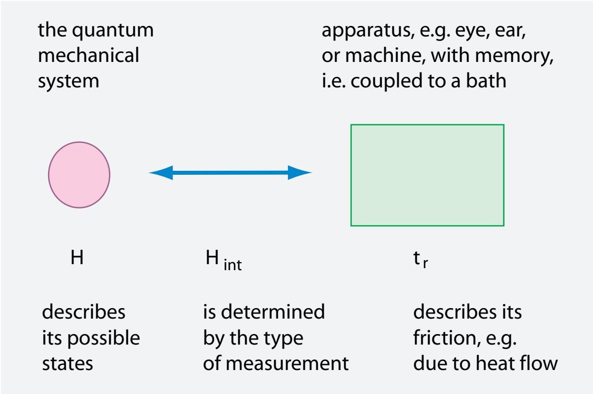


FIGURE 68 The concepts used in the description of measurements.

would not produce a record! Memory is a classical effect. (More precisely, memory is an effect that only appears in the classical limit.) Nevertheless, let us see what happens if we describe the measurement system quantum mechanically.

Let us call A the observable which is measured in the experiment and its eigenfunctions φ_n. We describe the quantum mechanical system under observation – often a particle – by a state ψ. The full state of the system can always be written as

$$\psi = \psi_p \, \psi_{\text{other}} = \sum_n c_n \varphi_n \, \psi_{\text{other}} \; . \tag{76}$$

Here, ψ_p is the aspect of the (particle or system) state that we want to measure, and ψ_{other} represents all other degrees of freedom, i.e., those not described – *spanned*, in mathematical language – by the operator A corresponding to the observable we want to measure. The numbers $c_n = |\varphi_n^\dagger \psi_p|$ give the expansion of the state ψ_p, which is taken to be normalized, in terms of the basis φ_n. For example, in a typical position measurement, the functions φ_n would be the position eigenfunctions and ψ_{other} would contain the information about the momentum, the spin and all other properties of the particle.

How does the system–detector interaction look like? Let us call the state of the apparatus before the measurement χ_{start}. The measurement apparatus itself, by definition, is a device which, when it is hit by a particle in the state $\varphi_n \psi_{\text{other}}$, changes from the state χ_{start} to the state χ_n. One then says that the apparatus has *measured* the eigenvalue a_n corresponding to the eigenfunction φ_n of the operator A. The index n is thus the record of the measurement; it is called the *pointer* index or variable. This index tells us in which state the microscopic system was before the interaction. The important point, taken from our previous discussion, is that the states χ_n, being records, are macroscopically distinct, precisely in the sense of the previous section. Otherwise they would not be records, and the interaction with the detector would not be a measurement.

Of course, during measurement, the apparatus sensitive to φ_n changes the part ψ_{other} of the particle state to some other situation $\psi_{\text{other},n}$, which depends on the measurement and on the apparatus; we do not need to specify it in the following discussion.* But let us have an intermediate check of our reasoning. Do apparatuses as described here exist?

* How does the interaction look like mathematically? From the description we just gave, we specified the

Yes, they do. For example, any photographic plate is a detector for the position of ionizing particles. A plate, and in general any apparatus measuring position, does this by changing its momentum in a way depending on the measured position: the electron on a photographic plate is stopped. In this case, χ_{start} is a white plate, φ_n would be a particle localized at spot n, χ_n is the function describing a plate blackened at spot n and $\psi_{\text{other,n}}$ describes the momentum and spin of the particle after it has hit the photographic plate at the spot n.

Now we are ready to look at the measurement process itself. For the moment, let us disregard the bath in the detector, and let us just describe it with a state as well, which we call χ_{start}. In the time before the interaction between the particle and the detector, the combined system (including the detector) was in the initial state ψ_i given simply by

$$\psi_i = \psi_p \chi_{\text{start}} = \sum_n c_n \varphi_n \psi_{\text{other}} \chi_{\text{start}} , \qquad (79)$$

where ψ_p is the (particle or system) state. After the interaction, using the just mentioned, experimentally known characteristics of the apparatus, the combined state ψ_a is

$$\psi_a = \sum_n c_n \varphi_n \psi_{\text{other},n} \chi_n . \qquad (80)$$

This evolution from ψ_i to ψ_a follows from the evolution equation applied to the particle–detector combination. Now, the combined state ψ_a is a superposition of macroscopically distinct states: it is a superposition of distinct macroscopic states of the detector. In our example ψ_a could correspond to a superposition of one state where a spot on the left upper corner is blackened on an otherwise white plate with another state where a spot on the right lower corner of the otherwise white plate is blackened. Such a situation is never observed. Let us see why.

The density matrix ρ_a of the combined state ψ_a after the measurement, given by

$$\rho_a = \psi_a \otimes \psi_a^\dagger = \sum_{n,m} c_n c_m^* (\varphi_n \psi_{\text{other},n} \chi_n) \otimes (\varphi_m \psi_{\text{other},m} \chi_m)^\dagger , \qquad (81)$$

contains large non-diagonal terms, i.e., terms for $n \neq m$, whose numerical coefficients are different from zero. Now let us take the bath back in. From the previous section we know the effect of a bath on such a macroscopic superposition. We found that a density matrix such as ρ_a decoheres extremely rapidly. We assume here that the decoherence

final state for every initial state. Since the two density matrices are related by

$$\rho_{\text{f}} = T \rho_{\text{i}} T^\dagger \qquad (77)$$

Challenge 122 ny
we can deduce the Hamiltonian from the matrix T. Are you able to see how?

By the way, one can say in general that an apparatus measuring an observable A has a system interaction Hamiltonian depending on the pointer variable A, and for which one has

$$[H + H_{\text{int}}, A] = 0 . \qquad (78)$$

time is negligibly small, in practice thus instantaneous,* so that the off-diagonal terms vanish, and only the final, diagonal density matrix ρ_{f}, given by

$$\rho_{\mathrm{f}} = \sum_n |c_n|^2 (\varphi_n \psi_{\mathrm{other},n} \chi_n) \otimes (\varphi_n \psi_{\mathrm{other},n} \chi_n)^{\dagger} \qquad (82)$$

has experimental relevance. As explained above, such a density matrix describes a mixed state and the numbers $P_n = |c_n|^2 = |\varphi_n^{\dagger} \psi_p|^2$ give the probability of measuring the value a_n and of finding the particle in the state $\varphi_n \psi_{\mathrm{other,n}}$ as well as the detector in the state χ_n. But this is precisely what the two properties of quantum measurements state.

We therefore find that describing a measurement as an evolution of a quantum system interacting with a macroscopic detector, itself containing a bath, we can *deduce* the two properties of quantum measurements, and thus the collapse of the wave function, from the quantum mechanical evolution equation. The decoherence time t_{d} of the previous section becomes the time of collapse in the case of a measurement; in addition we find

$$t_{\mathrm{collapse}} = t_{\mathrm{d}} < t_{\mathrm{r}} \; . \qquad (83)$$

In other words, the collapse time is always smaller than the relaxation time of the bath. We thus have a formula for the time the wave function takes to collapse. The first experimental measurements of the time of collapse have appeared and confirmed this result. Ref. 108

Hidden variables

A large number of people are not satisfied with the arguments just presented. They long for more mystery in quantum theory. They do not like the idea that probabilities are due to baths. The most famous prejudice they cultivate is the idea that the probabilities are due to some hidden aspect of nature which is still unknown to humans. These imagined, unknown aspects are called *hidden variables*.

The beautiful thing about quantum mechanics is that it allows both conceptual and experimental tests on whether such hidden variables exist without the need of knowing them. Hidden variables controlling the evolution of microscopic system would contradict the result that action values below $\hbar/2$ cannot be detected. This minimum observable action is the reason for the random behaviour of microscopic systems. A minimal action thus excludes hidden variables.

Historically, the first argument against hidden variables was given by John von Neumann.** An additional no-go theorem for hidden variables was published by Kochen and Specker in 1967, (and independently by Bell in 1969). The theorem states that non- Ref. 109

* Note however, that an *exactly* vanishing decoherence time, which would mean a *strictly* infinite number of degrees of freedom of the environment, is in contradiction with the evolution equation, and in particular with unitarity, locality and causality. It is essential in the whole argument not to confuse the logical consequences of a extremely small decoherence time with those of an exactly vanishing decoherence time.

** János von Neumann (b. 1903 Budapest, d. 1957 Washington DC) Hungarian mathematician. One of the greatest and clearest minds of the twentieth century, he settled already many questions, especially in applied mathematics and quantum theory, that others still struggle with today. He worked on the atomic and the hydrogen bomb, on ballistic missiles, and on general defence problems. In another famous project, he build the first US-American computer, building on his extension of the ideas of Konrad Zuse.

contextual hidden variables are impossible, if the Hilbert space has a dimension equal or larger than three. The theorem is about non-contextual variables, i.e., about hidden variables *inside* the quantum mechanical system. The Kochen–Specker theorem thus states that there is no non-contextual hidden variables model, because mathematics forbids it. This result essentially eliminates all possibilities for hidden variables, because usual quantum mechanical systems have Hilbert space dimensions much larger than three.

Of course, one cannot avoid noting that about *contextual* hidden variables, i.e., variables in the environment, there are no restricting theorems; indeed, their necessity was shown earlier in this section.

But also common sense eliminates hidden variables, without any recourse to mathematics, with an argument often overlooked. If a quantum mechanical system had internal hidden variables, the measurement apparatus would have zillions of them.* And that would mean that it could not work as a measurement system.

Despite all these results, people have also looked for experimental tests on hidden variables. Most tests are based on the famed *Bell's equation*, a beautifully simple relation published by John Bell** in the 1960s.

The starting idea is to distinguish quantum theory and locally realistic theories using hidden variables by measuring the polarizations of two correlated photons. Quantum theory says that the polarization of the photons is fixed only at the time it is measured, whereas local realistic theories say that it is fixed already in advance. Experiment can be used to decide which approach is correct.

Imagine the polarization is measured at two distant points A and B, each observer can measure 1 or -1 in each of his favourite direction. Let each observer choose two directions, 1 and 2, and call their results a_1, a_2, b_1 and b_2. Since the measurement results all are either 1 or -1, the value of the specific expression $(a_1 + a_2)b_1 + (a_2 - a_1)b_2$ has always the value ± 2.

Ref. 110 Imagine you repeat the experiment many times, assuming that the hidden variables appear statistically. You then can deduce (a special case of) Bell's inequality for two hidden variables; it predicts that

Challenge 123 e

$$|(a_1 b_1) + (a_2 b_1) + (a_2 b_2) - (a_1 b_2)| \leqslant 2 \,. \tag{84}$$

Here, the expressions in brackets are the averages of the measurement products over a large number of samples. This result holds independently of the directions of the involved polarizers.

On the other hand, for the case that the polarizers 1 and 2 at position A and the corresponding ones at position B are chosen with angles of $\pi/4$, quantum theory predicts that

$$|(a_1 b_1) + (a_2 b_1) + (a_2 b_2) - (a_1 b_2)| = 2\sqrt{2} > 2 \,. \tag{85}$$

This prediction is in complete contradiction with the hidden variable result.

All experimental checks of Bell's equation have confirmed standard quantum mechan-

* Which leads to the definition: one zillion is 10^{23}.

** John Stewart Bell (1928–1990), theoretical physicist who worked mainly on the foundations of quantum theory.

ics. No evidence for hidden variables has ever been found. This is not really surprising, since the search for such variables is based on a misunderstanding of quantum mechanics or on personal desires on how the world should be, instead of relying on experimental evidence.

Another measurable contradiction between quantum theory and locally realistic theories has been predicted by Greenberger, Horn and Zeilinger in systems with three en-
Ref. 111 tangles particles. The various predictions have been confirmed in all experiments.

Summary on probabilities and determinism

> “Geometric demonstramus quia facimus; si physics demonstrare possemus, faceremus.”
>
> Giambattista Vico*

From the arguments presented here we draw a number of conclusions which we need for the rest of our mountain ascent. Note that these conclusions are not yet shared by all physicists! The whole topic is still touchy.

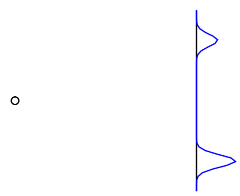
— Probabilities do not appear in measurements because the state of the quantum system is unknown or fuzzy, but because the detailed state of the bath in the environment is unknown. *Quantum mechanical probabilities are of statistical origin and are due to baths in the environment (or in the measurement apparatus).* The probabilities are due to the large number of degrees of freedom contained in any bath. These large numbers make the outcome of experiments unpredictable. If the state of the bath were known, the outcome of an experiment could be predicted. The probabilities of quantum theory are ‘thermodynamic’ in origin.

  In other words, there are *no* fundamental probabilities in nature. All probabilities in nature are due to decoherence; in particular, all probabilities are due to the statistics of the many particles – some of which may be virtual – that are part of the baths in the environment. Modifying well-known words by Albert Einstein, ‘nature really does not play dice.’ We therefore called ψ the *wave function* instead of ‘probability amplitude’, as is often done. An even better name would be *state function*.

— Any observation in everyday life is a special case of decoherence. What is usually called the ‘collapse of the wave function’ is a decoherence process due to the interaction with the baths present in the environment or in the measuring apparatus. Because humans are warm-blooded and have memory, humans themselves are thus measurement apparatuses. The fact that our body temperature is 37°C is thus the reason that we see only a single world, and no superpositions. (Actually, there are
Challenge 124 s many additional reasons; can you name a few?)

— A measurement is complete when the microscopic system has interacted with the bath in the measuring apparatus. Quantum theory as a description of nature does not require detectors; the evolution equation describes all examples of motion. However, *measurements* do require the existence of detectors. Detectors, being machines that

* ‘We are able to demonstrate geometrical matters because we make them; if we could prove physical matters we would be able to make them.’ Giovanni Battista Vico (b. 1668 Napoli, d. 1744 Napoli) important Italian philosopher and thinker. In this famous statement he points out a fundamental distinction between mathematics and physics.



record observations, have to include a bath, i.e., have to be classical, macroscopic objects. In this context one speaks also of a *classical apparatus*. This necessity of the measurement apparatus to be classical had been already stressed in the very early stages of quantum theory.

— All measurements, being decoherence processes that involve interactions with baths, are irreversible processes and increase entropy.

— A measurement is a special case of quantum mechanical evolution, namely the evolution for the combination of a quantum system, a macroscopic detector and the environment. Since the evolution equation is relativistically invariant, no causality problems appear in measurements; neither do locality problems and logical problems appear.

— Since both the evolution equation and the measurement process does not involve quantities other than space-time, Hamiltonians, baths and wave-functions, no other quantity plays a role in measurement. In particular, no human observer nor any consciousness are involved or necessary. Every measurement is complete when the microscopic system has interacted with the bath in the apparatus. The decoherence inherent in every measurement takes place even if nobody is looking. This trivial consequence is in agreement with the observations of everyday life, for example with the fact that the Moon is orbiting the Earth even if nobody looks at it.* Similarly, a tree falling in the middle of a forest makes noise even if nobody listens. Decoherence is independent of human observation, of the human mind and of human existence.

— In every measurement the quantum system interacts with the detector. Since there is a minimum value for the magnitude of action, every *observation influences the observed*. Therefore every measurement *disturbs* the quantum system. Any precise description of observations must also include the description of this disturbance. In the present section the disturbance was modelled by the change of the state of the system from ψ_{other} to $\psi_{\text{other,n}}$. Without such a change of state, without a disturbance of the quantum system, a measurement is impossible.

— Since the complete measurement is described by quantum mechanics, unitarity is and remains the basic property of evolution. There are no non-unitary processes in quantum mechanics.

— The description of the collapse of the wave function as a decoherence process is an explanation exactly in the sense in which the term 'explanation' was defined earlier
Vol. III, page 240
on; it describes the relation between an observation and all the other aspects of reality, in this case the bath in the detector or the environment. The collapse of the wave function has been both calculated and *explained*. The collapse is not a question of 'interpretation', i.e., of opinion, as unfortunately often is suggested.**

— It is not useful to speculate whether the evolution for a *single* quantum measurement could be determined if the state of the environment around the system were known. Measurements need baths. But a bath, being irreversible, cannot be described by a wave function, which behaves reversibly.*** Quantum mechanics is deterministic.

* The opposite view is sometimes falsely attributed to Niels Bohr. The Moon is obviously in contact with
Challenge 125 s
many radiation baths. Can you list a few?

** This implies that the so-called 'many worlds' interpretation is wishful thinking. The conclusion is con-
Ref. 112
firmed when studying the details of this religious approach. It is a belief system, not based on facts.

*** This very strong type of determinism will be very much challenged in the last part of this text, in which

Baths are probabilistic.

In summary, there is *no* irrationality in quantum theory. Whoever uses quantum theory as argument for superstitions, irrational behaviour, new age beliefs or ideologies is guilty of disinformation. The statement by Gell-Mann at the beginning of this chapter is thus such an example. Another is the following well-known, but incorrect statement by Richard Feynman:

Ref. 113

> ... nobody understands quantum mechanics.

Nobel Prizes obviously do not prevent views distorted by ideology.

The process of decoherence allows understanding many other issues. We explore a few interesting ones.

What is the difference between space and time?

Space and time differ. Objects are localized in space but not in time. Why is this the case? In nature, most bath–system interactions are mediated by a potential. All potentials are by definition position dependent. Therefore, every potential, being a function of the position $\boldsymbol{x}$, commutes with the position observable (and thus with the interaction Hamiltonian). The decoherence induced by baths – except if special care is taken – thus first of all destroys the non-diagonal elements for every superposition of states centred at different locations. In short, *objects are localized because they interact with baths via potentials.*

For the same reason, objects also have only one spatial orientation at a time. If the system–bath interaction is spin-dependent, the bath leads to 'localization' in the spin variable. This occurs for all microscopic systems interacting with magnets. As a result, macroscopic superpositions of magnetization are almost never observed. Since electrons, protons and neutrons have a magnetic moment and a spin, this conclusion can even be extended: everyday objects are never seen in superpositions of different rotation states because their interactions with baths are spin-dependent.

As a counter-example, most systems are not localized in time, but on the contrary exist for very long times, because practically all system–bath interactions do *not* commute with time. In fact, this is the way a bath is defined to begin with. In short, *objects are permanent because they interact with baths.*

Are you able to find an interaction which is momentum-dependent instead of position-dependent? What is the consequence for macroscopic systems?

Challenge 126 s

In other words, in contrast to general relativity, quantum theory produces a distinction between space and time. In fact, we can *define* position as the observable that commutes with interaction Hamiltonians. This distinction between space and time is due to the properties of matter and its interactions. We could not have deduced this distinction in general relativity.

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it will be shown that time is not a fundamental concept, and therefore that the debate around determinism looses most of its interest.

Are we good observers?

Are humans classical apparatuses? Yes, they are. Even though several prominent physicists claim that free will and probabilities are related, a detailed investigation shows that Ref. 114 this in not the case. Our senses are classical machines in the sense described above: they record observations by interaction with a bath. Our brain is also a classical apparatus: the neurons are embedded in baths. Quantum probabilities do not play a determining role in the brain.

Any observing entity needs a bath and a memory to record its observations. This means that observers have to be made of matter; an observer cannot be made of radiation. Our description of nature is thus severely biased: we describe it from the standpoint of matter. That is a bit like describing the stars by putting the Earth at the centre of the universe. Can we eliminate this basic anthropomorphism? We will find out as we Vol. VI, page 75 continue.

What relates information theory, cryptology and quantum theory?

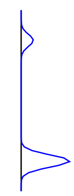
Physics means talking about observations of nature. Like any observation, also measurements produce information. It is thus possible to translate much (but not all) of quantum theory into the language of information theory. In particular, the existence of a minimal change in nature implies that the information about a physical system can never be complete, that information transport has its limits and that information can never be fully trusted. The details of these studies form a fascinating way to look at the microscopic world.

The analogy between quantum theory and information theory becomes even more Ref. 115 interesting when the statements are translated into the language of cryptology. Cryptology is the science of transmitting hidden messages that only the intended receiver can decrypt. In our modern times of constant surveillance, cryptology is an important tool to protect personal freedom.*

The quantum of action implies that messages can be sent in an (almost) safe way. Listening to a message is a measurement process. Since there is a smallest action, one can detect whether somebody has tried to listen to a sent message. A man in the middle attack – somebody who pretends to be the receiver and then sends a copy of the message to the real, intended receiver – can be avoided by using *entangled* systems as signals to transmit the information. Quantum cryptologists therefore usually use communication systems based on entangled photons.

The major issue of *quantum cryptology*, a large modern research field, is the key distribution problem. All secure communication is based on a secret key that is used to decrypt the message. Even if the communication channel is of the highest security – like entangled photons – one still has to find a way to send the communication partner the secret key necessary for the decryption of the messages. Finding such methods is the main

* Cryptology consists of the field of *cryptography*, the art of coding messages, and the field of *cryptoanalysis*, the art of deciphering encrypted messages. For a good introduction to cryptology, see the text by Albrecht Beutelspacher, Jörg Schwenk & Klaus-Dieter Wolfenstätter, *Moderne Verfahren der Kryptographie*, Vieweg 1995.



aspect of quantum cryptology. However, close investigation shows that all key exchange methods are limited in their security.

In short, due to the quantum of action, nature provides limits on the possibility of sending encrypted messages. The statement of these limits is (almost) equivalent to the statement that change in nature is limited by the quantum of action.

Is the universe a computer?

The quantum of action provides a limit to secure information exchange. This connection also allows us to brush aside several incorrect statements often found in the media. Stating that 'the universe is information' or that 'the universe is a computer' is as reasonable Vol. VI, page 99 as saying that the universe is an observation or a chewing-gum dispenser. Any expert of motion should beware of these and similarly fishy statements; people who use them either deceive themselves or try to deceive others.

Does the universe have a wave function? And initial conditions?

The wave function of the universe is frequently invoked in discussions about quantum theory. Many deduce conclusions from this idea, for example on the irreversibility of time, on the importance of initial conditions, on changes required to quantum theory and much more. Are these arguments correct?

Vol. II, page 202 The first thing to clarify is the meaning of 'universe'. As explained already, the term can have two meanings: either the collection of all matter and radiation, or this collection *plus* all of space-time. Let us also recall the meaning of 'wave function': it describes the *state* of a system. The state distinguishes two otherwise identical systems; for example, position and velocity distinguish two otherwise identical ivory balls on a billiard table. Alternatively and equivalently, the state describes changes in time.

Does the universe have a state? If we take the wider meaning of universe, it does not. Vol. I, page 25 Talking about the state of the universe is a contradiction: by definition, the concept of state, defined as the non-permanent aspects of an object, is applicable only to *parts* of the universe.

We then can take the narrower sense of 'universe' – the sum of all matter and radiation only – and ask the question again. To determine the state of all matter and radiation, we need a possibility to measure it: we need an environment. But the environment of matter and radiation is space-time only; initial conditions cannot be determined since we need measurements to do this, and thus an apparatus. An apparatus is a material system with a bath attached to it; however, there is no such system outside the universe.

In short, quantum theory does not allow for measurements of the universe; therefore the universe has no state. Beware of anybody who claims to know something about the wave function of the universe. Just ask him: If you know the wave function of the universe, why aren't you rich?

Despite this conclusion, several famous physicists have proposed evolution equations Ref. 116 for the wave function of the universe. (The best-known is the Wheeler–DeWitt equation.) It seems a silly point, but the predictions of these equations have not been compared to experiments; the arguments just given even make this impossible in principle. Exploring this direction, so interesting it may seem, must therefore be avoided if we want to reach the top of Motion Mountain.

There are many additional twists to this story. One twist is that space-time itself, even without matter, might be a bath. This speculation will be shown to be correct later on; this result seems to allow speaking of the wave function of all matter. But then again, it turns out that time is *undefined* at the scales where space-time is an effective bath; this implies that the concept of state is not applicable there.

A lack of 'state' for the universe is a strong statement. It also implies a lack of initial conditions! The arguments are precisely the same. This is a tough result. We are so used to think that the universe has initial conditions that we never question the term. (Even in this text the mistake might appear every now and then.) But there are no initial conditions of the universe.

We can retain as summary, valid even in the light of the latest research: the universe has *no* wave function and *no* initial conditions, independently of what is meant by 'universe'.

CHAPTER 8

COLOURS AND OTHER INTERACTIONS BETWEEN LIGHT AND MATTER

> Rem tene; verba sequentur.*
>
> Cato

STONES have colours. Why? We know how matter and radiation move. The next step is to describe the interaction between them. In other words, what is the specific way in which charged quantum particles react to electromagnetic fields, and vice versa? In this chapter, we first give an overview of the ways that colours in nature result from the quantum of action, i.e., from the interaction between matter quantons and photons. Then we explore the simplest such system: we show how the quantum of action leads to the colours of hydrogen atoms. After this, we discover that the interaction between matter and radiation leads to other surprising effects, especially when special relativity is taken into account.

THE CAUSES OF COLOUR

Quantum theory explains all colours in nature. Indeed, all the colours that we observe are due to charged particles. More precisely, colours are due to the interactions of charged particles with photons. All colours are thus quantum effects.

The charged particles at the basis of most colours are electrons and nuclei, including their composites, from ions, atoms and molecules to fluids and solids. An overview of the specific mechanisms that generate colour is given in the following table. The table includes all colours that appear in everyday life. (Can you find one that is missing?) Some colour issues are still topic of research. For example, until recently it was unclear why exactly asphalt is black. The exact structure of the chemical compounds, the *asphaltenes*, that produce the very dark brown colour was unknown. Only recent research has settled this question.

Ref. 117

Challenge 127 s

Ref. 118

* 'Know the subject and the words will follow.' Marcus Porcius Cato, (234–149 BCE) or Cato the elder, Roman politician famous for his speeches and his integrity.

TABLE 7 Causes of colour.

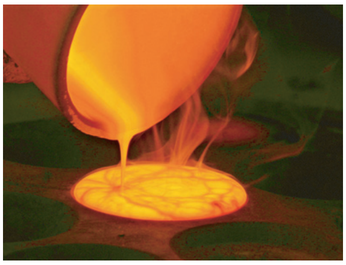
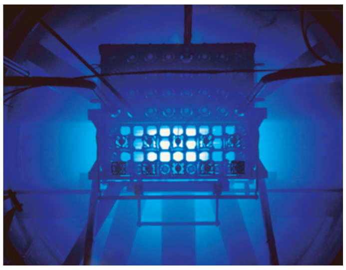
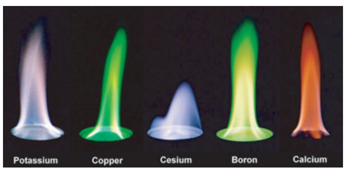
Colour type	Example	Details
Class I: Colours due to simple excitations		
	**1. Incandescence and free charge radiation**	
	Carbon arc lamp, hot steel, lightbulb wire, most stars, magma, lava, hot melts	Colours are due to continuous spectrum emitted by all hot matter; colour sequence, given by Wien's rule, is black, red, orange, yellow, white, blue-white (molten lead and silver © Graela)
	Wood fire, candle	Wood and wax flames are yellow due to incandescence if carbon-rich and oxygen-poor
	White fireworks, flashlamp, sparklers	Due to metals burning to oxide at high temperature, such as magnesium, zinc, iron, aluminium or zirconium (sparkler © Sarah Domingos)
	Nuclear reactors, synchroton light sources, free electron lasers	Due to fast free charges: Vavilov–Čerenkov radiation is due to speed of particle larger than the speed of light in matter, Bremsstrahlung is due to the deceleration of charged particles (nuclear reactor core under water, image from NASA)
	**2. Atomic gas excitations**	
	Red neon lamp, blue argon lamp, UV mercury lamp, yellow sodium street lamps, most gas lasers, metal vapour lasers, some fluorescence	Colours are due to transitions between atomic energy levels (gas discharges © Pslawinski)
	Aurora, triboluminescence in scotch tape, crystalloluminescence in strontium bromate	In air, blue and red colours are due to atomic and molecular energy levels of nitrogen, whereas green, yellow, orange colours are due to oxygen (aurora © Jan Curtis)
	Lightning, arcs, sparks, coloured fireworks, most coloured flames, some electroluminescence	Colour lines are due to energy levels of highly excited atoms (flames of K, Cu, Cs, B, Ca © Philip Evans)

TABLE 7 Causes of colour (continued).

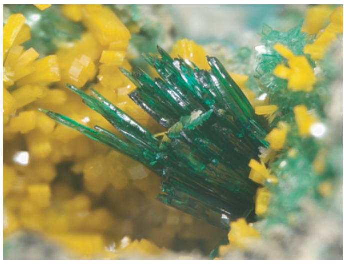
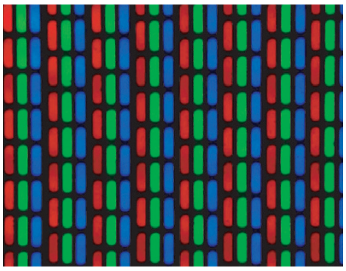
Colour type	Example	Details
	3. Vibrations and rotations of molecules	
	Bluish water, blue ice when clear, violet iodine, red-brown bromine, yellow-green chlorine, red flames from CN or blue-green flames from CH, some gas lasers	Colours are due to quantized levels of rotation and vibrations in molecules (blue iceberg © Marc Shandro)
Class II: Colours due to ligand field effects		
	4. Transition metal compounds	
 	Green malachite $Cu_2CO_3(OH)_2$, blue cobalt oxide, blue azurite $Cu_3(CO_3)_2(OH)_2$, red to brown hematite Fe_2O_3, green MnO, white $Mn(OH)_2$, brown manganite, chrome green Cr_2O_3, green praesodymium, pink europium and yellow samarium compounds, piezochromic and thermochromic $Cr_2O_3 - Al_2O_3$ UV and electron phosphors, scintillation, some fluorescence, some lasers	Colours are due to electronic states of the ions; phosphors are used in cathodes tubes for TV/computer displays and on fluorescent lamp tubes (green malachite on yellow kasolite, a uranium mineral, picture width 5 mm, found in Kolwezi, Zaire/Congo, © Stephan Wolfsried, television shadow mask photo © Planemad)
	5. Transition metal impurities	
	Ruby, emerald, alexandrite, perovskites, corresponding lasers	Electronic states of transition metal ions are excited by light and thus absorb specific wavelengths (ruby on calcite from Mogok, Myanmar, picture width 3 cm, © Rob Lavinsky)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
Class III: Colours due to molecular orbitals		
   	**6. Organic compounds** Red haemoglobin in blood, green chlorophyll in plants, yellow or orange carotenes in carrots, flowers and yellow autumn leaves, red or purple anthocyanins in berries, flowers and red autumn leaves, blue indigo, red lycopene in tomatoes, red meat from iron-containing myoglobin, brown glucosamine in crust of baked food, brown tannins, black eumelanin in human skin, hair and eye, iron-rich variation pheomelanin in redheads, black melanin also in cut apples and bananas as well as in movable sacks in chameleons, brown-black asphalt, some fluorescence, chemiluminescence, phosphorescence, halochromism, electrochromism and thermochromism, dye lasers	Colours are due to conjugated π-bonds, i.e. to alternating single and double bonds in molecules; floral pigments are almost all anthocyanins, betalains or carotenes; used in colourants for foods and cosmetics, in textile dyes, in electrochromic displays, in inks for colour printers, in photosensitizers (narcissus © Thomas Lüthi, blood on finger © Ian Humes, berries © Nathan Wall, hair courtesy dusdin)
	Glow-worms, some bacteria and funghi, most deep-sea fish, octopi, jellyfish, and other deep-sea animals	Bioluminescence is due to excited molecules, generally called *luciferines* (angler fish, length 4.5 cm, © Steve Haddock)

TABLE 7 Causes of colour (continued).

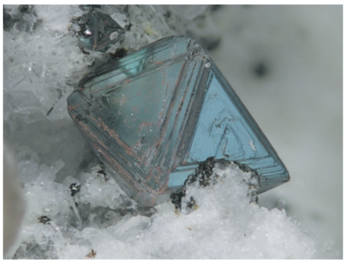
Colour type	Example	Details
 	**7. Inorganic charge transfer** Blue sapphire, blue lapis lazuli, green amazonite, brown-black magnetite Fe_3O_4 and most other iron minerals (colouring basalt black, beer bottles brown, quartz sand yellow, and many other rocks with brown or red tones), black graphite, purple permanganate, orange potassium dichromate, yellow molybdates, red hematite Fe_2O_3, some fluorescence	Light induces change of position of an electron from one atom to another; for example, in blue sapphire the transition is between Ti and Fe impurities; many paint pigments use charge transfer colours; fluorescent analytical reagents are used in molecular medicine and biology (magnetite found in Laach, Germany, picture width 10 mm, © Stephan Wolfsried, Evelien Willemsen)
Class IV: Colours due to energy band effects		
	**8. Metallic bands** Gold (green in transmission), pyrite, iron, brass, alloys, silver, copper, ruby glass	Colours in reflection and in transmission are due to transitions of electrons between overlapping bands (saxophone © Selmer)
	**9. Pure semiconductor bands** Silicon, GaAs, black galena PbS, red cinnabar HgS, cadmium yellow CdS, black CdSe, red CdS_xSe_{1-x}, white ZnO, orange vermillion HgS, colourless diamond, black to gold piezochromic SmS	Colours are due to electron transitions between separate bands; colour series is black, red, orange, yellow, white/colourless; some used as pigments (zinc oxide courtesy Walkerma)
	**10. Doped semiconductor bands** Blue, yellow, green and black diamond; LEDs; semiconductor lasers; solar cells; ZnS and $Zn_xCd_{1-x}S$ based and other phosphors	Colours are due to transitions between dopants and semiconductor bands (e.g. blue diamond: boron accepters, black diamond: nitrogen donors) (quantum dots © Andrey Rogach)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
	11. Colour centres	
	Amethyst, smoky quartz, fluorite, green diamonds, blue, yellow and brown topaz, brown salt, purple colour of irradiated glass containing Mn^{2+}, lyoluminescence, some fluorescence, F-centre lasers	Colours are due to colour centres, i.e. to electrons or to holes bound at crystal vacancies; colour centres are usually are created by radiation (amethyst © Rob Lavinsky)
	Some light-dependent sunglasses	The photochromic colouring is due to colour centres formed by the UV light of the Sun
Class V: Colours due to physical and geometrical optics		
	12. Dispersive refraction and polarization	
	Cut diamond, cut zirconia, halos and sun dogs formed by ice crystals in the air	Spectral decomposition (sparkle or 'fire' of gemstones) is due to dispersion in crystals (zirconia photo © Gregory Phillips)
	Rainbow	Colours of primary and secondary bow are due to dispersion in water droplets
	Green flash	dispersion in the atmosphere shifts the sun colours
	13. Scattering	
 	Blue sky, blue colouring of distant mountains, red sunset; colour intensification by pollution; blue quartz	Blue light is scattered more than red light by Rayleigh scattering, when scatterers (molecules, dust) are smaller than the wavelength of light (Tokyo sunset © Altus Plunkett, blue quartz © David Lynch)

TABLE 7 Causes of colour (continued).

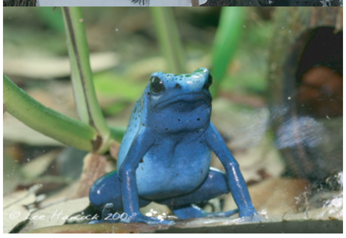
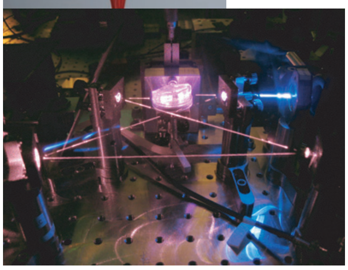
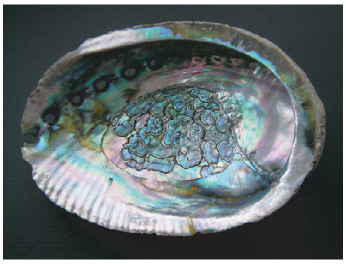
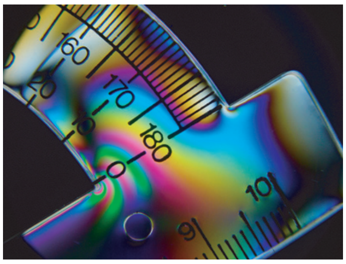
Colour type	Example	Details
	White colour of hair, milk, beer foam, clouds, fog, cigarette smoke coming out of lungs, snow, whipped cream, shampoo, stars in gemstones	The white colour is due to wavelength-independent Mie scattering, i.e. scattering at particles larger than the wavelength of light (snow man © Andreas Kostner)
	Blue human skin colour in cold weather, blue and green eyes in humans, blue monkey skin, blue turkey necks, most blue fish, blue reptiles, blue cigarette smoke	Tyndall blue colours are due to scattering on small particles in front of a dark background (blue poison frog *Dendrobates azureus* © Lee Hancock)
	Ruby glass	The red colour of Murano glass is due to scattering by tiny colloidal gold particles included in the glass in combination with the metallic band structure of gold (ruby glass © murano-glass-shop.it)
	Nonlinearities, Raman effect, potassium dihydrogen phosphate (KDP)	Frequency-shifting scattering, second harmonic generation and other nonlinearities of certain materials change the colour of light impinging with high intensities (800 nm to 400 nm frequency doubling ring laser © Jeff Sherman)
	14. Interference (without diffraction)	
	Nacre, oil films, soap bubbles, coatings on camera lenses, eyes of cats in the dark, wings of flies and dragonflies, fish scales, some snakes, pearls	Thin film interference produces a standard colour sequence that allows precise thickness determination (abalone shell © Anne Elliot)
	Polarization colours of thin layers of birefringent crystals or thicker layers of stressed polymers	Colours are due to interference, as shown by the dependence on layer thickness (photoelasticity courtesy Nevit Dilmen)

TABLE 7 Causes of colour (continued).

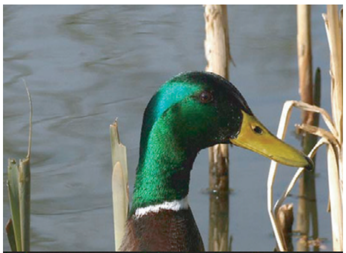
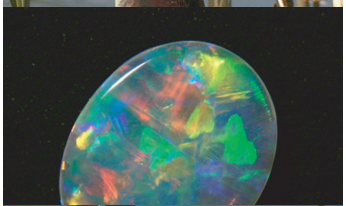
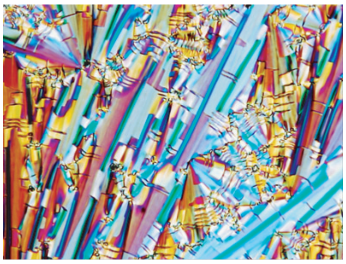
Colour type	Example	Details
	Supernumerary rainbows (see page 88)	Due to interference, as shown by the dependence on drop size
	Iridescent beetles, butterflies and bird feathers, iridescent colours on banknotes and on cars	Due to scattering at small structures or at nanoparticles, as shown by the angular dependence of the colour (mallard duck © Simon Griffith)
	15. Diffraction (with interference)	
	Opal	Colours are due to the tiny spheres included in the water inside the opal; colours can change if the opal dries out (polished Brazilian opal © Opalsnopals)
	Aureole, glory, corona	Colours are due to diffraction at the tiny mist droplets (aeroplane condensation cloud iridescence © Franz Kerschbaum)
	Diffraction gratings, CDs, vinyl records, some beetles and snakes	Colours are due to diffraction and interference at tiny, regular pits (CD illuminated by flashlamp © Alfons Reichert)
	photonic crystals	A modern research topic
	Cholesteric liquid crystals, certain beetles	Colours are due to diffraction and interference in internal material layers (liquid crystal colours © Ingo Dierking)
Class VI: Colours due to eye limitations		
Fechner colours, as on lite.bu.edu/vision/applets/Color/Benham/Benham.html	Benham's top	Colours are due to different speed response of different photoreceptors

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
Internal colour production when eyes are stimulated	Phosphenes	Occur through pressure (rubbing, sneeze), or with electric or magnetic fields
Polarization colours	Haidinger's brush	See page 90
Colour illusions, as on www.psy.ritsumei.ac.jp/~akitaoka/color9e.html	Appearing and disappearing colours	Effects are due to combinations of brain processing and eye limitations
False colour output, as described on page 140	Red light can be seen as green	Observable with adaptive optics, if red light is focused on a green-sensitive cone
Colour-blind or 'daltonic' person, see page 144, with reduced colour spectrum	Protan, deutan or tritan	Each type limits colour perception in a different way

Colours fascinate. Fascination always also means business; indeed, a large part of the chemical industry is dedicated to synthesizing colourants for paints, clothes, food and cosmetics. Also evolution uses the fascination of colours for its own business: propagating life. The specialists in this domain are the flowering plants. The chemistry of colour production in plants is extremely involved and at least as interesting as the production of colours in factories. Practically all flower colourants, from white, yellow, orange, red to blue, are from three chemical classes: the carotenoids, the anthocyanins (flavonoids) and the betalains. These colourants are stored in petals inside dedicated containers, the
Ref. 119 *vacuoles*. There are many good review articles giving more details.

Even though colours are common in plants and animals, most higher animals do not produce many colourants themselves. For example, humans produce only one colourant: *melanin*. (*Hemoglobin*, which colours blood red, is not a dedicated colourant, but transports the oxygen from the lungs through the body. Also the pink *myoglobin* in the muscles is not a dedicated colourant.) Many higher animals, such as birds, need to *eat* the colourants that are so characteristic for their appearance. The yellow colour of legs of pigeons is an example. It has been shown that the connection between colour and nutrition is regularly used by potential mates to judge from the body colours whether a proposing
Ref. 120 partner is sufficiently healthy, and thus sufficiently sexy.

In summary, an exploration of the causes of colours found in nature confirms that all colours are due to quantum effects. We therefore explore the simplest coloured systems found in nature: atomic gases.

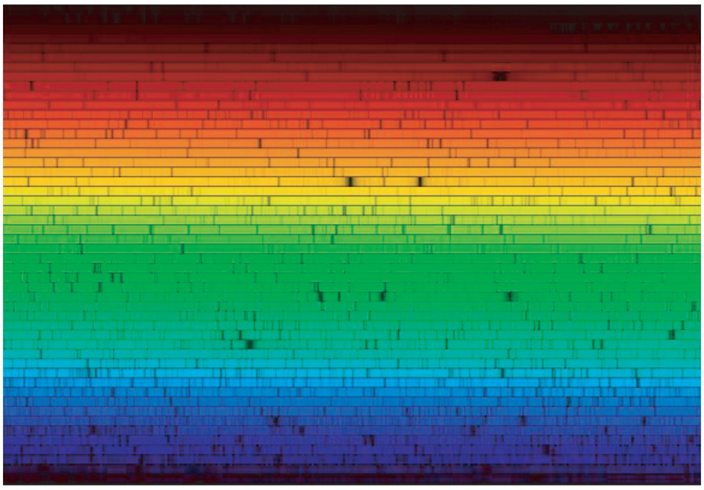


FIGURE 69 The spectrum of daylight: a *stacked* image of an extended rainbow, showing its Fraunhofer lines (© Nigel Sharp, NOAO, FTS, NSO, KPNO, AURA, NSF).

Using the rainbow to determine what stars are made of

Near the beginning of the eighteenth century, Bavarian instrument-maker Joseph Fraunhofer* and the English physicist William Wollaston noted that the rainbow *lacks* certain colours. These colours appear as black lines when the rainbow is spread out in sufficient breadth. Figure 69 shows the lines in detail; they are called *Fraunhofer lines* today. In 1860, Gustav Kirchhoff and Robert Bunsen showed that the colours *missing* in the rainbow were exactly those colours that certain elements *emit* when heated. In this way they managed to show that sodium, calcium, barium, nickel, magnesium, zinc, copper and iron are present in the Sun. Looking at the rainbow thus tells us what the Sun is made of.

Of the 476 Fraunhofer lines that Kirchhoff and Bunsen observed, 13 did not correspond to any known element. In 1868, Jules Janssen and Joseph Lockyer independently predicted that these unknown lines were from a new element. It was eventually found on Earth, in an uranium mineral called cleveite, in 1895. This new element was called *helium*, from the Greek word ἥλιος 'helios' – Sun. Today we know that it is the second most widespread ingredient of the Sun and of the universe, after hydrogen. Despite being so common, helium is rare on Earth because it is a light noble gas that does not form chemical compounds. Helium thus tends to rise in the atmosphere and escape into space.

Understanding the colour lines produced by each element had started to become interesting already before the discovery of helium; but afterwards the interest increased

* Joseph Fraunhofer (b. 1787 Straubing, d. 1826 München). Bavarian. Having been orphaned at the age of 11, he learned lens-polishing. He taught himself optics from books. He entered an optical company at the age of 19, ensuring the success of the business by producing the best available lenses, telescopes, micrometers, optical gratings and optical systems of his time. He invented the spectroscope and the heliometer. He discovered and counted 476 lines in the spectrum of the Sun; these lines are now named after him. (Today, Fraunhofer lines are still used as measurement standards: the second and the metre are defined in terms of them.) Physicists from all over the world would buy their equipment from him, visit him, and ask for copies of his publications. Even after his death, his instruments remained unsurpassed for generations. With his telescopes, in 1837 Bessel was able to make the first measurement of parallax of a star, and in 1846 Johann Gottfried Galle discovered Neptune. Fraunhofer became a professor in 1819. He died young, from the consequences of the years spent working with lead and glass powder.

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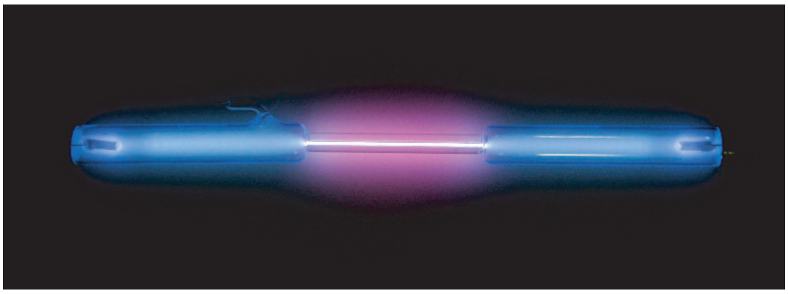

FIGURE 70 A low-pressure hydrogen discharge in a 20 cm long glass tube (© Jürgen Bauer at www.smart-elements.com).

further, thanks to the increasing number of applications of colour knowledge in chemistry, physics, technology, crystallography, biology and lasers. Colours are big business, as the fashion industry shows.

Colours are specific light frequencies. Light is an electromagnetic wave and is emitted by moving charges. For a physicist, colours thus result from the interaction of charged matter with the electromagnetic field. However, *sharp* colour lines cannot be explained by classical electrodynamics. Indeed, only quantum theory can explain them – or any other colour. In fact, every colour in nature is formed with the help of $\hbar$, the quantum of action.

What determines the colours of atoms?

The simplest colours to study are the sharp colour lines emitted by single atoms. Single atoms are mainly found in gases. The simplest atom to study is that of hydrogen. Hot hydrogen gas, shown in Figure 70, emits light consisting of a handful of sharp spectral lines, as shown on the left of Figure 71. Already in 1885, the Swiss schoolteacher Johann Balmer (1828–1898) had discovered that the wavelengths of visible hydrogen lines obey the formula:

$$\frac{1}{\lambda_m} = R\left(\frac{1}{4} - \frac{1}{m^2}\right) . \tag{86}$$

Careful measurements, which included the hydrogen's spectral lines in the infrared and in the ultraviolet, allowed Johannes Rydberg (1854–1919) to generalize this formula to:

$$\frac{1}{\lambda_{mn}} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right) , \tag{87}$$

where n and $m > n$ are positive integers, and the so-called *Rydberg constant* R has the value $10.97\,\mu\mathrm{m}^{-1}$; easier to remember, the inverse value is $1/R = 91.16\,\mathrm{nm}$. All the colour lines emitted by hydrogen satisfy this simple formula. Classical physics cannot explain this result at all. Thus, quantum theory has a clearly defined challenge here: to explain the formula and the value of R.

Incidentally, the transition λ_{21} for hydrogen is called the *Lyman-alpha line*. Its wavelength, 121.6 nm, lies in the ultraviolet. It is easily observed with telescopes, since most of the visible stars consist of excited hydrogen. The Lyman-alpha line is routinely used to determine the speed of distant stars or galaxies, since the *Doppler effect* changes the

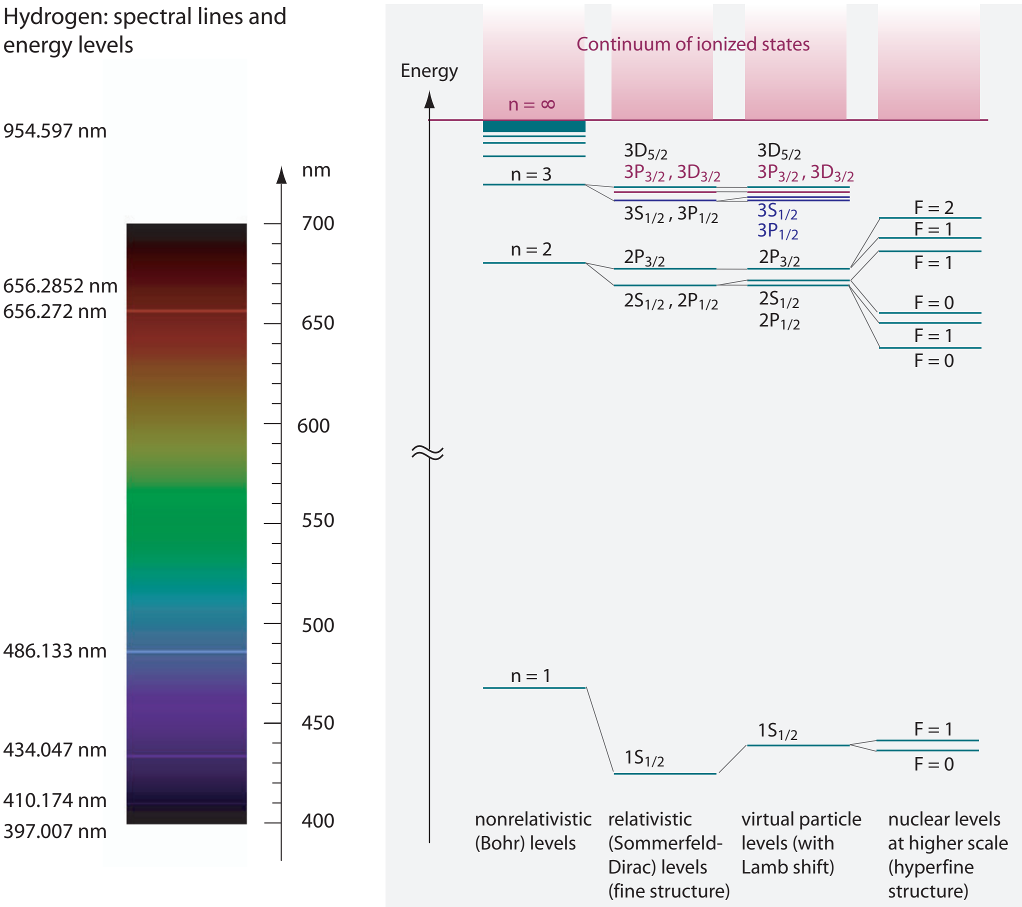

FIGURE 71 Atomic hydrogen: the visible spectrum of hydrogen (NASA) and its calculated energy levels, in four approximations of increasing precision. Can you associate the visible lines to the correct level transitions?

Ref. 121 wavelength when the speed is large. The record so far is a galaxy found in 2004, with a Lyman-alpha line shifted to 1337 nm. Can you calculate the speed with which it is moving away from the Earth? Challenge 128 ny

There are many ways to deduce Balmer's formula from the minimum action. The first way was found by Niels Bohr. Then, in 1926, Erwin Schrödinger solved his equation of motion for an electron moving in the electrostatic potential $V(r) = e^2/4\pi\varepsilon_0 r$ of a point-like proton. By doing so, Schrödinger deduced Balmer's formula and became famous in the world of physics. However, this important calculation is long and complex. In order to understand hydrogen colours, it is not necessary to solve an equation of motion; it is sufficient to compare the energies of the initial and final states. This can be done most easily by noting that a specific form of the action must be a multiple of $\hbar/2$. This approach was developed by Einstein, Brillouin and Keller, and is now named after them. It relies

Ref. 122 on the fact that the action S of any quantum system obeys

$$S = \frac{1}{2\pi} \oint \mathrm{d}q_i p_i = \left(n_i + \frac{\mu_i}{4}\right)\hbar \tag{88}$$

for every coordinate q_i and its conjugate momentum p_i. Here, n_i can be zero or any positive integer, and μ_i is the so-called *Maslov index*, an even integer, which in the case of atoms has the value 2 for the radial and azimuthal coordinates r and θ, and 0 for the rotation angle φ. The integral is to be taken along a full orbit. In simple words, *the action S is a half-integer multiple of the quantum of action.* This result can be used to calculate the energy levels.

Any rotational motion in a spherical potential $V(r)$ is characterized by a constant energy E and constant angular momenta L and L_z. Therefore the conjugate momenta for the coordinates r, θ and φ are Challenge 129 ny

$$\begin{aligned} p_r &= \sqrt{2m(E - V(r)) - \frac{L^2}{r^2}} \\ p_\theta &= \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}} \\ p_\varphi &= L_z \,. \end{aligned} \tag{89}$$

Using these expressions in equation (88) and setting $n = n_r + n_\theta + n_\varphi + 1$, we get* the result

$$E_n = -\frac{1}{n^2}\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} = -\frac{R}{n^2} \approx -\frac{2.19\,\mathrm{aJ}}{n^2} \approx -\frac{13.6\,\mathrm{eV}}{n^2} \,. \tag{92}$$

These *energy levels E_n* are shown in Figure 71.

Using the idea that a hydrogen atom emits a single photon when its electron changes from state E_n to E_m, we get exactly the formula deduced by Balmer and Rydberg from observations! The match between observation and calculation is about four digits. For the first time ever, a material property, the colour of hydrogen atoms, had been explained from a fundamental principle of nature. Key to this explanation was the quantum of action. (This whole discussion assumes that the electrons in hydrogen atoms that emit light are in eigenstates. Can you argue why this is the case?) Challenge 131 e Challenge 132 ny

In short, the quantum of action implies that only certain specific energy values for

* The calculation is straightforward. After insertion of $V(r) = e/4\pi\varepsilon_0 r$ into equation (89) one needs to perform the (tricky) integration. Using the general result Challenge 130 ny

$$\frac{1}{2\pi}\oint \frac{\mathrm{d}z}{z}\sqrt{Az^2 + 2Bz - C} = -\sqrt{C} + \frac{B}{\sqrt{-A}} \tag{90}$$

one gets

$$\left(n_r + \frac{1}{2}\right)\hbar + L = n\hbar = \frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{m}{-2E}} \,. \tag{91}$$

This leads to the energy formula (92).

an electron are allowed inside an atom. The lowest energy level, for $n = 1$, is called the *ground state*. Its energy value 2.19 aJ is the *ionization energy* of hydrogen; if that energy is added to the ground state, the electron is no longer bound to the nucleus. The ionization energy thus plays the same role for electrons around atoms as does the escape velocity for satellites around planets.

The calculation also yields the effective radius of the electron orbit in hydrogen. It is given by

$$r_n = n^2 \frac{\hbar^2 4\pi\varepsilon_0}{m_e e^2} = \frac{\hbar}{m_e c\alpha} = n^2 a_0 \approx n^2\, 52.918\,937\,\mathrm{pm}\,. \tag{93}$$

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In contrast to classical physics, quantum theory allows only certain specific orbits around the nucleus. (For more details about the fine structure constant α, see below.) The smallest value, 53 pm for $n = 1$, is called the *Bohr radius*, and is denoted by a_0. To be more precise, these radii are the average sizes of the electron clouds surrounding the nucleus. Quantum theory thus implies that a hydrogen atom excited to the level $n = 500$ is about 12 μm in size: larger than many bacteria! Such blown-up atoms, usually called *Rydberg atoms*, have indeed been observed in the laboratory, although they are extremely sensitive to perturbations.

Ref. 123

The orbital frequency of electrons in hydrogen is

$$f_n = \frac{1}{n^3} \frac{e^4 m_e}{4\varepsilon_0^2 h^3} = \frac{1}{n^3} \frac{m_e c^2 \alpha^2}{h} \approx \frac{6.7\,\mathrm{PHz}}{n^3} \tag{94}$$

and the electron speed is

$$v_n = \frac{1}{n} \frac{e^2}{4\pi\varepsilon_0 \hbar} = \frac{\alpha c}{n} \approx \frac{2.2\,\mathrm{Mm/s}}{n} \approx \frac{0.007\ c}{n}\,. \tag{95}$$

As expected, the further the electron's orbit is from the nucleus, the more slowly it moves. This result can also be checked by experiment: exchanging the electron for a muon allows us to measure the time dilation of its lifetime. Measurements are in full agreement with the calculations.

Ref. 124

In 1915, Arnold Sommerfeld understood that the analogy of electron motion with orbital motion could be continued in two ways. First of all, electrons can move in ellipses. The quantization of angular momentum implies that only selected eccentricities are possible. The higher the angular momentum, the larger the number of possibilities: the first are shown in Figure 72. The highest eccentricity corresponds to the minimum value $l = 0$ of the so-called azimuthal quantum number, whereas the case $l = n - 1$ correspond to circular orbits. In addition, the ellipses can have different orientations in space.

Ref. 125

The second point Sommerfeld noted was that the speeds of the electron in hydrogen are slightly relativistic. The above calculation did not take into account relativistic effects. However, high-precision measurements show slight differences between the non-relativistically calculated energy levels and the measured ones.

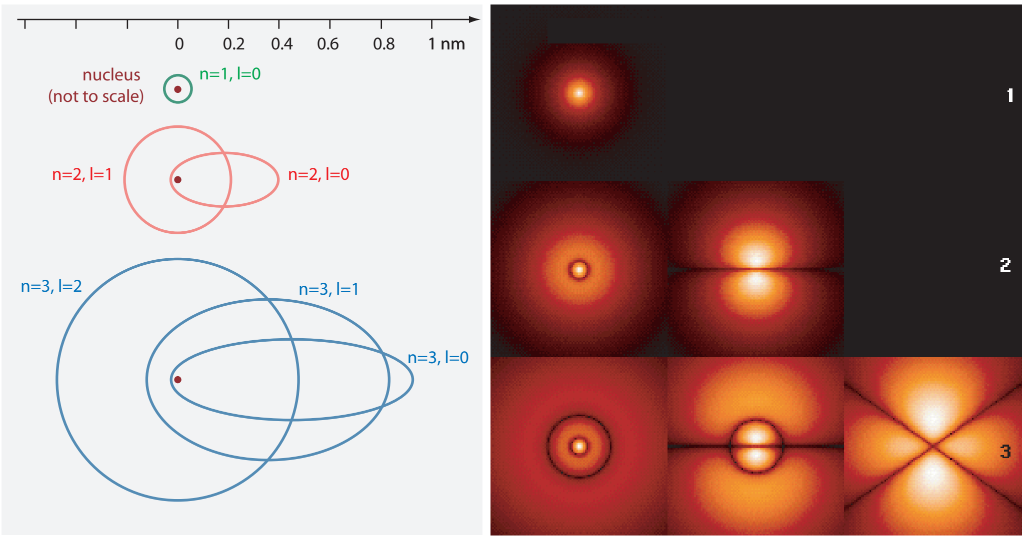

FIGURE 72 The imagined, but not existing and thus false electron orbits of the Bohr–Sommerfeld model of the hydrogen atom (left) and the correct description, using the probability density of the electron in the various states (right) (© Wikimedia).

Relativistic hydrogen

In the relativistic case, too, the EBK action has to be a multiple of $\hbar/2$. From the relativistic expression for the kinetic energy of the electron

Ref. 122

$$E + mc^2 = \sqrt{p^2 c^2 + m^2 c^4} - \frac{e^2}{4\pi\varepsilon_0 r} \tag{96}$$

Challenge 133 ny

we get the expression

$$p_r^2 = 2mE\left(1 + \frac{E}{2mc^2}\right) + \frac{2me^2}{4\pi\varepsilon_0 r}\left(1 + \frac{E}{mc^2}\right) . \tag{97}$$

We now introduce, for convenience, the so-called *fine structure constant*, as $\alpha = e^2/(4\pi\varepsilon_0 \hbar c) = \sqrt{4\pi\hbar R/mc} \approx 1/137.036$. ($\alpha$ is a dimensionless constant; $R = 10.97\,\mu\text{m}^{-1}$ is the Rydberg constant.) The radial EBK action then implies that

Challenge 134 ny

$$E_{nl} + mc^2 = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{\left(n - l - \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - \alpha^2}\right)^2}}} . \tag{98}$$

This result, first found by Arnold Sommerfeld in 1915, is correct for point-like electrons. In reality, the electron has spin 1/2; the correct relativistic energy levels thus appear when

FIGURE 73 Paul Dirac (1902–1984)

we set $l = j \pm 1/2$ in the above formula. The result can be approximated by

$$E_{nj} = -\frac{R}{n^2}\left(1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + ...\right) . \tag{99}$$

It reproduces the hydrogen spectrum to an extremely high accuracy. If we compare the result with the non-relativistic one, we note that each non-relativistic level n is split in n different levels. This is shown in Figure 71. In practice, in precision experiments the lines of the hydrogen spectrum have a so-called *fine structure*. The magnitude of the fine structure depends on α, a fundamental constant of nature. Since the importance of this fundamental constant was discovered in this context, the name chosen by Arnold Sommerfeld, the *fine structure constant*, has been taken over across the world. The fine structure constant describes the *strength* of the electromagnetic interaction.

Of course, it is possible to do even better. The introduction of virtual-particle effects and the coupling of the proton spin gives additional corrections, as shown in Figure 71. But that is still not all, isotope effects, Doppler shifts and level shifts to environmental electric or magnetic fields also influence the hydrogen spectrum.

Relativistic wave equations – again

> “The equation was more intelligent than I was.”
> Paul Dirac, about his equation, repeating a statement made by Heinrich Hertz.

What is the evolution equation for the wave function in the case that relativity, spin and interactions with the electromagnetic field are taken into account? We could try to gener- Page 88 alize the representation of relativistic motion given by Foldy and Wouthuysen to the case of particles with electromagnetic interactions. Unfortunately, this is not a simple matter. The simple identity between the classical and quantum-mechanical descriptions is lost if electromagnetism is included.

Charged quantum particles are best described by another, equivalent representation of the Hamiltonian, which was discovered much earlier, in 1926, by the British physicist Paul Dirac.* Dirac found a neat trick to take the square root appearing in the relativistic

* Paul Adrien Maurice Dirac (b. 1902 Bristol, d. 1984 Tallahassee), British physicist, son of a Francophone Swiss immigrant. He studied electrotechnics in Bristol, then went to Cambridge, where he later became a

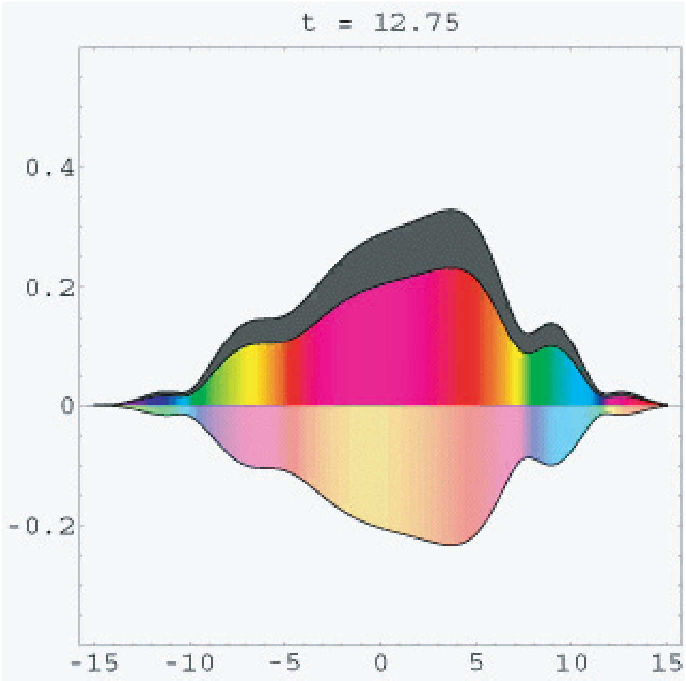

FIGURE 74 The famous Zitterbewegung: the superposition of positive and negative energy states leads to an oscillation around a mean vale. Colour indicates phase; two coloured curves are shown, as the Dirac equation in one dimension has only two components (not four); the grey curve is the probability density. (QuickTime film © Bernd Thaller).

energy operator. In Dirac's representation, the Hamilton operator is given by

$$H_{\text{Dirac}} = \beta m + \boldsymbol{\alpha} \cdot p \ . \tag{100}$$

The quantities β and the three components $(\alpha_1, \alpha_2, \alpha_3) = \boldsymbol{\alpha}$ turn out to be complex 4×4 matrices.

In Dirac's representation, the position operator x is not the position of a particle, but has additional terms; its velocity operator has only the eigenvalues plus or minus the velocity of light; the velocity operator is not simply related to the momentum operator; the equation of motion contains the famous 'Zitterbewegung' term; orbital angular momentum and spin are not separate constants of motion.

So why use this horrible Hamiltonian? Because only the Dirac Hamiltonian can easily be used for charged particles. Indeed, it is transformed to the Hamiltonian coupled to the

professor, holding the chair that Newton had once held. In the years from 1925 to 1933 he published a stream of papers, of which several were worth a Nobel Prize; he received this in 1933. He unified special relativity and quantum theory, predicted antimatter, worked on spin and statistics, predicted magnetic monopoles, speculated on the law of large numbers, and more besides. His introversion, friendliness and shyness, and his deep insights into nature, combined with a dedication to beauty in theoretical physics, made him a legend all over the world during his lifetime. For the latter half of his life he tried, unsuccessfully, to find an alternative to quantum electrodynamics, of which he was the founder, as he was repelled by the problems of infinities. He died in Florida, where he lived and worked after his retirement from Cambridge.

Vol. III, page 71 electromagnetic field by the so-called *minimal coupling*, i.e., by the substitution

$$\boldsymbol{p} \rightarrow \boldsymbol{p} - q\boldsymbol{A} , \tag{101}$$

that treats electromagnetic momentum like particle momentum. With this prescription, Dirac's Hamiltonian describes the motion of charged particles interacting with an electromagnetic field $\boldsymbol{A}$. The minimal coupling substitution is not possible in the Foldy–Wouthuysen Hamiltonian. In the Dirac representation, particles are pure, point-like, structureless electric charges; in the Foldy–Wouthuysen representation they acquire a charge radius and a magnetic-moment interaction. (We will come to the reasons below, in the section on QED.) Ref. 126

In more detail, the simplest description of an electron (or any other elementary, stable, electrically-charged particle of spin 1/2) is given by the action S and Lagrangian

$$\begin{aligned} S &= \int \mathcal{L}_{\text{QED}} \, d^4x \quad \text{where} \\ \mathcal{L}_{\text{QED}} &= \overline{\psi}\,(i\hbar c \not{D} - mc^2)\,\psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \quad \text{and where} \\ \not{D}_\mu &= \gamma^\mu (\partial_\mu - ieA_\mu) \end{aligned} \tag{102}$$

The first, matter term in the Lagrangian leads to the Dirac equation: it describes how elementary, charged, spin 1/2 particles are moved by electromagnetic fields. The second, radiation term leads to Maxwell's equations, and describes how electromagnetic fields are moved by the charged particle wave function. Together with a few calculating tricks, these equations describe what is usually called *quantum electrodynamics*, or QED for short.

As far as is known today, the relativistic description of the motion of charged matter and electromagnetic fields given the QED Lagrangian (102) is *perfect*: no differences between theory and experiment have ever been found, despite intensive searches and despite a high reward for anybody who would find one. All known predictions completely correspond with the measurements. In the most spectacular cases, the correspondence between theory and measurement extends to more than thirteen digits. But even more interesting than the precision of QED are certain of its features that are missing in classical electrodynamics. Let's have a quick tour.

Getting a feeling for the Dirac equation

The QED Lagrangian implies that the wave function of a charged particle in a potential follows the Dirac equation:

$$i\hbar c \gamma^\mu (\partial_\mu - ieA_\mu)\psi = mc\psi . \tag{103}$$

The many indices should not make us forget that this equation simply states that the eigenvalue of the energy–momentum operator is the rest mass (times the speed of light c). In other words, the equation states that the wave ψ moves with a phase velocity c.

The wave function ψ has four complex components. Two describe the motion of particles, and two the motion of antiparticles. Each type of particle needs two complex components, because the equation describes spin and particle density. Spin is a rotation, and a rotation requires three real parameters. Spin and density thus require four real parameters; they can be combined into two complex numbers, both for particles and for antiparticles.

Challenge 135 e Each of the four components of the wave function of a relativistic spinning particle follows the relativistic Schrödinger–Klein–Gordon equation. This means that the relativistic energy–momentum relation is followed by each component separately.

Challenge 136 e The relativistic wave function ψ has the important property that a rotation by 2π changes its sign. Only a rotation by 4π leaves the wave function unchanged. This is the typical behaviour of spin 1/2 particles. For this reason, the four-component wave function of a spin 1/2 particle is called a *spinor*.

Antimatter

'Antimatter' is now a household term. Interestingly, the concept appeared *before* there was any experimental evidence for it. The relativistic expression for the energy E of an electron with charge e in the field of a charge Q is

$$E + \frac{Qe}{4\pi\varepsilon_0 r} = \sqrt{m^2 c^4 + p^2 c^2} \ . \tag{104}$$

This expression also allows solutions with *negative* energy and opposite charge $-e$. Quantum theory shows that this is a general property, and these solutions correspond to what is called *antimatter*.

Indeed, the antimatter companion of the electron was predicted in the 1920s by Paul Dirac from his equation. Unaware of this prediction, Carl Anderson discovered the anti-electron in 1932, and called it the *positron*. (The correct name would have been 'positon', without the 'r'. This correct form is used in the French language.) Anderson was studying cosmic rays, and noticed that some 'electrons' were turning the wrong way in the magnetic field he had applied to his apparatus. He checked his apparatus thoroughly, and finally deduced that he had found a particle with the same mass as the electron but with positive electric charge.

The existence of positrons has many strange implications. Already in 1928, before their discovery, the Swedish theorist Oskar Klein had pointed out that Dirac's equation for electrons makes a strange prediction: when an electron hits a sufficiently steep potential wall, the reflection coefficient is larger than unity. Such a wall will reflect *more* than is thrown at it. In addition, a large part of the wave function is transmitted through the wall. In 1935, after the discovery of the positron, Werner Heisenberg and Hans Euler Ref. 127 explained the paradox. They found that the Dirac equation predicts that whenever an electric field exceeds the critical value of

$$E_{\mathrm{c}} = \frac{m_{\mathrm{e}} c^2}{e\lambda_{\mathrm{e}}} = \frac{m_{\mathrm{e}}^2 c^3}{e\hbar} = 1.3\,\mathrm{EV/m} \ , \tag{105}$$

the vacuum will spontaneously generate electron–positron pairs, which are then sepa-

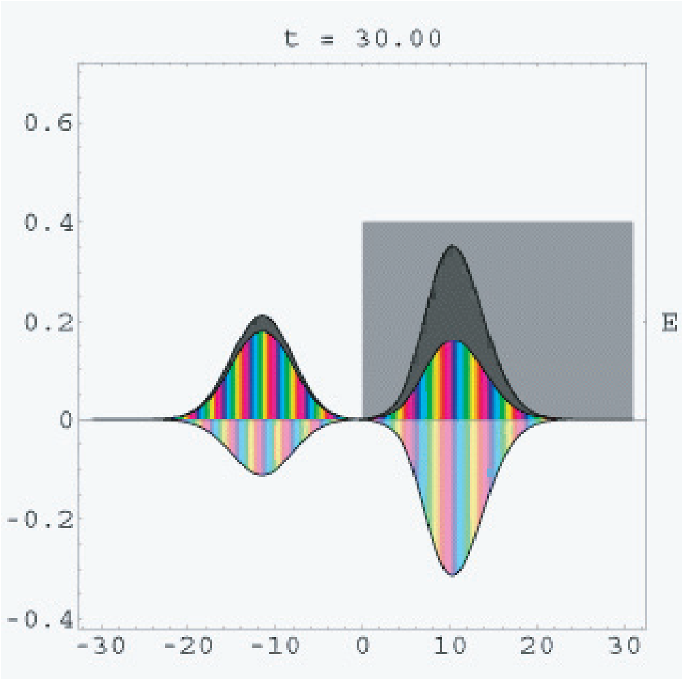

FIGURE 75 Klein's paradox: the motion of a relativistic wave function that encounters a very steep potential. Part of the wave function is transmitted; this part is antimatter, as the larger lower component shows. (QuickTime film © Bernd Thaller).

rated by the field. As a result, the original field is reduced. This so-called *vacuum polarization* is the reason for the reflection coefficient greater than unity found by Klein. Indeed, steep potentials correspond to high electric fields.

Vacuum polarization is a weak effect. It has been only observed in collisions of high energy, where it the effectively increases the fine structure constant. Later on we will describe truly gigantic examples of vacuum polarization that are postulated around charged black holes.

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Vacuum polarization shows that, in contrast to everyday life, the *number* of particles is not a constant in the microscopic domain. Only the *difference* between particle number and antiparticle number turns out to be conserved. Vacuum polarization thus limits our possibility to count particles in nature!

Of course, the generation of electron–positron pairs is not a *creation* out of nothing, but a *transformation* of energy into matter. Such processes are part of every relativistic description of nature. Unfortunately, physicists have a habit of calling this transformation 'pair creation', thus confusing the issue somewhat.

Virtual particles

Despite what was said so far, actions smaller than the minimal action do have a role to play. We have already encountered one example: in a collision between two electrons, there is an exchange of virtual photons. We learned that the exchanged virtual photon

Page 53

cannot be observed. Indeed, the action S for this exchange obeys

$$S \leqslant \hbar . \tag{106}$$

In short, *virtual* particles appear only as mediators in interactions. They cannot be observed. Virtual particles, in contrast to ordinary, real particles, do not obey the relation $E^2 - p^2c^2 = m^2c^4$. For example, the kinetic energy can be negative. Indeed, virtual particles are the opposite of 'free' or real particles. They may be observed in a vacuum if the measurement time is very short. They are intrinsically short-lived.

Virtual photons are the cause for electrostatic potentials, for magnetic fields, for the Casimir effect, for spontaneous emission, for the van der Waals force, and for the Lamb shift in atoms. A more detailed treatment shows that in every situation with virtual photons there are also, with even lower probability, virtual electrons and virtual positrons.

Massive virtual particles are essential for vacuum polarization, for the limit in the number of the elements, for black-hole radiation and for Unruh radiation. Massive virtual particles also play a role in the strong interaction, where they hold the nucleons together in nuclei, and in weak nuclear interaction, where they explain why beta decay happens and why the Sun shines.

In particular, virtual particle–antiparticles pairs of matter and virtual radiation particles together form what we call the *vacuum*. In addition, virtual radiation particles form what are usually called static fields. Virtual particles are needed for a full description of all interactions. In particular, virtual particles are responsible for every decay process.

Curiosities and fun challenges about colour

Where is the sea bluest? Sea water is blue because it absorbs red and green light. Sea water can also be of bright colour if the sea floor reflects light. In addition, sea water can be green, when it contains small particles that scatter or absorb blue light. Most often, these particles are soil or plankton. (Satellites can determine plankton content from the 'greenness' of the sea.) Thus the sea is especially blue if it is deep, quiet and cold; in that case, the ground is distant, soil is not mixed into the water, and the plankton content is low. The Sargasso Sea is deep, quiet and cold for most of the year. It is often called the bluest of the Earth's waters.

* *

If atoms contain orbiting electrons, the rotation of the Earth, via the Coriolis acceleration, should have an effect on their motion. This beautiful prediction is due to Mark Silverman; the effect is so small, however, that is has not yet been measured. Ref. 124

* *

Light is diffracted by material gratings of light. Can matter be diffracted by light gratings? Surprisingly, it actually can, as predicted by Dirac and Kapitza in 1937. This was accomplished for the first time in 1986, using atoms. For free electrons, the feat is more difficult; the clearest confirmation came in 2001, when new laser technology was used to perform a beautiful measurement of the typical diffraction maxima for electrons diffracted by a light grating. Ref. 128

* *

Light is totally reflected when it is directed to a dense material at a large enough angle so that it cannot enter the material. A group of Russian physicists have shown that if the dense material is excited, the intensity of the totally-reflected beam can be *amplified*. It is unclear whether this will ever lead to applications. Ref. 124

* *

The ways people handle *single* atoms with electromagnetic fields provide many beautiful examples of modern applied technologies. Nowadays it is possible to levitate, to trap, to excite, to photograph, to deexcite and to move single atoms just by shining light onto them. In 1997, the Nobel Prize in Physics has been awarded to the originators of the field, Steven Chu, Claude Cohen-Tannoudji and William Philips. Vol. I, page 273 Ref. 130

* *

An example of modern research is the study of *hollow* atoms, i.e., atoms missing a number of inner electrons. They have been discovered in 1990 by J.P. Briand and his group. They appear when a completely ionized atom, i.e., one without any electrons, is brought in contact with a metal. The acquired electrons then orbit on the outside, leaving the inner shells empty, in stark contrast with usual atoms. Such hollow atoms can also be formed by intense laser irradiation. Ref. 131

* *

Ref. 132 Given two mirrors and a few photons, it is possible to capture an atom and keep it floating between the two mirrors. This feat, one of several ways to isolate single atoms, is now standard practice in laboratories. Can you imagine how it is done? Challenge 137 ny

Material properties

The quantum of action determines the colours of the hydrogen atoms. In the same way, it determines the colours of all the other atoms whose Fraunhofer lines are observed in the infrared, visible and ultraviolet. In fact, also the colour of solids and liquids are determined by the quantum of action.

The quantum of action also determines all other material properties. The elasticity, the plasticity, the brittleness, the magnetic and electric properties of materials are equally fixed by the quantum of action. Many details of this general statement are, however, still a subject of research. Many material properties are not completely understood, though none is in contradiction with the quantum of action. Material research is among the most important fields of modern science, and most advances in the standard of living result from it. However, we will not explore it much in the following.

In summary, material research has confirmed that quantum physics is the correct description of materials. And it has confirmed that all material properties of everyday life are of electromagnetic origin.

The strength of electromagnetism

The great physicist Wolfgang Pauli used to say that after his death, the first thing he would ask the devil would be to explain Sommerfeld's fine-structure constant. (Others used to comment that after the devil had explained it to him, he would think a little, and then snap: 'Wrong!')

Page 158
Ref. 129
The *fine-structure constant* was introduced by Arnold Sommerfeld. It is the dimensionless constant of nature given by

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137.035\,999\,679(94)} \approx 0.007\,297\,352\,5376(50)\ . \qquad (107)$$

This number first appeared in explanations of the fine structure of atomic colour spectra; hence its strange name. Sommerfeld was the first to understand its general importance. It is central to quantum electrodynamics for several reasons. First of all, it describes the *strength* of electromagnetism. Since all charges are multiples of the electron charge, a higher value for the fine structure constant α would mean a stronger attraction or repulsion between charged bodies. Thus the value of α determines the sizes of atoms, and indeed of all things, as well as all colours in nature.

Secondly, it is only because the number α is so small that we are able to talk about particles at all. The argument is somewhat involved; it will be given in detail later on. In any case, the small value of the fine-structure constant makes it possible to distinguish particles from each other. If the number were near to or larger than one, particles would interact so strongly that it would not be possible to observe them or to talk about them at all.

This leads on to the third reason for the importance of the fine-structure constant. Since it is a dimensionless number, it implies some yet-unknown mechanism that fixes its value. Uncovering this mechanism is one of the challenges remaining in our adventure. As long as the mechanism remains unknown – as was the case in 2007 – we do not understand the colour and size of a single thing around us!

Small changes in the strength of electromagnetic attraction between electrons and protons would have numerous important consequences. Can you describe what would happen to the size of people, to the colour of objects, to the colour of the Sun, or to the workings of computers, if the strength were to double? And what if it were to gradually drop to half its usual value?

Challenge 138 s

Since the 1920s, explaining the value of α has been seen as one of the toughest challenges facing modern physics. That is the reason for Pauli's request to the devil. In 1946, during his Nobel Prize lecture, he repeated the statement that a theory that does not determine this number cannot be complete. Since that time, physicists seem to have fallen into two classes: those who did not dare to take on the challenge, and those who had no clue. This fascinating story still awaits us.

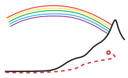
Ref. 133

The problem of the fine-structure constant is so deep that it leads many astray. For example, it is sometimes said that it is impossible to change physical units in such a way that $\hbar$, c and e are all equal to 1 at the same time, because to do so would change the number $\alpha = 1/137.036...$. Can you show that the argument is wrong?

Challenge 139 s

A summary on colours and materials

In summary, the interaction of electromagnetic fields and the electrons inside atoms, molecules, liquids and solids determines the site, the shape, the colour and the material properties of all things around us.



CHAPTER 9

QUANTUM PHYSICS IN A NUTSHELL

Compared to classical physics, quantum theory is remarkably more complex. The basic idea however, is simple: in nature there is a minimum change, or a minimum action, with the value $\hbar = 1.1 \cdot 10^{-34}$ Js. The minimum action leads to all the strange observations made in the microscopic domain, such as wave behaviour of matter, tunnelling, indeterminacy relations, randomness in measurements, quantization of angular momentum, pair creation, decay, indistinguishability and particle reactions.

The essence of quantum theory is thus the lack of the infinitely small. The mathematics of quantum theory is often disturbingly involved. Was this part of our walk worth the effort? It was; the results are profound and the accuracy of the description is excellent. We first give an overview of these results and then turn to the questions that are still left open.

Physical results of quantum theory

> “Deorum offensae diis curae.”
>
> Voltaire, *Traité sur la tolérance.*

All of quantum theory can be resumed in one sentence:

▷ *In nature, actions or changes smaller than* $\hbar = 1.1 \cdot 10^{-34}$ Js *are not observed.*

The existence of a smallest action in nature directly leads to the main lesson we learned about motion in the quantum part of our adventure:

▷ *If it moves, it is made of quantons, or quantum particles.*

This statement applies to every physical system, thus to all objects and to all images, i.e., to all matter and radiation. Moving stuff is made of *quantons*. Stones, water waves, light, sound waves, earthquakes, gelatine and everything else we can interact with is made of quantum particles.

Once we asked: what is matter and what are interactions? Now we know: they are composites of elementary quantum particles. An *elementary quantum particle* is a countable entity, smaller than its own Compton wavelength, described by energy–momentum, mass, spin, C, P and T parity. As we will see in the next volume however, this is not yet the complete list of particle properties. About the *intrinsic* particle properties, i.e., those that do not depend on the observer, quantum theory makes a simple statement:

> ▷ *In nature, all intrinsic properties – with the exception of mass – such as electric charge, spin, parities, etc., appear as integer numbers. Since all physical systems are made of quantons, in composed systems all intrinsic properties – with the exception of mass – either add or multiply.*

In summary, all moving entities are made of quantum particles described by intrinsic properties. To see how deep this result is, you can apply it to all those moving entities for which it is usually forgotten, such as ghosts, spirits, angels, nymphs, daemons, devils, gods, goddesses and souls. You can check yourself what happens when their particle nature is taken into account.

Challenge 140 e

Motion of quantum particles

Quantons, or quantum particles, differ from everyday particles: quantum particles *interfere*: they behave like a mixture of particles and waves. This property follows directly from the existence of $\hbar$, the smallest action in nature. From the existence of $\hbar$, quantum theory deduces all its statements about quantum particle motion. We now summarize the main ones.

There is *no rest* in nature. All objects obey the indeterminacy principle, which states that the indeterminacies in position x and momentum p follow

$$\Delta x \Delta p \geqslant \hbar/2 \quad \text{with} \quad \hbar = 1.1 \cdot 10^{-34}\,\text{Js} \tag{108}$$

and making rest an impossibility. The state of quantum particles is defined by the same observables as in classical physics, with the difference that observables do not commute. Classical physics appears in the limit that the Planck constant $\hbar$ can effectively be set to zero.

Quantum theory introduces a *probabilistic element* into motion. It results from the minimum action value through the interactions with the baths in the environment of any system.

Quantum particles *behave like waves*. The associated de Broglie wavelength λ is given by the momentum p through

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p} \tag{109}$$

both in the case of matter and of radiation. This relation is the origin of the wave behaviour of light and matter. The light particles are called photons; their observation is now standard practice. Quantum theory states that particle waves, like all waves, interfere, refract, disperse, dampen, can be dampened and can be polarized. This applies to photons, electrons, atoms and molecules. All waves being made of quantum particles, all waves can be seen, touched and moved. Light for example, can be 'seen' in photon-photon scattering, can be 'touched' using the Compton effect, and can be 'moved' by gravitational bending. Matter particles, such as molecules or atoms, can be seen in electron microscopes and can be touched and moved with atomic force microscopes. The interference and diffraction of wave particles is observed daily in the electron microscope.

Matter waves can be imagined as clouds that rotate locally. In the limit of negligible cloud size, quantum particles can be imagined as rotating little arrows.

Particles *cannot be enclosed.* Even though matter is impenetrable, quantum theory shows that tight boxes or insurmountable obstacles do not exist. Waiting long enough always allows us to overcome any boundary, since there is a finite probability to overcome any obstacle. This process is called *tunnelling* when seen from the spatial point of view and is called *decay* when seen from the temporal point of view. Tunnelling explains the working of television tubes as well as radioactive decay.

All particles and all particle beams *can be rotated.* Particles possess an intrinsic angular momentum called *spin*, specifying their behaviour under rotations. Bosons have integer spin, fermions have half integer spin. An even number of bound fermions or any number of bound bosons yield a composite boson; an odd number of bound fermions or an infinite number of interacting bosons yield a low-energy fermion. Solids are impenetrable because of the fermion character of its electrons in the atoms.

Identical particles are *indistinguishable*. Radiation is made of indistinguishable particles called bosons, matter of fermions. Under exchange, fermions commute at space-like separations, whereas bosons anticommute. All other properties of quantum particles are the same as for classical particles, namely countability, interaction, mass, charge, angular momentum, energy, momentum, position, as well as impenetrability for matter and penetrability for radiation. Perfect copying machines do not exist.

In collisions, particles *interact locally*, through the exchange of other particles. When matter particles collide, they interact through the exchange of virtual bosons, i.e., off-shell bosons. Motion change is thus due to particle exchange. Exchange bosons of even spin mediate only attractive interactions. Exchange bosons of odd spin mediate repulsive interactions as well.

The properties of collisions imply the existence of *antiparticles*, as regularly observed in experiments. Elementary fermions, in contrast to many elementary bosons, differ from their antiparticles; they can be created and annihilated only in pairs. Apart from neutrinos, elementary fermions have non-vanishing mass and move slower than light.

Images, made of radiation, are described by the same properties as matter. Images can only be localized with a precision of the wavelength λ of the radiation producing them.

The appearance of Planck's constant $\hbar$ implies that *length scales* and *time scales* exist in nature. Quantum theory introduces a fundamental jitter in every example of motion. Thus the infinitely small is eliminated. In this way, lower limits to structural dimensions and to many other measurable quantities appear. In particular, quantum theory shows that it is impossible that on the electrons in an atom small creatures live in the same way that humans live on the Earth circling the Sun. Quantum theory shows the impossibility of Lilliput.

Clocks and metre bars have *finite precision*, due to the existence of a smallest action and due to their interactions with baths. On the other hand, all measurement apparatuses must contain baths, since otherwise they would not be able to record results.

Ref. 134 Quantum physics leaves no room for cold fusion, astrology, teleportation, telekinesis, supernatural phenomena, multiple universes, or faster than light phenomena – the EPR paradox notwithstanding.

Achievements in precision

Apart from the conceptual changes, quantum theory improved the accuracy of predictions from the few – if any – digits common in classical mechanics to the full number of digits – sometimes thirteen – that can be measured today. The limited precision is usually *not* given by the inaccuracy of theory, it is given by the measurement accuracy. In other words, the agreement is only limited by the amount of money the experimenter is willing to spend. Table 8 shows this in more detail.

TABLE 8 Selected comparisons between classical physics, quantum theory and experiment.

Observable	Classical prediction	Prediction of quantum theory[a]	Measurement	Cost estimate
Simple motion of bodies				
Indeterminacy	0	$\Delta x \Delta p \geqslant \hbar/2$	$(1 \pm 10^{-2})\,\hbar/2$	10 k€
Matter wavelength	none	$\lambda p = 2\pi\hbar$	$(1 \pm 10^{-2})\,\hbar$	10 k€
Tunnelling rate in alpha decay	0	$1/\tau$ is finite	$(1 \pm 10^{-2})\,\tau$	5 k€
Compton wavelength	none	$\lambda_c = h/m_e c$	$(1 \pm 10^{-3})\,\lambda$	20 k€
Pair creation rate	0	σE	agrees	100 k€
Radiative decay time in hydrogen	none	$\tau \sim 1/n^3$	(1 ± 10^{-2})	5 k€
Smallest angular momentum	0	$\hbar/2$	$(1 \pm 10^{-6})\,\hbar/2$	10 k€
Casimir effect/pressure	0	$p = (\pi^2 \hbar c)/(240 r^4)$	(1 ± 10^{-3})	30 k€
Colours of objects				
Spectrum of hot objects	diverges	$\lambda_{max} = hc/(4.956\,kT)$	$(1 \pm 10^{-4})\ \Delta\lambda$	10 k€
Lamb shift	none	$\Delta\lambda = 1057.86(1)$ MHz	$(1 \pm 10^{-6})\,\Delta\lambda$	50 k€
Rydberg constant	none	$R_\infty = m_e c \alpha^2/2h$	$(1 \pm 10^{-9})\,R_\infty$	50 k€
Stefan–Boltzmann constant	none	$\sigma = \pi^2 k^4/60 \hbar^3 c^2$	$(1 \pm 3 \cdot 10^{-8})\,\sigma$	20 k€
Wien's displacement constant	none	$b = \lambda_{max} T$	$(1 \pm 10^{-5})\,b$	20 k€
Refractive index of water	none	1.34	a few %	1 k€
Photon-photon scattering	0	from QED: finite	agrees	50 M€
Electron gyromagnetic ratio	1 or 2	2.002 319 304 3(1)	2.002 319 304 3737(82)	30 M€
Composite matter properties				
Atom lifetime	$\approx 1\,\mu$s	∞	$> 10^{20}$ a	1 €

Observable	Classical prediction	Prediction of quantum theory[a]	Measurement	Cost[b] estimate
Molecular size and shape	none	from QED	within 10^{-3}	20 k€

Page 174 *a*. All these predictions are calculated from the fundamental quantities given in Appendix A.

We notice that the predicted values are not noticeably different from the measured ones. If we remember that classical physics does not allow us to calculate any of the measured values, we get an idea of the progress quantum physics has brought. But despite this impressive agreement, there still are *unexplained* observations: the one we have encountered so far is the fine structure constant. The measured value is $\alpha = 1/137.035\,9991(1)$, but no reason for this value has appeared. This is the main open problem – some would even say, the only open problem – of the electromagnetic interaction.

In summary, in the microscopic domain we are left with the impression that quantum theory is in *perfect* correspondence with nature; despite prospects of fame and riches, despite the largest number of researchers ever, no contradiction with observation has been found yet.

Is quantum theory magic?

Studying nature is like experiencing magic. Nature often looks different from what it is. During magic we are fooled – but only if we forget our own limitations. Once we start to see ourselves as part of the game, we start to understand the tricks. That is the fun of it. The same happens in physics.

* *

The world looks irreversible, even though it isn't. We never remember the future. We are fooled because we are macroscopic.

* *

The world looks decoherent, even though it isn't. We are fooled again because we are macroscopic.

* *

There are no clocks possible in nature. We are fooled because we are surrounded by a huge number of particles.

* *

Motion seems to disappear, even though it is eternal. We are fooled again, because our senses cannot experience the microscopic domain.

* *

The world seems dependent on the choice of the frame of reference, even though it is not. We are fooled because we are used to live on the surface of the Earth.

* *

Objects seem distinguishable, even though the statistical properties of their components show that they are not. We are fooled because we live at low energies.

* *

Matter looks continuous, even though it isn't. We are fooled because of the limitations of our senses.

In short, our human condition permanently fools us. The answer to the title question is affirmative: quantum theory is magic. That is its main attraction.

Quantum theory can do more

We can summarize this part of our adventure with a simple statement: *quantum physics is the description of matter and radiation* without *the concept of infinitely small.* All change in nature is described by *finite* quantities, above all, by the smallest change possible in nature, the quantum of action $\hbar$.

If we turn back to the start of our exploration of quantum theory, we cannot hide a certain disappointment. We know that classical physics cannot explain life. Searching for the details of microscopic motion, we encountered so many interesting aspects that we have *not* finished the explanation of life. For example, we know what determines the speed of electrons in atoms, but we do not know what determines the running speed of an athlete. In fact, we have not even discussed the properties of any solid or liquid, let alone those of more complex structures like living beings.

In short, after this introduction into quantum theory, we must still connect it to the everyday world. Therefore, the topic of the next volume will be the exploration of the motion of living things and of the properties of composite materials, including solids and stars, using the quantum of action as a starting point.

Appendix A

UNITS, MEASUREMENTS AND CONSTANTS

Measurements are comparisons with standards. Standards are based on a *unit*. Many different systems of units have been used throughout the world. Most standards confer power to the organization in charge of them. Such power can be misused; this is the case today, for example in the computer industry, and was so in the distant past. The solution is the same in both cases: organize an independent and global standard. For units, this happened in the eighteenth century: to avoid misuse by authoritarian institutions, to eliminate problems with differing, changing and irreproducible standards, and – this is not a joke – to simplify tax collection, a group of scientists, politicians and economists agreed on a set of units. It is called the *Système International d'Unités*, abbreviated *SI*, and is defined by an international treaty, the 'Convention du Mètre'. The units are maintained by an international organization, the 'Conférence Générale des Poids et Mesures', and its daughter organizations, the 'Commission Internationale des Poids et Mesures' and the 'Bureau International des Poids et Mesures' (BIPM), which all originated in the times just before the French revolution. Ref. 135

SI units

All SI units are built from seven *base units*, whose official definitions, translated from French into English, are given below, together with the dates of their formulation:

- 'The *second* is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.' (1967)*
- 'The *metre* is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.' (1983)
- 'The *kilogram* is the unit of mass; it is equal to the mass of the international prototype of the kilogram.' (1901)*
- 'The *ampere* is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to $2 \cdot 10^{-7}$ newton per metre of length.' (1948)
- 'The *kelvin*, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.' (1967)*
- 'The *mole* is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.' (1971)*
- 'The *candela* is the luminous intensity, in a given direction, of a source that emits

monochromatic radiation of frequency $540 \cdot 10^{12}$ hertz and has a radiant intensity in that direction of (1/683) watt per steradian.' (1979)*

Note that both time and length units are defined as certain properties of a standard example of motion, namely light. In other words, also the Conférence Générale des Poids et Mesures makes the point that the observation of motion is a *prerequisite* for the definition and construction of time and space. *Motion is the fundament each observation and measurements.* By the way, the use of light in the definitions had been proposed already in 1827 by Jacques Babinet.*

From these basic units, all other units are defined by multiplication and division. Thus, all SI units have the following properties:

- SI units form a system with *state-of-the-art precision*: all units are defined with a precision that is higher than the precision of commonly used measurements. Moreover, the precision of the definitions is regularly being improved. The present relative uncertainty of the definition of the second is around 10^{-14}, for the metre about 10^{-10}, for the kilogram about 10^{-9}, for the ampere 10^{-7}, for the mole less than 10^{-6}, for the kelvin 10^{-6} and for the candela 10^{-3}.
- SI units form an *absolute* system: all units are defined in such a way that they can be reproduced in every suitably equipped laboratory, independently, and with high precision. This avoids as much as possible any misuse by the standard-setting organization. (The kilogram, still defined with the help of an artefact, is the last exception to this requirement; extensive research is under way to eliminate this artefact from the definition – an international race that will take a few more years. There are two approaches: counting particles, or fixing $\hbar$. The former can be achieved in crystals, the latter using any formula where $\hbar$ appears, such as the formula for the de Broglie wavelength or that of the Josephson effect.)
- SI units form a *practical* system: the base units are quantities of everyday magnitude. Frequently used units have standard names and abbreviations. The complete list includes the seven base units, the supplementary units, the derived units and the admitted units.

The *supplementary* SI units are two: the unit for (plane) angle, defined as the ratio of arc length to radius, is the *radian* (rad). For solid angle, defined as the ratio of the subtended area to the square of the radius, the unit is the *steradian* (sr).

The *derived* units with special names, in their official English spelling, i.e., without capital letters and accents, are:

Vol. I, page 88
Ref. 136

* The respective symbols are s, m, kg, A, K, mol and cd. The international prototype of the kilogram is a platinum–iridium cylinder kept at the BIPM in Sèvres, in France. For more details on the levels of the caesium atom, consult a book on atomic physics. The Celsius scale of temperature θ is defined as: $\theta/°C = T/K - 273.15$; note the small difference with the number appearing in the definition of the kelvin. SI also states: 'When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.' In the definition of the mole, it is understood that the carbon 12 atoms are unbound, at rest and in their ground state. In the definition of the candela, the frequency of the light corresponds to 555.5 nm, i.e., green colour, around the wavelength to which the eye is most sensitive.

* Jacques Babinet (1794–1874), French physicist who published important work in optics.

Name	Abbreviation	Name	Abbreviation
hertz	Hz = 1/s	newton	N = kg m/s^2
pascal	Pa = N/m^2 = kg/m s^2	joule	J = Nm = kg m^2/s^2
watt	W = kg m^2/s^3	coulomb	C = As
volt	V = kg m^2/As^3	farad	F = As/V = A^{2}s^4/kg m^2
ohm	Ω = V/A = kg m^2/A^{2}s^3	siemens	S = 1/Ω
weber	Wb = Vs = kg m^2/As^2	tesla	T = Wb/m^2 = kg/As^2 = kg/Cs
henry	H = Vs/A = kg m^2/A^{2}s^2	degree Celsius	°C (see definition of kelvin)
lumen	lm = cd sr	lux	lx = lm/m^2 = cd sr/m^2
becquerel	Bq = 1/s	gray	Gy = J/kg = m^2/s^2
sievert	Sv = J/kg = m^2/s^2	katal	kat = mol/s

We note that in all definitions of units, the kilogram only appears to the powers of 1, 0 and −1. The final explanation for this fact appeared only recently. Can you try to formulate the reason?

Challenge 141 ny

The *admitted* non-SI units are *minute*, *hour*, *day* (for time), *degree* $1° = \pi/180$ rad, *minute* $1' = \pi/10\,800$ rad, *second* $1'' = \pi/648\,000$ rad (for angles), *litre* and *tonne*. All other units are to be avoided.

All SI units are made more practical by the introduction of standard names and abbreviations for the powers of ten, the so-called *prefixes*:*

Power	Name		Power	Name		Power	Name		Power	Name	
10^1	deca	da	10^{-1}	deci	d	10^{18}	Exa	E	10^{-18}	atto	a
10^2	hecto	h	10^{-2}	centi	c	10^{21}	Zetta	Z	10^{-21}	zepto	z
10^3	kilo	k	10^{-3}	milli	m	10^{24}	Yotta	Y	10^{-24}	yocto	y
10^6	Mega	M	10^{-6}	micro	µ	unofficial:			Ref. 137		
10^9	Giga	G	10^{-9}	nano	n	10^{27}	Xenta	X	10^{-27}	xenno	x
10^{12}	Tera	T	10^{-12}	pico	p	10^{30}	Wekta	W	10^{-30}	weko	w
10^{15}	Peta	P	10^{-15}	femto	f	10^{33}	Vendekta	V	10^{-33}	vendeko	v
						10^{36}	Udekta	U	10^{-36}	udeko	u

▪ SI units form a *complete* system: they cover in a systematic way the complete set of observables of physics. Moreover, they fix the units of measurement for all other sciences

* Some of these names are invented (yocto to sound similar to Latin *octo* 'eight', zepto to sound similar to Latin *septem*, yotta and zetta to resemble them, exa and peta to sound like the Greek words ἑξάκις and πεντάκις for 'six times' and 'five times', the unofficial ones to sound similar to the Greek words for nine, ten, eleven and twelve); some are from Danish/Norwegian (atto from *atten* 'eighteen', femto from *femten* 'fifteen'); some are from Latin (from *mille* 'thousand', from *centum* 'hundred', from *decem* 'ten', from *nanus* 'dwarf'); some are from Italian (from *piccolo* 'small'); some are Greek (micro is from μικρός 'small', deca/deka from δέκα 'ten', hecto from ἑκατόν 'hundred', kilo from χίλιοι 'thousand', mega from μέγας 'large', giga from γίγας 'giant', tera from τέρας 'monster').

Translate: I was caught in such a traffic jam that I needed a microcentury for a picoparsec and that my car's fuel consumption was two tenths of a square millimetre.

Challenge 142 e

as well.

- SI units form a *universal* system: they can be used in trade, in industry, in commerce, at home, in education and in research. They could even be used by extraterrestrial civilizations, if they existed.
- SI units form a *coherent* system: the product or quotient of two SI units is also an SI unit. This means that in principle, the same abbreviation, e.g. 'SI', could be used for every unit.

The SI units are not the only possible set that could fulfil all these requirements, but they are the only existing system that does so.*

Challenge 143 e

Since every measurement is a comparison with a standard, any measurement requires *matter* to realize the standard (even for a speed standard), and *radiation* to achieve the comparison. The concept of measurement thus assumes that matter and radiation exist and can be clearly separated from each other.

Planck's natural units

Since the exact form of many equations depends on the system of units used, theoretical physicists often use unit systems optimized for producing simple equations. The chosen units and the values of the constants of nature are related. In microscopic physics, the system of *Planck's natural units* is frequently used. They are defined by setting $c = 1$, $\hbar = 1$, $G = 1$, $k = 1$, $\varepsilon_0 = 1/4\pi$ and $\mu_0 = 4\pi$. Planck units are thus defined from combinations of fundamental constants; those corresponding to the fundamental SI units are given in Table 10.** The table is also useful for converting equations written in natural units back to SI units: just substitute every quantity X by X/X_{Pl}.

Challenge 144 e

TABLE 10 Planck's (uncorrected) natural units.

Name	Definition			Value
Basic units				
the Planck length	l_{Pl}	=	$\sqrt{\hbar G/c^3}$	$= 1.616\,0(12) \cdot 10^{-35}$ m
the Planck time	t_{Pl}	=	$\sqrt{\hbar G/c^5}$	$= 5.390\,6(40) \cdot 10^{-44}$ s
the Planck mass	m_{Pl}	=	$\sqrt{\hbar c/G}$	$= 21.767(16)$ µg
the Planck current	I_{Pl}	=	$\sqrt{4\pi\varepsilon_0 c^6/G}$	$= 3.479\,3(22) \cdot 10^{25}$ A

* Apart from international units, there are also *provincial* units. Most provincial units still in use are of Roman origin. The mile comes from *milia passum*, which used to be one thousand (double) strides of about 1480 mm each; today a nautical mile, once defined as minute of arc on the Earth's surface, is exactly 1852 m). The inch comes from *uncia/onzia* (a twelfth – now of a foot). The pound (from *pondere* 'to weigh') is used as a translation of *libra* – balance – which is the origin of its abbreviation lb. Even the habit of counting in dozens instead of tens is Roman in origin. These and all other similarly funny units – like the system in which all units start with 'f', and which uses furlong/fortnight as its unit of velocity – are now officially defined as multiples of SI units.

** The natural units x_{Pl} given here are those commonly used today, i.e., those defined using the constant $\hbar$, and not, as Planck originally did, by using the constant $h = 2\pi\hbar$. The electromagnetic units can also be defined with other factors than $4\pi\varepsilon_0$ in the expressions: for example, using $4\pi\varepsilon_0\alpha$, with the *fine structure constant* α, gives $q_{\text{Pl}} = e$. For the explanation of the numbers between brackets, the standard deviations, see below.

Name	Definition			Value	
the Planck temperature	T_{Pl}	=	$\sqrt{\hbar c^5/Gk^2}$	=	$1.417\,1(91) \cdot 10^{32}$ K
Trivial units					
the Planck velocity	v_{Pl}	=	c	=	0.3 Gm/s
the Planck angular momentum	L_{Pl}	=	$\hbar$	=	$1.1 \cdot 10^{-34}$ Js
the Planck action	S_{aPl}	=	$\hbar$	=	$1.1 \cdot 10^{-34}$ Js
the Planck entropy	S_{ePl}	=	k	=	13.8 yJ/K
Composed units					
the Planck mass density	ρ_{Pl}	=	$c^5/G^2\hbar$	=	$5.2 \cdot 10^{96}\ \text{kg/m}^3$
the Planck energy	E_{Pl}	=	$\sqrt{\hbar c^5/G}$	=	$2.0\ \text{GJ} = 1.2 \cdot 10^{28}$ eV
the Planck momentum	p_{Pl}	=	$\sqrt{\hbar c^3/G}$	=	6.5 Ns
the Planck power	P_{Pl}	=	c^5/G	=	$3.6 \cdot 10^{52}$ W
the Planck force	F_{Pl}	=	c^4/G	=	$1.2 \cdot 10^{44}$ N
the Planck pressure	p_{Pl}	=	$c^7/G\hbar$	=	$4.6 \cdot 10^{113}$ Pa
the Planck acceleration	a_{Pl}	=	$\sqrt{c^7/\hbar G}$	=	$5.6 \cdot 10^{51}\ \text{m/s}^2$
the Planck frequency	f_{Pl}	=	$\sqrt{c^5/\hbar G}$	=	$1.9 \cdot 10^{43}$ Hz
the Planck electric charge	q_{Pl}	=	$\sqrt{4\pi\varepsilon_0 c\hbar}$	=	$1.9\ \text{aC} = 11.7$ e
the Planck voltage	U_{Pl}	=	$\sqrt{c^4/4\pi\varepsilon_0 G}$	=	$1.0 \cdot 10^{27}$ V
the Planck resistance	R_{Pl}	=	$1/4\pi\varepsilon_0 c$	=	$30.0\ \Omega$
the Planck capacitance	C_{Pl}	=	$4\pi\varepsilon_0\sqrt{\hbar G/c^3}$	=	$1.8 \cdot 10^{-45}$ F
the Planck inductance	L_{Pl}	=	$(1/4\pi\varepsilon_0)\sqrt{\hbar G/c^7}$	=	$1.6 \cdot 10^{-42}$ H
the Planck electric field	E_{Pl}	=	$\sqrt{c^7/4\pi\varepsilon_0\hbar G^2}$	=	$6.5 \cdot 10^{61}$ V/m
the Planck magnetic flux density	B_{Pl}	=	$\sqrt{c^5/4\pi\varepsilon_0\hbar G^2}$	=	$2.2 \cdot 10^{53}$ T

The natural units are important for another reason: whenever a quantity is sloppily called 'infinitely small (or large)', the correct expression is 'as small (or as large) as the corresponding corrected Planck unit'. As explained throughout the text, and especially in the Vol. VI, page 32 final part, this substitution is possible because almost all Planck units provide, within a correction factor of order 1, the extremal value for the corresponding observable – some an upper and some a lower limit. Unfortunately, these correction factors are not yet widely known. The exact extremal value for each observable in nature is obtained when G is substituted by $4G$ and $4\pi\varepsilon_0$ by $4\pi\varepsilon_0\alpha$ in all Planck quantities. These extremal values, or *corrected Planck units*, are the *true natural units*. To exceed the extremal values Challenge 145 s is possible only for some extensive quantities. (Can you find out which ones?)

Other unit systems

A central aim of research in high-energy physics is the calculation of the strengths of all interactions; therefore it is not practical to set the gravitational constant G to unity, as in the Planck system of units. For this reason, high-energy physicists often only set

$c = \hbar = k = 1$ and $\mu_0 = 1/\varepsilon_0 = 4\pi$,* leaving only the gravitational constant G in the equations.

In this system, only one fundamental unit exists, but its choice is free. Often a standard length is chosen as the fundamental unit, length being the archetype of a measured quantity. The most important physical observables are then related by

$$\begin{aligned} 1/[l^2] &= [E]^2 = [F] = [B] = [E_{\text{electric}}] \,, \\ 1/[l] &= [E] = [m] = [p] = [a] = [f] = [I] = [U] = [T] \,, \\ 1 &= [v] = [q] = [e] = [R] = [S_{\text{action}}] = [S_{\text{entropy}}] = \hbar = c = k = [\alpha] \,, \\ [l] &= 1/[E] = [t] = [C] = [L] \quad \text{and} \\ [l]^2 &= 1/[E]^2 = [G] = [P] \end{aligned} \tag{110}$$

where we write $[x]$ for the unit of quantity x. Using the same unit for time, capacitance and inductance is not to everybody's taste, however, and therefore electricians do not use this system.**

Often, in order to get an impression of the energies needed to observe an effect under study, a standard energy is chosen as fundamental unit. In particle physics the most common energy unit is the *electronvolt* (eV), defined as the kinetic energy acquired by an electron when accelerated by an electrical potential difference of 1 volt ('protonvolt' would be a better name). Therefore one has $1\,\text{eV} = 1.6 \cdot 10^{-19}\,\text{J}$, or roughly

$$1\,\text{eV} \approx \tfrac{1}{6}\,\text{aJ} \tag{111}$$

which is easily remembered. The simplification $c = \hbar = 1$ yields $G = 6.9 \cdot 10^{-57}\,\text{eV}^{-2}$ and allows one to use the unit eV also for mass, momentum, temperature, frequency, time
Challenge 146 e
and length, with the respective correspondences $1\,\text{eV} \equiv 1.8 \cdot 10^{-36}\,\text{kg} \equiv 5.4 \cdot 10^{-28}\,\text{Ns} \equiv 242\,\text{THz} \equiv 11.6\,\text{kK}$ and $1\,\text{eV}^{-1} \equiv 4.1\,\text{fs} \equiv 1.2\,\mu\text{m}$.

To get some feeling for the unit eV, the following relations are useful. Room temperature, usually taken as 20°C or 293 K, corresponds to a kinetic energy per particle of 0.025 eV or 4.0 zJ. The highest particle energy measured so far belongs to a cosmic ray
Ref. 139
with an energy of $3 \cdot 10^{20}$ eV or 48 J. Down here on the Earth, an accelerator able to produce an energy of about 105 GeV or 17 nJ for electrons and antielectrons has been built, and one able to produce an energy of 14 TeV or 2.2 μJ for protons will be finished soon. Both are owned by CERN in Geneva and have a circumference of 27 km.

The lowest temperature measured up to now is 280 pK, in a system of rhodium nuclei

* Other definitions for the proportionality constants in electrodynamics lead to the Gaussian unit system often used in theoretical calculations, the Heaviside–Lorentz unit system, the electrostatic unit system, and
Ref. 138
the electromagnetic unit system, among others.

** In the list, l is length, E energy, F force, E_{electric} the electric and B the magnetic field, m mass, p momentum, a acceleration, f frequency, I electric current, U voltage, T temperature, v speed, q charge, R resistance, P power, G the gravitational constant.

The web page www.chemie.fu-berlin.de/chemistry/general/units_en.html provides a tool to convert various units into each other.

Researchers in general relativity often use another system, in which the *Schwarzschild radius* $r_s = 2Gm/c^2$ is used to measure masses, by setting $c = G = 1$. In this case, mass and length have the *same* dimension, and $\hbar$ has the dimension of an area.

Ref. 140 held inside a special cooling system. The interior of that cryostat may even be the coolest point in the whole universe. The kinetic energy per particle corresponding to that temperature is also the smallest ever measured: it corresponds to 24 feV or 3.8 vJ = $3.8 \cdot 10^{-33}$ J. For isolated particles, the record seems to be for neutrons: kinetic energies as low as 10^{-7} eV have been achieved, corresponding to de Broglie wavelengths of 60 nm.

Curiosities and fun challenges about units

Not using SI units can be expensive. In 1999, NASA lost a satellite on Mars because some software programmers had used provincial units instead of SI units in part of the code. As a result, the Mars Climate Orbiter crashed into the planet, instead of orbiting it; the loss was around 100 million euro.*

* *

The Planck length is roughly the de Broglie wavelength $\lambda_B = h/mv$ of a man walking comfortably (m = 80 kg, v = 0.5 m/s); this motion is therefore aptly called the 'Planck stroll.' Ref. 141

* *

The Planck mass is equal to the mass of about 10^{19} protons. This is roughly the mass of a human embryo at about ten days of age.

* *

The most precisely measured quantities in nature are the frequencies of certain millisecond pulsars, the frequency of certain narrow atomic transitions, and the Rydberg constant of *atomic* hydrogen, which can all be measured as precisely as the second is defined. The caesium transition that defines the second has a finite linewidth that limits the achievable precision: the limit is about 14 digits. Ref. 143

* *

The most precise clock ever built, using microwaves, had a stability of 10^{-16} during a running time of 500 s. For longer time periods, the record in 1997 was about 10^{-15}; but values around 10^{-17} seem within technological reach. The precision of clocks is limited for short measuring times by noise, and for long measuring times by drifts, i.e., by systematic effects. The region of highest stability depends on the clock type; it usually lies between 1 ms for optical clocks and 5000 s for masers. Pulsars are the only type of clock for which this region is not known yet; it certainly lies at more than 20 years, the time elapsed at the time of writing since their discovery. Ref. 142 Ref. 144

* *

The shortest times measured are the lifetimes of certain 'elementary' particles. In particular, the lifetime of certain D mesons have been measured at less than 10^{-23} s. Such times are measured using a bubble chamber, where the track is photographed. Can you esti- Ref. 145

* This story revived an old (and false) urban legend that states that only three countries in the world do not use SI units: Liberia, the USA and Myanmar.

Challenge 147 s mate how long the track is? (This is a trick question – if your length cannot be observed with an optical microscope, you have made a mistake in your calculation.)

* *

The longest times encountered in nature are the lifetimes of certain radioisotopes, over 10^{15} years, and the lower limit of certain proton decays, over 10^{32} years. These times are thus much larger than the age of the universe, estimated to be fourteen thousand million years. Ref. 146

* *

Variations of quantities are often much easier to measure than their values. For example, in gravitational wave detectors, the sensitivity achieved in 1992 was $\Delta l/l = 3 \cdot 10^{-19}$ for lengths of the order of 1 m. In other words, for a block of about a cubic metre of metal it is possible to measure length changes about 3000 times smaller than a proton radius. These set-ups are now being superseded by ring interferometers. Ring interferometers measuring frequency differences of 10^{-21} have already been built; and they are still being improved. Ref. 147 Ref. 148

Precision and accuracy of measurements

Measurements are the basis of physics. Every measurement has an *error*. Errors are due to lack of precision or to lack of accuracy. *Precision* means how well a result is reproduced when the measurement is repeated; *accuracy* is the degree to which a measurement corresponds to the actual value. Lack of precision is due to accidental or *random errors*; they are best measured by the *standard deviation*, usually abbreviated σ; it is defined through

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 , \qquad (112)$$

where $\bar{x}$ is the average of the measurements x_i. (Can you imagine why $n-1$ is used in the formula instead of n?) Challenge 148 s

For most experiments, the distribution of measurement values tends towards a normal distribution, also called *Gaussian distribution*, whenever the number of measurements is increased. The distribution, shown in Figure 226, is described by the expression

$$N(x) \approx \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}} . \qquad (113)$$

The square σ^2 of the standard deviation is also called the *variance*. For a Gaussian distribution of measurement values, 2.35σ is the full width at half maximum. Challenge 149 e

Lack of accuracy is due to *systematic errors*; usually these can only be estimated. This estimate is often added to the random errors to produce a *total* experimental error, sometimes also called *total uncertainty*. Ref. 149

The tables below give the values of the most important physical constants and particle properties in SI units and in a few other common units, as published in the standard references. The values are the world averages of the best measurements made up to the Ref. 150

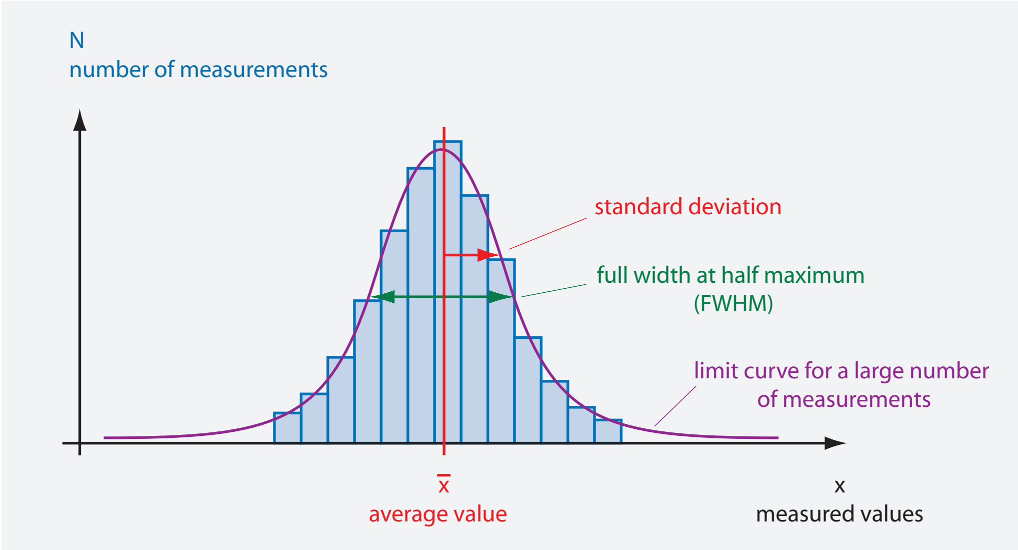


FIGURE 76 A precision experiment and its measurement distribution.

present. As usual, experimental errors, including both random and estimated systematic errors, are expressed by giving the standard deviation in the last digits; e.g. 0.31(6) means – roughly speaking – 0.31 ± 0.06. In fact, behind each of the numbers in the following tables there is a long story which is worth telling, but for which there is not enough room here.

Ref. 151

Limits to precision

What are the limits to accuracy and precision? There is no way, even in principle, to measure a length x to a *precision* higher than about 61 digits, because the ratio between the largest and the smallest measurable length is $\Delta x / x > l_{\mathrm{Pl}} / d_{\mathrm{horizon}} = 10^{-61}$. (Is this ratio valid also for force or for volume?) In the final volume of our text, studies of clocks and metre bars strengthen this theoretical limit.

Challenge 150 e
Vol. VI, page 85

But it is not difficult to deduce more stringent practical limits. No imaginable machine can measure quantities with a higher precision than measuring the diameter of the Earth within the smallest length ever measured, about 10^{-19} m; that is about 26 digits of precision. Using a more realistic limit of a 1000 m sized machine implies a limit of 22 digits. If, as predicted above, time measurements really achieve 17 digits of precision, then they are nearing the practical limit, because apart from size, there is an additional practical restriction: cost. Indeed, an additional digit in measurement precision often means an additional digit in equipment cost.

Physical constants

In principle, all quantitative properties of matter can be calculated with quantum theory. For example, colour, density and elastic properties can be predicted using the values of the following constants using the equations of the standard model of high-energy physics.

Ref. 150
Vol. V, page 190

TABLE 11 Basic physical constants.

Quantity	Symbol	Value in SI units	Uncert.[a]
number of space-time dimensions		3 + 1	0[b]
vacuum speed of light[c]	c	299 792 458 m/s	0
vacuum permeability[c]	μ_0	$4\pi \cdot 10^{-7}$ H/m	0
		= 1.256 637 061 435 ... μH/m	0
vacuum permittivity[c]	$\varepsilon_0 = 1/\mu_0 c^2$	8.854 187 817 620 ... pF/m	0
original Planck constant	h	$6.626\,068\,76(52) \cdot 10^{-34}$ Js	$7.8 \cdot 10^{-8}$
reduced Planck constant	$\hbar$	$1.054\,571\,596(82) \cdot 10^{-34}$ Js	$7.8 \cdot 10^{-8}$
positron charge	e	0.160 217 646 2(63) aC	$3.9 \cdot 10^{-8}$
Boltzmann constant	k	$1.380\,650\,3(24) \cdot 10^{-23}$ J/K	$1.7 \cdot 10^{-6}$
gravitational constant	G	$6.673(10) \cdot 10^{-11}$ Nm²/kg²	$1.5 \cdot 10^{-3}$
gravitational coupling constant	$\kappa = 8\pi G/c^4$	$2.076(3) \cdot 10^{-43}$ s²/kg m	$1.5 \cdot 10^{-3}$
fine structure constant,[d]	$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$	1/137.035 999 76(50)	$3.7 \cdot 10^{-9}$
e.m. coupling constant	$= \alpha_{em}(m_e^2 c^2)$	= 0.007 297 352 533(27)	$3.7 \cdot 10^{-9}$
Fermi coupling constant,[d]	$G_F/(\hbar c)^3$	$1.166\,39(1) \cdot 10^{-5}$ GeV^{-2}	$8.6 \cdot 10^{-6}$
weak coupling constant	$\alpha_w(M_Z) = g_w^2/4\pi$	1/30.1(3)	$1 \cdot 10^{-2}$
weak mixing angle	$\sin^2\theta_W(\overline{MS})$	0.231 24(24)	$1.0 \cdot 10^{-3}$
weak mixing angle	$\sin^2\theta_W$ (on shell)	0.2224(19)	$8.7 \cdot 10^{-3}$
	$= 1 - (m_W/m_Z)^2$		
strong coupling constant[d]	$\alpha_s(M_Z) = g_s^2/4\pi$	0.118(3)	$25 \cdot 10^{-3}$

a. Uncertainty: standard deviation of measurement errors.
b. Only down to 10^{-19} m and up to 10^{26} m.
c. Defining constant.
d. All coupling constants depend on the 4-momentum transfer, as explained in the section on renormalization. *Fine structure constant* is the traditional name for the electromagnetic coupling constant α in the case of a 4-momentum transfer of $Q^2 = m_e^2 c^2$, which is the smallest one possible. At higher momentum transfers it has larger values, e.g., $\alpha_{em}(Q^2 = M_W^2 c^2) \approx 1/128$. In contrast, the strong coupling constant has lover values at higher momentum transfers; e.g., $\alpha_s(34\,\text{GeV}) = 0.14(2)$.

Vol. V, page 88

Why do all these constants have the values they have? For any constant *with a dimension*, such as the quantum of action $\hbar$, the numerical value has only historical meaning. It is $1.054 \cdot 10^{-34}$ Js because of the SI definition of the joule and the second. The question why the value of a dimensional constant is not larger or smaller therefore always requires one to understand the origin of some dimensionless number giving the ratio between the constant and the corresponding natural unit that is defined with c, G, $\hbar$ and α. Understanding the sizes of atoms, people, trees and stars, the duration of molecular and atomic processes, or the mass of nuclei and mountains, implies understanding the ratios between these values and the corresponding natural units. The key to understanding nature is thus the understanding of all ratios, and thus of all dimensionless constants. The quest of understanding all ratios, all dimensionless constants, including the fine structure constant α itself, is completed only in the final volume of our adventure.

The basic constants yield the following useful high-precision observations.

TABLE 12 Derived physical constants.

Quantity	Symbol	Value in SI units	Uncert.
Vacuum wave resistance	$Z_0 = \sqrt{\mu_0/\varepsilon_0}$	376.730 313 461 77... Ω	0
Avogadro's number	N_A	$6.022\,141\,99(47) \cdot 10^{23}$	$7.9 \cdot 10^{-8}$
Rydberg constant [a]	$R_\infty = m_e c\alpha^2/2h$	$10\,973\,731.568\,549(83)\,\mathrm{m}^{-1}$	$7.6 \cdot 10^{-12}$
conductance quantum	$G_0 = 2e^2/h$	77.480 916 96(28) μS	$3.7 \cdot 10^{-9}$
magnetic flux quantum	$\varphi_0 = h/2e$	2.067 833 636(81) pWb	$3.9 \cdot 10^{-8}$
Josephson frequency ratio	$2e/h$	483.597 898(19) THz/V	$3.9 \cdot 10^{-8}$
von Klitzing constant	$h/e^2 = \mu_0 c/2\alpha$	25 812.807 572(95) Ω	$3.7 \cdot 10^{-9}$
Bohr magneton	$\mu_B = e\hbar/2m_e$	9.274 008 99(37) yJ/T	$4.0 \cdot 10^{-8}$
cyclotron frequency of the electron	$f_c/B = e/2\pi m_e$	27.992 4925(11) GHz/T	$4.0 \cdot 10^{-8}$
classical electron radius	$r_e = e^2/4\pi\varepsilon_0 m_e c^2$	2.817 940 285(31) fm	$1.1 \cdot 10^{-8}$
Compton wavelength	$\lambda_c = h/m_e c$	2.426 310 215(18) pm	$7.3 \cdot 10^{-9}$
of the electron	$\lambda\!\!\!^{-}_c = \hbar/m_e c = r_e/\alpha$	0.386 159 264 2(28) pm	$7.3 \cdot 10^{-9}$
Bohr radius [a]	$a_\infty = r_e/\alpha^2$	52.917 720 83(19) pm	$3.7 \cdot 10^{-9}$
nuclear magneton	$\mu_N = e\hbar/2m_p$	$5.050\,783\,17(20) \cdot 10^{-27}$ J/T	$4.0 \cdot 10^{-8}$
proton–electron mass ratio	m_p/m_e	1 836.152 667 5(39)	$2.1 \cdot 10^{-9}$
Stefan–Boltzmann constant	$\sigma = \pi^2 k^4/60\hbar^3 c^2$	$56.704\,00(40)\,\mathrm{nW/m^2K^4}$	$7.0 \cdot 10^{-6}$
Wien's displacement constant	$b = \lambda_{max} T$	2.897 768 6(51) mmK	$1.7 \cdot 10^{-6}$
bits to entropy conversion const.		10^{23} bit = 0.956 994 5(17) J/K	$1.7 \cdot 10^{-6}$
TNT energy content		3.7 to 4.0 MJ/kg	$4 \cdot 10^{-2}$

a. For infinite mass of the nucleus.

Some useful properties of our local environment are given in the following table.

TABLE 13 Astronomical constants.

Quantity	Symbol	Value
tropical year 1900 [a]	a	31 556 925.974 7 s
tropical year 1994	a	31 556 925.2 s
mean sidereal day	d	$23^h 56' 4.090\,53''$
astronomical unit [b]	AU	149 597 870.691(30) km
light year	al	9.460 528 173 ... Pm
parsec	pc	30.856 775 806 Pm = 3.261 634 al
Earth's mass	$M_♁$	$5.973(1) \cdot 10^{24}$ kg
Geocentric gravitational constant	GM	$3.986\,004\,418(8) \cdot 10^{14}\,\mathrm{m^3/s^2}$
Earth's gravitational length	$l_♁ = 2GM/c^2$	8.870 056 078(16) mm
Earth's equatorial radius [c]	$R_{♁eq}$	6378.1366(1) km
Earth's polar radius [c]	$R_{♁p}$	6356.752(1) km

TABLE 13 (Continued) Astronomical constants.

Quantity	Symbol	Value
Equator–pole distance [c]		10 001.966 km (average)
Earth's flattening [c]	$e_{♁}$	1/298.25642(1)
Earth's av. density	$\rho_{♁}$	5.5 Mg/m^3
Earth's age	$T_{♁}$	4.50(4) Ga = 142(2) Ps
Moon's radius	$R_{☾\mathrm{v}}$	1738 km in direction of Earth
Moon's radius	$R_{☾\mathrm{h}}$	1737.4 km in other two directions
Moon's mass	$M_{☾}$	$7.35 \cdot 10^{22}$ kg
Moon's mean distance [d]	$d_{☾}$	384 401 km
Moon's distance at perigee [d]		typically 363 Mm, historical minimum 359 861 km
Moon's distance at apogee [d]		typically 404 Mm, historical maximum 406 720 km
Moon's angular size [e]		average $0.5181° = 31.08'$, minimum $0.49°$, maximum - shortens line $0.55°$
Moon's average density	$\rho_{☾}$	3.3 Mg/m^3
Jupiter's mass	$M_{♃}$	$1.90 \cdot 10^{27}$ kg
Jupiter's radius, equatorial	$R_{♃}$	71.398 Mm
Jupiter's radius, polar	$R_{♃}$	67.1(1) Mm
Jupiter's average distance from Sun	$D_{♃}$	778 412 020 km
Sun's mass	$M_{\odot}$	$1.988\,43(3) \cdot 10^{30}$ kg
Sun's gravitational length	$l_{\odot} = 2GM_{\odot}/c^2$	2.953 250 08 km
Sun's luminosity	$L_{\odot}$	384.6 YW
Solar equatorial radius	$R_{\odot}$	695.98(7) Mm
Sun's angular size		$0.53°$ average; minimum on fourth of July (aphelion) $1888''$, maximum on fourth of January (perihelion) $1952''$
Sun's average density	$\rho_{\odot}$	1.4 Mg/m^3
Sun's average distance	AU	149 597 870.691(30) km
Sun's age	$T_{\odot}$	4.6 Ga
Solar velocity around centre of galaxy	$v_{\odot\mathrm{g}}$	220(20) km/s
Solar velocity against cosmic background	$v_{\odot\mathrm{b}}$	370.6(5) km/s
Distance to Milky Way's centre		8.0(5) kpc = 26.1(1.6) kal
Milky Way's age		13.6 Ga
Milky Way's size		*c.* 10^{21} m or 100 kal
Milky Way's mass		10^{12} solar masses, *c.* $2 \cdot 10^{42}$ kg
Most distant galaxy cluster known	SXDF-XCLJ 0218-0510	$9.6 \cdot 10^9$ al

a. Defining constant, from vernal equinox to vernal equinox; it was once used to define the second. (Remember: π seconds is about a nanocentury.) The value for 1990 is about 0.7 s less, corresponding to a slowdown
Challenge 151 s of roughly 0.2 ms/a. (Watch out: why?) There is even an empirical formula for the change of the length of
Ref. 152 the year over time.
b. Average distance Earth–Sun. The truly amazing precision of 30 m results from time averages of signals sent from Viking orbiters and Mars landers taken over a period of over twenty years.
c. The shape of the Earth is described most precisely with the World Geodetic System. The last edition dates from 1984. For an extensive presentation of its background and its details, see the www.wgs84.com website. The International Geodesic Union refined the data in 2000. The radii and the flattening given here are those for the 'mean tide system'. They differ from those of the 'zero tide system' and other systems by about 0.7 m. The details constitute a science in itself.
d. Measured centre to centre. To find the precise position of the Moon at a given date, see the www.fourmilab.ch/earthview/moon_ap_per.html page. For the planets, see the page www.fourmilab.ch/solar/solar.html and the other pages on the same site.
e. Angles are defined as follows: 1 degree = $1^\circ = \pi/180$ rad, 1 (first) minute = $1' = 1^\circ/60$, 1 second (minute) = $1'' = 1'/60$. The ancient units 'third minute' and 'fourth minute', each 1/60th of the preceding, are not in use any more. ('Minute' originally means 'very small', as it still does in modern English.)

Some properties of nature at large are listed in the following table. (If you want a chal-
Challenge 152 s lenge, can you determine whether any property of the universe itself is listed?)

TABLE 14 Astrophysical constants.

Quantity	Symbol	Value
gravitational constant	G	$6.672\,59(85)\cdot10^{-11}\,\mathrm{m^3/kg\,s^2}$
cosmological constant	Λ	*c.* $1\cdot10^{-52}\,\mathrm{m^{-2}}$
age of the universe [a]	t_0	$4.333(53)\cdot10^{17}\,\mathrm{s} = 13.73(0.17)\cdot10^9\,\mathrm{a}$
(determined from space-time, via expansion, using general relativity)		
age of the universe [a]	t_0	over $3.5(4)\cdot10^{17}\,\mathrm{s} = 11.5(1.5)\cdot10^9\,\mathrm{a}$
(determined from matter, via galaxies and stars, using quantum theory)		
Hubble parameter [a]	H_0	$2.3(2)\cdot10^{-18}\,\mathrm{s^{-1}} = 0.73(4)\cdot10^{-10}\,\mathrm{a^{-1}}$
		$= h_0\cdot100\,\mathrm{km/s\,Mpc} = h_0\cdot1.0227\cdot10^{-10}\,\mathrm{a^{-1}}$
reduced Hubble parameter [a]	h_0	$0.71(4)$
deceleration parameter	$q_0 = -(\ddot{a}/a)_0/H_0^2$	$-0.66(10)$
universe's horizon distance [a]	$d_0 = 3ct_0$	$40.0(6)\cdot10^{26}\,\mathrm{m} = 13.0(2)\,\mathrm{Gpc}$
universe's topology		trivial up to $10^{26}\,\mathrm{m}$
number of space dimensions		3, for distances up to $10^{26}\,\mathrm{m}$
critical density of the universe	$\rho_c = 3H_0^2/8\pi G$	$h_0^2\cdot1.878\,82(24)\cdot10^{-26}\,\mathrm{kg/m^3}$ $= 0.95(12)\cdot10^{-26}\,\mathrm{kg/m^3}$
(total) density parameter [a]	$\Omega_0 = \rho_0/\rho_c$	$1.02(2)$
baryon density parameter [a]	$\Omega_{B0} = \rho_{B0}/\rho_c$	$0.044(4)$
cold dark matter density parameter [a]	$\Omega_{CDM0} = \rho_{CDM0}/\rho_c$	$0.23(4)$
neutrino density parameter [a]	$\Omega_{\nu0} = \rho_{\nu0}/\rho_c$	0.001 to 0.05
dark energy density parameter [a]	$\Omega_{X0} = \rho_{X0}/\rho_c$	$0.73(4)$
dark energy state parameter	$w = p_X/\rho_X$	$-1.0(2)$
baryon mass	m_b	$1.67\cdot10^{-27}\,\mathrm{kg}$

TABLE 14 (Continued) Astrophysical constants.

Quantity	Symbol	Value
baryon number density		$0.25(1)\,/\mathrm{m}^3$
luminous matter density		$3.8(2)\cdot 10^{-28}\,\mathrm{kg/m}^3$
stars in the universe	n_s	$10^{22\pm1}$
baryons in the universe	n_b	$10^{81\pm1}$
microwave background temperature [b]	T_0	$2.725(1)\,\mathrm{K}$
photons in the universe	n_γ	10^{89}
photon energy density	$\rho_\gamma = \pi^2 k^4/15T_0^4$	$4.6\cdot 10^{-31}\,\mathrm{kg/m}^3$
photon number density		$410.89\,/\mathrm{cm}^3$ or $400\,/\mathrm{cm}^3(T_0/2.7\,\mathrm{K})^3$
density perturbation amplitude	$\sqrt{S}$	$5.6(1.5)\cdot 10^{-6}$
gravity wave amplitude	$\sqrt{T}$	$< 0.71\sqrt{S}$
mass fluctuations on 8 Mpc	σ_8	0.84(4)
scalar index	n	0.93(3)
running of scalar index	$\mathrm{d}n/\mathrm{d}\ln k$	-0.03(2)
Planck length	$l_\mathrm{Pl} = \sqrt{\hbar G/c^3}$	$1.62\cdot 10^{-35}\,\mathrm{m}$
Planck time	$t_\mathrm{Pl} = \sqrt{\hbar G/c^5}$	$5.39\cdot 10^{-44}\,\mathrm{s}$
Planck mass	$m_\mathrm{Pl} = \sqrt{\hbar c/G}$	$21.8\,\mu\mathrm{g}$
instants in history [a]	t_0/t_Pl	$8.7(2.8)\cdot 10^{60}$
space-time points inside the horizon [a]	$N_0 = (R_0/l_\mathrm{Pl})^3\cdot(t_0/t_\mathrm{Pl})$	$10^{244\pm1}$
mass inside horizon	M	$10^{54\pm1}\,\mathrm{kg}$

a. The index 0 indicates present-day values.
b. The radiation originated when the universe was 380 000 years old and had a temperature of about 3000 K; the fluctuations ΔT_0 which led to galaxy formation are today about $16 \pm 4\,\mu\mathrm{K} = 6(2)\cdot 10^{-6}\,T_0$.

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Useful numbers

Ref. 153

π	$3.14159\,26535\,89793\,23846\,26433\,83279\,50288\,41971\,69399\,37510_5$
e	$2.71828\,18284\,59045\,23536\,02874\,71352\,66249\,77572\,47093\,69995_9$
γ	$0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,10421\,59335\,93992_3$
ln 2	$0.69314\,71805\,59945\,30941\,72321\,21458\,17656\,80755\,00134\,36025_5$
ln 10	$2.30258\,50929\,94045\,68401\,79914\,54684\,36420\,76011\,01488\,62877_2$
$\sqrt{10}$	$3.16227\,76601\,68379\,33199\,88935\,44432\,71853\,37195\,55139\,32521_6$

If the number π is *normal*, i.e., if all digits and digit combinations in its decimal expansion appear with the same limiting frequency, then every text ever written or yet to be written, as well as every word ever spoken or yet to be spoken, can be found coded in its sequence. The property of normality has not yet been proven, although it is suspected to hold. Does this mean that all wisdom is encoded in the simple circle? No. The property is nothing special: it also applies to the number 0.123456789101112131415161718192021... and many others. Can you specify a few examples? Challenge 153 s

By the way, in the graph of the exponential function e^x, the point $(0, 1)$ is the only point with two rational coordinates. If you imagine painting in blue all points on the plane with two rational coordinates, the plane would look quite bluish. Nevertheless, the graph goes through only one of these points and manages to avoid all the others.

Appendix B

NUMBERS AND VECTOR SPACES

> A mathematician is a machine that transforms coffee into theorems.
> Paul Erdős (b. 1913 Budapest, d. 1996 Warsaw)

Mathematical concepts can all be expressed in terms of 'sets' and 'relations.' Many fundamental concepts were presented in the last chapter. Why does mathematics, given this simple basis, grow into a passion for certain people? The following pages present a few more advanced concepts as simplyand vividly as possible, for all those who want to smell the passion for mathematics.

Ref. 154

In particular, in this appendix we shall introduce the simplest *algebraic* structures. The appendix in the next volume will present some more involved algebraic structures and the most important *topological* structures; the third basic type of mathematical structures, *order* structures, are not so important in physics.

Mathematicians are concerned not only with the exploration of concepts, but also with their *classification*. Whenever a new mathematical concept is introduced, mathematicians try to classify all the possible cases and types. This has been achieved most spectacularly for the different types of numbers, for finite simple groups, and for many types of spaces and manifolds.

Numbers as mathematical structures

Challenge 154 ny

> A person who can solve $x^2 - 92y^2 = 1$ in less than a year is a mathematician.
> Brahmagupta (b. 598 Sindh, d. 668) (implied: solve in *integers*)

We start with a short introduction to the vocabulary. Any mathematical system with the same basic properties as the *natural* numbers is called a *semi-ring*. Any mathematical system with the same basic properties as the *integers* is called a *ring*. (The term is due to David Hilbert. Both structures can also be finite rather than infinite.) More precisely, a *ring* $(R, +, \cdot)$ is a set R of elements with two binary operations, called *addition* and *multiplication*, usually written $+$ and $\cdot$ (the latter may simply be understood without notation), for which the following properties hold for all elements $a, b, c \in R$:

— R is a commutative group with respect to addition, i.e.
$a+b \in R,\ a+b = b+a,\ a+0 = a,\ a+(-a) = a-a = 0$ and $a+(b+c) = (a+b)+c$;
— R is closed under multiplication, i.e., $ab \in R$;

— multiplication is associative, i.e., $a(bc) = (ab)c$;
— distributivity holds, i.e., $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Many authors add the axiom
— a multiplicative unit exists, i.e., $1a = a1 = a$.
Defining properties such as these are called *axioms*. Note that axioms are *not* basic beliefs, as is often stated; axioms are the basic properties used in the definition of a concept: in this case, of a ring. With the last axiom, one also speaks of a *unital ring*.

A *semi-ring* is a set satisfying all the axioms of a ring, except that the existence of neutral and negative elements for addition is replaced by the weaker requirement that if $a + c = b + c$ then $a = b$. Sloppily, a semi-ring is a ring 'without' negative elements.

To incorporate division and define the rational numbers, we need another concept. A *field* K is a ring with

— a multiplicative identity 1, such that all elements a obey $1a = a$;
— at least one element different from zero; and most importantly
— a (multiplicative) inverse a^{-1} for every element $a \neq 0$.

A ring or field is said to be *commutative* if the multiplication is commutative. A non-commutative field is also called a *skew field*. Fields can be finite or infinite. (A field or a ring is characterized by its *characteristic* p. This is the smallest number of times one has to add 1 to itself to give zero. If there is no such number the characteristic is set to 0. p is always a prime number or zero.) All finite fields are commutative. In a field, all equations of the type $cx = b$ and $xc = b$ ($c \neq 0$) have solutions for x; there is a unique solution if $b \neq 0$. To sum up sloppily by focusing on the most important property, a field is a set of elements for which, together with addition, subtraction and multiplication, a *division* (by non-zero elements) is also defined. The *rational numbers* are the simplest field that incorporates the integers.

The system of the *real numbers* is the minimal extension of the rationals which is complete and totally ordered.*

However, the concept of 'number' is not limited to these examples. It can be general-
Ref. 155 ized in several ways. The simplest generalization is achieved by extending the real numbers to manifolds of more than one dimension.

* A set is mathematically *complete* if physicists call it continuous. More precisely, a set of numbers is *complete* if every non-empty subset that is bounded above has a least upper bound.

A set is *totally ordered* if there exists a binary relation $\leqslant$ between pairs of elements such that for all elements a and b

— if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$;
— if $a \leqslant b$ and $b \leqslant a$, then $a = b$;
— $a \leqslant b$ or $b \leqslant a$ holds.

In summary, a set is totally ordered if there is a binary relation that allows saying about any two elements which one is the predecessor of the other in a consistent way.

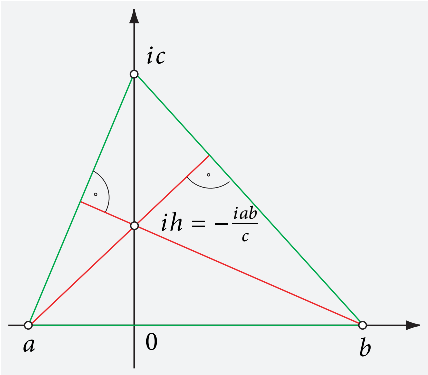

FIGURE 77 A property of triangles easily provable with complex numbers.

Complex numbers

A complex number is defined by $z = a + ib$, where a and b are real numbers, and i is a new symbol. Under multiplication, the generators of the complex numbers, 1 and i, obey

$$\begin{array}{c|cc} \cdot & 1 & i \\ \hline 1 & 1 & i \\ i & i & -1 \end{array} \qquad (114)$$

often summarized as $i = +\sqrt{-1}$.

The *complex conjugate* z^*, also written $\bar{z}$, of a complex number $z = a + ib$ is defined as $z^* = a - ib$. The *absolute value* $|z|$ of a complex number is defined as $|z| = \sqrt{zz^*} = \sqrt{z^*z} = \sqrt{a^2 + b^2}$. It defines a *norm* on the vector space of the complex numbers. From $|wz| = |w|\,|z|$ follows the *two-squares theorem*

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1 b_1 - a_2 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 \qquad (115)$$

valid for all real numbers a_i, b_i. It was already known, in its version for integers, to Diophantus of Alexandria.

Complex numbers can also be written as ordered pairs (a, A) of real numbers, with their addition defined as $(a, A) + (b, B) = (a + b, A + B)$ and their multiplication defined as $(a, A) \cdot (b, B) = (ab - AB, aB + bA)$. This notation allows us to identify the complex numbers with the *points* on a plane or, if we prefer, to *arrows* in a plane. Translating the definition of multiplication into geometrical language allows us to rapidly prove certain geometrical theorems, such as the one of Figure 77. Challenge 155 e

Complex numbers $a + ib$ can also be represented as 2×2 matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with} a, b \in \mathbb{R} \,. \qquad (116)$$

Matrix addition and multiplication then correspond to complex addition and multiplication. In this way, complex numbers can be represented by a special type of real matrix.

Challenge 156 s What is $|z|$ in matrix language?

The set $\mathbb{C}$ of complex numbers with addition and multiplication as defined above forms both a commutative two-dimensional field and a vector space over $\mathbb{R}$. In the field of complex numbers, quadratic equations $az^2 + bz + c = 0$ for an unknown z always have two solutions (for $a \neq 0$ and counting multiplicity).

Page 200 Challenge 157 ny

Complex numbers can be used to describe the points of a plane. A rotation around the origin can be described by multiplication by a complex number of unit length. Other two-dimensional quantities can also be described with complex numbers. Electrical engineers use complex numbers to describe quantities with phases, such as alternating currents or electrical fields in space.

Writing complex numbers of unit length as $\cos\theta + i\sin\theta$ is a useful method for remembering angle addition formulae. Since one has $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$, one can easily deduce formulae $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

Challenge 158 e

The complex exponential function is periodic in $2\pi i$; in other words, one has

$$\mathrm{e}^1 = \mathrm{e}^{1+2\pi i} . \tag{117}$$

If one uses this equation twice, one gets

$$\mathrm{e}^1 = (\mathrm{e}^{1+2\pi i})^{1+2\pi i} = \mathrm{e}^{(1+2\pi i)(1+2\pi i)} = \mathrm{e}^{1-4\pi^2+4\pi i} = \mathrm{e}^{1-4\pi^2} . \tag{118}$$

Challenge 159 e Oops, that would imply $\pi = 0$! What is wrong here?

By the way, there are exactly as many complex numbers as there are real numbers. Can you show this?

Challenge 160 s

Challenge 161 e The unit complex numbers form the group SO(2)=U(1).

> “Love is complex: it has real and imaginary parts.”
> Anonymous

Quaternions

The positions of the points on a line can be described by real numbers. Complex numbers can be used to describe the positions of the points of a plane. It is natural to try to generalize the idea of a number to higher-dimensional spaces. However, it turns out that no useful number system can be defined for *three*-dimensional space. A new number system, the *quaternions*, can be constructed which corresponds the points of *four*-dimensional space, but only if the commutativity of multiplication is sacrificed. No useful number system can be defined for dimensions other than 1, 2 and 4.

The quaternions were discovered by several mathematicians in the nineteenth century, among them Hamilton,* who studied them for much of his life. In fact, Maxwell's theory of electrodynamics was formulated in terms of quaternions before three-dimensional vectors were used.

Ref. 156

Vol. V, page 278 Under multiplication, the quaternions $\mathbb{H}$ form a 4-dimensional algebra over the reals

* William Rowan Hamilton (b. 1805 Dublin, d. 1865 Dunsink), Irish child prodigy and famous mathematician, named the quaternions after an expression from the Vulgate (Acts. 12: 4).

with a basis $1, i, j, k$ satisfying

$$\begin{array}{c|cccc} \cdot & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} \; . \tag{119}$$

These relations are also often written $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The quaternions $1, i, j, k$ are also called *basic units* or *generators*. The lack of symmetry across the diagonal of the table shows the non-commutativity of quaternionic multiplication. With the quaternions, the idea of a non-commutative product appeared for the first time in mathematics. However, the multiplication of quaternions *is* associative. As a consequence of non-commutativity, polynomial equations in quaternions have many more solutions than in complex numbers: just search for all solutions of the equation $X^2 + 1 = 0$ to convince yourself of it. Challenge 162 s

Every quaternion X can be written in the form

$$X = x_0 + x_1 i + x_2 j + x_3 k = x_0 + \boldsymbol{v} = (x_0, x_1, x_2, x_3) = (x_0, \boldsymbol{v}) \; , \tag{120}$$

where x_0 is called the *scalar* part and $\boldsymbol{v}$ the *vector* part. The multiplication is thus defined as $(x, \boldsymbol{v})(y, \boldsymbol{w}) = (xy - \boldsymbol{v}\cdot\boldsymbol{w}, x\boldsymbol{w} + y\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{w})$. The multiplication of two general quaternions can be written as

$$\begin{aligned}(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (&a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \, , \; a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2, \\ &a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2, a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2) \, .\end{aligned} \tag{121}$$

The conjugate quaternion $\overline{X}$ is defined as $\overline{X} = x_0 - \boldsymbol{v}$, so that $\overline{XY} = \overline{Y}\,\overline{X}$. The *norm* $|X|$ of a quaternion X is defined as $|X|^2 = X\overline{X} = \overline{X}X = x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 + \boldsymbol{v}^2$. The norm is multiplicative, i.e., $|XY| = |X|\,|Y|$.

Unlike complex numbers, every quaternion is related to its complex conjugate by

$$\overline{X} = -\tfrac{1}{2}(X + iXi + jXj + kXk) \; . \tag{122}$$

No relation of this type exists for complex numbers. In the language of physics, a complex number and its conjugate are independent variables; for quaternions, this is not the case. As a result, functions of quaternions are less useful in physics than functions of complex variables.

The relation $|XY| = |X|\,|Y|$ implies the *four-squares theorem*

$$\begin{aligned}&(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) \\ &= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2 + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)^2 \\ &+ (a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4)^2 + (a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)^2\end{aligned} \tag{123}$$

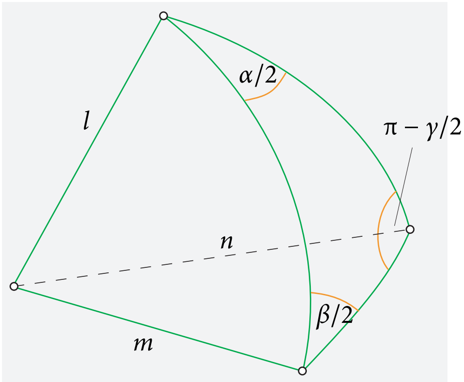

FIGURE 78 Combinations of rotations.

valid for all real numbers a_i and b_i, and thus also for any set of eight integers. It was discovered in 1748 by Leonhard Euler (1707–1783) when trying to prove that each integer is the sum of four squares. (That fact was proved only in 1770, by Joseph Lagrange.)

Hamilton thought that a quaternion with zero scalar part, which he simply called a *vector* (a term which he invented), could be identified with an ordinary three-dimensional translation vector; but this is wrong. Such a quaternion is now called a *pure*, or *homogeneous*, or *imaginary* quaternion. The product of two pure quaternions $V = (0, \boldsymbol{v})$ and $W = (0, \boldsymbol{w})$ is given by $VW = (-\boldsymbol{v} \cdot \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w})$, where $\cdot$ denotes the scalar product and $\times$ denotes the vector product. Note that any quaternion can be written as the ratio of two pure quaternions.

In reality, a pure quaternion $(0, \boldsymbol{v})$ does not behave like a translation vector under coordinate transformations; in fact, a pure quaternion represents a *rotation* by the angle
Challenge 163 ny π or 180° around the axis defined by the direction $\boldsymbol{v} = (v_x, v_y, v_z)$.

It turns out that in three-dimensional space, a *general* rotation about the origin can be described by a *unit* quaternion Q, also called a *normed* quaternion, for which $|Q| = 1$. Such a quaternion can be written as $(\cos\theta/2, \boldsymbol{n}\sin\theta/2)$, where $\boldsymbol{n} = (n_x, n_y, n_z)$ is the normed vector describing the direction of the rotation axis and θ is the rotation angle. Such a unit quaternion $Q = (\cos\theta/2, \boldsymbol{n}\sin\theta/2)$ rotates a pure quaternion $V = (0, \boldsymbol{v})$ into another pure quaternion $W = (0, \boldsymbol{w})$ given by

$$W = QVQ^* . \qquad (124)$$

Thus, if we use pure quaternions such as V or W to describe positions, we can use unit quaternions to describe rotations and to calculate coordinate changes. The concatenation of two rotations is then given by the product of the corresponding unit quaternions. Indeed, a rotation by an angle α about the axis $\boldsymbol{l}$ followed by a rotation by an angle β about the axis $\boldsymbol{m}$ gives a rotation by an angle γ about the axis $\boldsymbol{n}$, with the values determined by

$$(\cos\gamma/2, \sin\gamma/2\boldsymbol{n}) = (\cos\beta/2, \sin\beta/2\boldsymbol{m})(\cos\alpha/2, \sin\alpha/2\boldsymbol{l}) . \qquad (125)$$

One way to show the result graphically is given in Figure 78. By drawing a triangle on a unit sphere, and taking care to remember the factor 1/2 in the angles, the combination

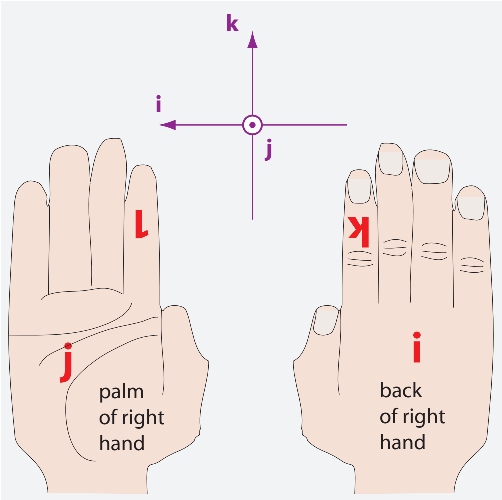

FIGURE 79 The top and back of the right hand, and the quaternions.

of two rotations can be simply determined.

The interpretation of quaternions as rotations is also illustrated, in a somewhat different way, in the motion of any hand. To see this, take a green marker and write the letters 1, i, j and k on your hand as shown in Figure 79. Defining the three possible 180° rotation axes as shown in the figure and taking concatenation as multiplication, the motion of the right hand follows the same 'laws' as those of pure unit quaternions. (One still needs to distinguish $+i$ and $-i$, and the same for the other units, by the sense of the arm twist. And the result of a multiplication is that letter that can be read by a person facing you.) You can show that $i^2 = j^2 = k^2 = -1$, that $i^4 = 1$, and all other quaternion relations.) The model also shows that the rotation angle of the arm is half the rotation angle of the corresponding quaternion. In other words, quaternions can be used to describe the belt trick, if the multiplication VW of two quaternions is taken to mean that rotation V is performed *after* rotation W. Quaternions, like human hands, thus behave like a spin 1/2 particle. Quaternions and spinors are isomorphic.

Ref. 157
Challenge 164 e
Challenge 165 s
Page 109

The reason for the half-angle behaviour of rotations can be specified more precisely using mathematical language. The rotations in three dimensions around a point form the 'special orthogonal group' in three dimensions, which is called SO(3). But the motions of a hand attached to a shoulder via an arm form a different group, isomorphic to the Lie group SU(2). The difference is due to the appearance of half angles in the parametrization of rotations; indeed, the above parametrizations imply that a rotation by 2π corresponds to a multiplication by −1. Only in the twentieth century was it realized that there exist fundamental physical observables that behaves like hands attached to arms: they are called *spinors*. More on spinors can be found in the section on permutation symmetry, where belts are used as an analogy as well as arms. In short, the group SU(2) of the quaternions is the *double cover* of the rotation group SO(3).

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Page 109
Ref. 158

The simple representation of rotations and positions with quaternions is used by computer programmes in robotics, in astronomy and in flight simulation. In the software used to create three-dimensional images and animations, visualization software, quaternions are often used to calculate the path taken by repeatedly reflected light rays and thus give surfaces a realistic appearance.

The algebra of the quaternions is the only associative, non-commutative, finite-dimensional normed algebra with an identity over the field of real numbers. Quaternions form a non-commutative field, i.e., a skew field, in which the inverse of a quaternion X is $\overline{X}/|X|$. We can therefore define division of quaternions (while being careful to distinguish XY^{-1} and $Y^{-1}X$). Therefore quaternions are said to form a *division algebra*. In fact, the quaternions $\mathbb{H}$, the complex numbers $\mathbb{C}$ and the reals $\mathbb{R}$ are the only three finite-dimensional associative division algebras. In other words, the skew-field of quaternions is the only finite-dimensional real associative non-commutative algebra without divisors of zero. The *centre* of the quaternions, i.e., the set of quaternions that commute with all other quaternions, is just the set of real numbers.

Quaternions can be represented as matrices of the form

$$\begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \text{ with } A, B \in \mathbb{C}, \quad \text{or as} \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \text{ with } a, b, c, d \in \mathbb{R}, \tag{126}$$

where $A = a + ib, B = c + id$ and the quaternion X is $X = A + Bj = a + ib + jc + kd$; matrix addition and multiplication then corresponds to quaternionic addition and multiplication.

The generators of the quaternions can be realized as

$$1 : \sigma_0 \quad , \quad i : -i\sigma_1 \quad , \quad j : -i\sigma_2 \quad , \quad k : -i\sigma_3 \tag{127}$$

where the σ_n are the Pauli spin matrices.*

Real 4×4 representations are not unique, as the alternative representation

$$\begin{pmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{pmatrix} \tag{129}$$

Challenge 166 ny

shows; however, no representation by 3×3 matrices is possible.

These matrices contain real and complex elements, which pose no special problems. In contrast, when matrices with quaternionic elements are constructed, care has to be taken, because quaternionic multiplication is not commutative, so that simple relations such as $\mathrm{tr}AB = \mathrm{tr}BA$ are *not* generally valid.

* The *Pauli spin matrices* are the complex Hermitean matrices

$$\sigma_0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{128}$$

all of whose eigenvalues are ± 1; they satisfy the relations $[\sigma_i, \sigma_k]_+ = 2\,\delta_{ik}$ and $[\sigma_i, \sigma_k] = 2i\,\varepsilon_{ikl}\,\sigma_l$. The linear combinations $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$ are also frequently used. By the way, another possible representation of the quaternions is $i : i\sigma_3, j : i\sigma_2, k : i\sigma_1$.

What can we learn from quaternions about the description of nature? First of all, we see that binary rotations are similar to positions, and thus to translations: all are represented by 3-vectors. Are rotations the basic operations of nature? Is it possible that translations are only 'shadows' of rotations? The connection between translations and rotations is investigated in the last volume of our mountain ascent.

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When Maxwell wrote down his equations of electrodynamics, he used quaternion notation. (The now usual 3-vector notation was introduced later by Hertz and Heaviside.) The equations can be written in various ways using quaternions. The simplest is achieved when one keeps a distinction between $\sqrt{-1}$ and the units i, j, k of the quaternions. One then can write all of electrodynamics in a single equation:

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Ref. 156

Challenge 167 s

$$\mathrm{d}F = -\frac{Q}{\varepsilon_0} \tag{130}$$

where F is the generalized electromagnetic field and Q the generalized charge. These are defined by

$$\begin{aligned} F &= E + \sqrt{-1}\,cB \\ E &= iE_x + jE_y + kE_z \\ B &= iB_x + jB_y + kB_z \\ \mathrm{d} &= \delta + \sqrt{-1}\,\partial_t/c \\ \delta &= i\partial_x + j\partial_y + k\partial_z \\ Q &= \rho + \sqrt{-1}\,J/c \end{aligned} \tag{131}$$

where the fields E and B and the charge distributions ρ and J have the usual meanings. The content of equation (130) for the electromagnetic field is exactly the same as the usual formulation.

Despite their charm, quaternions do not seem to be ready for the reformulation of special relativity; the main reason for this is the sign in the expression for their norm. Therefore, relativity and space-time are usually described using real numbers.

Octonions

In the same way that quaternions are constructed from complex numbers, *octonions* can be constructed from quaternions. They were first investigated by Arthur Cayley (1821–1895). Under multiplication, *octonions* (or *octaves*) are the elements of an eight-dimensional algebra over the reals with the generators $1, i_n$ with $n = 1 \ldots 7$ satisfying

$$
\begin{array}{c|cccccccc}
\cdot & 1 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\
\hline
1 & 1 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\
i_1 & i_1 & -1 & i_3 & -i_2 & i_5 & -i_4 & i_7 & -i_6 \\
i_2 & i_2 & -i_3 & -1 & i_1 & -i_6 & i_7 & i_4 & -i_5 \\
i_3 & i_3 & i_2 & -i_1 & -1 & i_7 & i_6 & -i_5 & -i_4 \\
i_4 & i_4 & -i_5 & i_6 & -i_7 & -1 & i_1 & -i_2 & i_3 \\
i_5 & i_5 & i_4 & -i_7 & -i_6 & -i_1 & -1 & i_3 & i_2 \\
i_6 & i_6 & -i_7 & -i_4 & i_5 & i_2 & -i_3 & -1 & i_1 \\
i_7 & i_7 & i_6 & i_5 & i_4 & -i_3 & -i_2 & -i_1 & -1
\end{array}
\tag{132}
$$

479 other, equivalent multiplication tables are also possible. This algebra is called the *Cayley algebra*; it has an identity and a unique division. The algebra is non-commutative, and also non-associative. It is, however, *alternative*, meaning that for all elements x and y, one has $x(xy) = x^2 y$ and $(xy)y = xy^2$: a property somewhat weaker than associativity. It is the only 8-dimensional real alternative algebra without zero divisors. Because it is not associative, the set $\mathbb{O}$ of all octonions does not form a field, nor even a ring, so that the old designation of 'Cayley numbers' has been abandoned. The octonions are the most general hypercomplex 'numbers' whose norm is multiplicative. Its generators obey $(i_n i_m) i_l = \pm i_n (i_m i_l)$, where the minus sign, which shows the non-associativity, is valid for combinations of indices, such as 1-2-4, which are not quaternionic.

Octonions can be represented as matrices of the form

$$
\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \text{where } A, B \in \mathbb{H} \,, \quad \text{or as real } 8 \times 8 \text{ matrices.} \tag{133}
$$

Matrix multiplication then gives the same result as octonionic multiplication.

The relation $|wz| = |w|\,|z|$ allows one to deduce the impressive *eight-squares theorem*

$$
\begin{aligned}
(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2)&(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2) \\
= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 - a_5 b_5 - a_6 b_6 - a_7 b_7 - a_8 b_8)^2 & \\
+ (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 + a_5 b_6 - a_6 b_5 + a_7 b_8 - a_8 b_7)^2 & \\
+ (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 - a_5 b_7 + a_6 b_8 + a_7 b_5 - a_8 b_6)^2 & \\
+ (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 + a_5 b_8 + a_6 b_7 - a_7 b_6 - a_8 b_5)^2 & \\
+ (a_1 b_5 - a_2 b_6 + a_3 b_7 - a_4 b_8 + a_5 b_1 + a_6 b_2 - a_7 b_3 + a_8 b_4)^2 & \\
+ (a_1 b_6 + a_2 b_5 - a_3 b_8 - a_4 b_7 - a_5 b_2 + a_6 b_1 + a_7 b_4 + a_8 b_3)^2 & \\
+ (a_1 b_7 - a_2 b_8 - a_3 b_5 + a_4 b_6 + a_5 b_3 - a_6 b_4 + a_7 b_1 + a_8 b_2)^2 & \\
+ (a_1 b_8 + a_2 b_7 + a_3 b_6 + a_4 b_5 - a_5 b_4 - a_6 b_3 - a_7 b_2 + a_8 b_1)^2 &
\end{aligned}
\tag{134}
$$

valid for all real numbers a_i and b_i and thus in particular also for all integers. (There are many variations of this expression, with different possible sign combinations.) The theorem was discovered in 1818 by Carl Ferdinand Degen (1766–1825), and then rediscovered in 1844 by John Graves and in 1845 by Arthur Cayley. There is no generalization to higher numbers of squares, a fact proved by Adolf Hurwitz (1859–1919) in 1898.

The octonions can be used to show that a vector product can be defined in more than three dimensions. A *vector product* or *cross product* is an operation × satisfying

$$\begin{aligned} u \times v &= -v \times u && \textit{anticommutativity} \\ (u \times v)w &= u(v \times w) && \textit{exchange rule.} \end{aligned} \tag{135}$$

Using the definition

$$X \times Y = \frac{1}{2}(XY - YX)\,, \tag{136}$$

the ×-products of imaginary quaternions, i.e., of quaternions of the type $(0, \boldsymbol{u})$, are again imaginary, and correspond to the usual vector product, thus fulfilling (135). Interestingly,
Ref. 155 it is possible to use definition (136) for *octonions* as well. In that case, the product of
Challenge 168 e imaginary octonions is also imaginary, and (135) is again satisfied. In fact, this is the only other non-trivial example of a vector product. Thus a vector product exists only in three and in seven dimensions.

Other types of numbers

The process of constructing new systems of hypercomplex 'numbers' or real algebras by 'doubling' a given one can be continued ad infinitum. However, octonions, *sedenions* and all the following doublings are neither rings nor fields, but only non-associative algebras with unity. Other finite-dimensional algebras with unit element over the reals, once called hypercomplex 'numbers', can also be defined: they include the so-called 'dual numbers', 'double numbers', 'Clifford–Lifshitz numbers' etc. They play no special role in physics.

Mathematicians have also defined number fields which have 'one and a bit' dimensions, such as algebraic number fields. There is also a generalization of the concept of
Ref. 159 integers to the complex domain: the *Gaussian integers*, defined as $n + im$, where n and m are ordinary integers. Gauss even defined what are now known as *Gaussian primes*. (Can
Challenge 169 s you find out how?) They are not used in the description of nature, but are important in number theory.

Physicists used to call quantum-mechanical operators 'q-numbers.' But this term has now fallen out of fashion.

Another way in which the natural numbers can be extended is to include numbers
Ref. 160 larger infinite numbers. The most important such classes of *transfinite number* are the
Vol. III, page 202 *ordinals*, the *cardinals* and the *surreals*. The ordinals are essentially an extension of the integers beyond infinity, whereas the surreals are a continuous extension of the reals, also beyond infinity. Loosely speaking, among the transfinites, the ordinals have a similar role as the integers have among the reals; the surreals fill in all the gaps between the ordinals, like the reals do for integers. Interestingly, many series that diverge in $\mathbb{R}$ converge in the
Challenge 170 ny surreals. Can you find one example?

The surreals include infinitely small numbers, as do the numbers of *nonstandard an-*

Ref. 155 *alysis*, also called *hyperreals*. In both number systems, in contrast to real numbers, the numbers 1 and 0.999 999 9... (where an infinite, but hyperfinite string of nines is implied) do not coincide, but are separated by infinitely many other numbers. Vol. III, page 204

Vector spaces

Vector spaces, also called linear spaces, are mathematical generalizations of certain aspects of the intuitive three-dimensional space. A set of elements any two of which can be added together and any one of which can be multiplied by a number is called a vector space, if the result is again in the set and the usual rules of calculation hold.

More precisely, a *vector space* over a number field K is a set of elements, called *vectors*, for which a vector addition and a *scalar multiplication* is defined, such that for all vectors a, b, c and for all numbers s and r from K one has

$$\begin{aligned} (a+b)+c = a+(b+c) &= a+b+c && \textit{associativity of vector addition} \\ n+a &= a && \textit{existence of null vector} \\ (-a)+a &= n && \textit{existence of negative vector} \\ 1a &= a && \textit{regularity of scalar multiplication} \\ (s+r)(a+b) &= sa+sb+ra+rb && \textit{complete distributivity of scalar multiplication} \end{aligned} \tag{137}$$

If the field K, whose elements are called *scalars* in this context, is taken to be the real (or complex, or quaternionic) numbers, one speaks of a real (or complex, or quaternionic) vector space. Vector spaces are also called *linear vector spaces* or simply *linear spaces*.

The complex numbers, the set of all real functions defined on the real line, the set of all polynomials, the set of matrices with a given number of rows and columns, all form vector spaces. In mathematics, a vector is thus a more general concept than in physics. Challenge 171 ny (What is the simplest possible mathematical vector space?)

In physics, the term 'vector' is reserved for elements of a more specialized type of vector space, namely normed inner product spaces. To define these, we first need the concept of a metric space.

A *metric space* is a set with a metric, i.e., a way to define distances between elements. A real function $d(a,b)$ between elements is called a metric if

$$\begin{aligned} d(a,b) &\geqslant 0 && \textit{positivity of metric} \\ d(a,b)+d(b,c) &\geqslant d(a,c) && \textit{triangle inequality} \\ d(a,b)=0 \quad \text{if and only if} \quad a &= b && \textit{regularity of metric} \end{aligned} \tag{138}$$

A non-trivial example is the following. We define a special distance d between cities. If the two cities lie on a line going through Paris, we use the usual distance. In all other cases, we define the distance d by the shortest distance from one to the other travelling via Paris. This strange method defines a metric between all cities in France. Challenge 172 s

A *normed* vector space is a linear space with a norm, or 'length', associated to each a vector. A *norm* is a non-negative number $\|a\|$ defined for each vector a with the properties

$$\begin{aligned} \|ra\| &= |r|\,\|a\| && \textit{linearity of norm} \\ \|a+b\| &\leqslant \|a\| + \|b\| && \textit{triangle inequality} \\ \|a\| = 0 \quad \text{only if} \quad a &= 0 && \textit{regularity} \end{aligned} \tag{139}$$

Challenge 173 ny Usually there are many ways to define a norm for a given space. Note that a norm can always be used to define a metric by setting

$$d(a,b) = \|a-b\| \tag{140}$$

so that all normed spaces are also metric spaces. This is the natural distance definition (in contrast to unnatural ones like that between French cities).

The norm is often defined with the help of an inner product. Indeed, the most special class of linear spaces are the *inner product spaces*. These are vector spaces with an *inner product*, also called *scalar product* · (not to be confused with the scalar multiplication!) which associates a number to each pair of vectors. An inner product space over $\mathbb{R}$ satisfies

$$\begin{aligned} a\cdot b &= b\cdot a && \textit{commutativity of scalar product} \\ (ra)\cdot(sb) &= rs(a\cdot b) && \textit{bilinearity of scalar product} \\ (a+b)\cdot c &= a\cdot c + b\cdot c && \textit{left distributivity of scalar product} \\ a\cdot(b+c) &= a\cdot b + a\cdot c && \textit{right distributivity of scalar product} \\ a\cdot a &\geqslant 0 && \textit{positivity of scalar product} \\ a\cdot a = 0 \quad \text{if and only if} \quad a &= 0 && \textit{regularity of scalar product} \end{aligned} \tag{141}$$

for all vectors a, b, c and all scalars r, s. A *real* inner product space of finite dimension is also called a *Euclidean* vector space. The set of all velocities, the set of all positions, or the set of all possible momenta form such spaces.

An inner product space over $\mathbb{C}$ satisfies*

$$\begin{aligned} a\cdot b = \overline{b\cdot a} &= \bar{b}\cdot\bar{a} && \textit{Hermitean property} \\ (ra)\cdot(sb) &= r\bar{s}(a\cdot b) && \textit{sesquilinearity of scalar product} \\ (a+b)\cdot c &= a\cdot c + b\cdot c && \textit{left distributivity of scalar product} \\ a\cdot(b+c) &= a\cdot b + a\cdot c && \textit{right distributivity of scalar product} \\ a\cdot a &\geqslant 0 && \textit{positivity of scalar product} \\ a\cdot a = 0 \quad \text{if and only if} \quad a &= 0 && \textit{regularity of scalar product} \end{aligned} \tag{142}$$

for all vectors a, b, c and all scalars r, s. A *complex* inner product space (of finite dimension) is also called a *unitary* or *Hermitean* vector space. If the inner product space is *complete*, it is called, especially in the infinite-dimensional complex case, a *Hilbert space*. Page 190

* Two inequivalent forms of the sesquilinearity axiom exist. The other is $(ra)\cdot(sb) = \bar{r}s(a\cdot b)$. The term *sesquilinear* is derived from Latin and means for 'one-and-a-half-linear'.

The space of all possible states of a quantum system forms a Hilbert space.

All inner product spaces are also metric spaces, and thus normed spaces, if the metric is defined by

$$d(a,b) = \sqrt{(a-b)\cdot(a-b)}\ . \tag{143}$$

Only in the context of an inner product spaces we can speak about angles (or phase differences) between vectors, as we are used to in physics. Of course, like in normed spaces, inner product spaces also allows us to speak about the length of vectors and to define a *basis*, the mathematical concept necessary to define a coordinate system.

The *dimension* of a vector space is the number of linearly independent basis vectors. Challenge 174 ny Can you define these terms precisely?

A *Hilbert space* is a real or complex inner product space that is also a *complete* metric space. In other terms, in a Hilbert space, distances vary continuously and behave as naively expected. Hilbert spaces can have an infinite number of dimensions.

Challenge 175 ny Which vector spaces are of importance in physics?

Mathematical curiosities and fun challenges

Mathematics provides many counter-intuitive results. Reading a book on the topic, such as Bernard R. Gelbaum & John M. H. Olmsted, *Theorems and Counterexamples in Mathematics*, Springer, 1993, can help you sharpen your mind.

* *

The distinction between one, two and three dimensions is blurred in mathematics. This is well demonstrated in the text Hans Sagan, *Space Filling Curves*, Springer Verlag, 1994.

* *

Show that two operators A and B obey

$$\mathrm{e}^A\mathrm{e}^B = \exp(A + B + \frac{1}{2}[A,B] \tag{144}$$
$$+ \frac{1}{12}[[A,B],B] - \frac{1}{12}[[A,B],A] \tag{145}$$
$$- \frac{1}{48}[B,[A,[A,B]]] - \frac{1}{48}[A,[B,[A,B]]] \tag{146}$$
$$+ \ldots \tag{147}$$

for most operators A and B. This result is often called the *Baker–Campbell–Hausdorff formula* or the *BCH formula*.

CHALLENGE HINTS AND SOLUTIONS

> Never make a calculation before you know the answer.
>
> John Wheeler's motto

Challenge 1, page 9: Do not hesitate to be demanding and strict. The next edition of the text will benefit from it.

Challenge 2, page 13: Classical physics fails in explaining any material property, such as colour or softness. Material properties result from nature's interactions; they are inevitably quantum. Explanations of material properties require, without exception, the use of particles and their quantum properties.

Challenge 3, page 15: Classical physics allows any observable to change *smoothly* with time. There is no minimum value for any observable physical quantity.

Challenge 4, page 17: The higher the mass, the smaller the motion fuzziness induced by the quantum of action, because action is mass times speed times distance: For a large mass, the speed and distance variations are small.

Challenge 5, page 17: The simplest time is $\sqrt{2G\hbar/c^5}$. The factor 2 is obviously not fixed; it is changed later on. With the correct factor 4, the time is the shortest time measurable in nature.

Challenge 6, page 17: The electron charge is special to the electromagnetic interactions; it does not take into account the nuclear interactions or gravity. It is unclear why the length defined with the elementary charge e should be of importance for neutral systems or for the vacuum. On the other hand, the quantum of action $\hbar$ is valid for all interactions and all observations. However, we can also argue that the two options to define a fundamental length – with the quantum of action and with the quantum of charge – are not too different, as the electron charge is related to the quantum of action by $e = \sqrt{4\pi\varepsilon_0 \alpha c\hbar}$. The two length scales defined by the two options differ only by a factor near 11.7.

Challenge 8, page 17: On purely dimensional grounds, the radius of an atom must be

$$r \approx \frac{\hbar^2 4\pi\varepsilon_0}{me^2} , \tag{148}$$

which is about 160 nm. Indeed, this guess is excellent: it is just π times the Bohr radius.

Challenge 9, page 18: Due to the quantum of action, atoms in all people, be they giants or dwarfs, have the same size. This implies that giants cannot exist, as was shown already by Galileo. The argument is based on the given strength of materials; and a same strength everywhere is equivalent to the same properties of atoms everywhere. That dwarfs cannot exist is due to a similar reason; nature is not able to make people smaller than usual (except in the womb) as this would require smaller atoms.

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Challenge 12, page 23: A disappearance of a mass m in a time Δt is an action change $mc^2\Delta t$. That is much larger than $\hbar$ for all objects of everyday life.

Challenge 14, page 24: Tunnelling of a lion would imply action values S of the order of $S = 100\,\mathrm{kgm^2/s} \gg \hbar$. This cannot happen spontaneously.

Challenge 16, page 25: Yes! Many beliefs and myths are due to the neglect of quantum effects.

Challenge 17, page 25: Continuous flow is in contrast to the fuzziness of motion induced by the quantum of action.

Challenge 18, page 26: The impossibility of following two particles along their path appears when their mutual distance d is smaller than their position indeterminacy due to their relative momentum p, thus when $d < \hbar/p$. Check the numbers with electrons, atoms, molecules, bacteria, people and galaxies.

Challenge 19, page 26: Also photons are indistinguishable. See page 53.

Challenge 21, page 30: The total angular momentum counts, including the orbital angular momentum. The orbital angular momentum $\boldsymbol{L}$ is given, using the radius and the linear momentum, $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$.

Challenge 22, page 30: Yes, we could have!

Challenge 23, page 31: That is just the indeterminacy relation. Bohr expanded this idea to all sort of other pairs of concepts, more in the philosophical domain, such as clarity and precision of explanations: both cannot be high at the same time.

Challenge 24, page 31: Growth is not proportional to light intensity or to light frequency, but shows both intensity and frequency thresholds. That is a quantum effects.

Challenge 25, page 31: All effects mentioned above, such as tunnelling, interference, decay, transformation, non-emptiness of the vacuum, indeterminacy and randomness, are also observed in the nuclear domain.

Challenge 26, page 31: This is not evident from what was said so far, but it turns out to be correct. In fact, there is no other option, as you will see when you try to find one.

Challenge 28, page 33: The big bang cannot have been an event, for example.

Challenge 31, page 38: Charged photons would be deflected by electric of magnetic fields; in particular, they would not cross undisturbed. This is not observed. Massive photons would be deflected by masses, such as the Sun, much more than is observed.

Challenge 35, page 41: Photons are elementary because they realize the minimum action, because they cannot decay, because they cannot be deformed or split, because they have no mass, no electric charge and no other quantum number, and because they appear in the Lagrangian of quantum electrodynamics.

Challenge 39, page 48: To be observable to the eye, the interference fringes need to be visible for around 0.1 s. That implies a maximum frequency difference between the two beams of around 10 Hz. This is achievable only if either a single beam is split into two or if the two beams come from high-precision, stabilized lasers.

Ref. 161 **Challenge 54**, page 59: The calculation is not easy, but not too difficult either. For an initial orientation *close* to the vertical, the fall time T turns out to be

$$T = \frac{1}{2\pi} T_0 \ln \frac{8}{\alpha} \tag{149}$$

where α is the starting angle, and a fall through angle π is assumed. Here T_0 is the oscillation time of the pencil for small angles. (Can you determine it?) The indeterminacy relation for the tip of the pencil yields a *minimum* starting angle, because the momentum indeterminacy cannot be made arbitrarily large. You should be able to provide an upper limit. Once this angle is known, you can calculate the maximum time.

Challenge 55, page 61: Use the temperature to calculate the average kinetic energy, and thus the average speed of atoms.

Challenge 56, page 61: The atoms cannot be fully distinguished; they form a state of matter with peculiar properties, called a *condensate*. The condensate is not at rest either; but due to its large mass, its fluctuations are greatly reduced, compared to those of a single atom.

Challenge 58, page 65: Only variables whose product has the same units as physical action – Js – can be complementary to each other.

Challenge 59, page 65: Use $\Delta E < E$ and $a\ \Delta t < c$.

Challenge 68, page 81: Terabyte chips would need to have small memory cells. Small cells imply thin barriers. Thin barriers imply high probabilities for tunnelling. Tunnelling implies lack of memory.

Challenge 75, page 91: The difficulties to see hydrogen atoms are due to their small size and their small number of electrons. As a result, hydrogen atoms produce only weak contrasts in X-ray images. For the same reasons it is difficult to image them using electrons; the Bohr radius of hydrogen is only slightly larger than the electron Compton wavelength.

For the first time, in 2008, a research team claimed to have imaged hydrogen atoms adsorbed on graphene with the help of a transmission electron microscope. For details, see J. C. Meyer, C. O. Grit, M. F. Crommle & A. Zetti, Imaging and dynamics of light atoms and molecules on graphene, *Nature* **454**, pp. 319–322, 2008. However, it seems that the report has not been confirmed by another group yet.

Challenge 78, page 92: $r = 86$ pm, thus $T = 12$ eV. That compares to the actual value of 13.6 eV. The trick for the derivation of the formula is to use $< \psi|r_x^2|\psi >= \frac{1}{3} < \psi|\boldsymbol{rr}|\psi >$, a relation valid for states with no orbital angular momentum. It is valid for all coordinates and also for the three momentum observables, as long as the system is non-relativistic.

Challenge 80, page 93: Point particles cannot be marked; nearby point particles cannot be distinguished, due to the quantum of action.

Challenge 81, page 93: The solution is *two* gloves. In other words, if two men and two women want to make love without danger and , they need only *two* condoms. You can deduce the procedure by yourself.

Challenge 85, page 94: The Sackur–Tetrode formula is best deduced in the following way. We start with an ideal monoatomic gas of volume V, with N particles, and total energy U. In phase space, *state sum* Z is given by

$$Z = \frac{V^N}{N!} \frac{1}{\Lambda^{3N}} . \tag{150}$$

We use Stirling's approximation $N! \approx N^N/e^N$, and the definition of the entropy as $S = \partial(kT \ln Z)/\partial T$. Inserting the definition of Λ, this gives the Sackur–Tetrode equation.

Challenge 86, page 96: For a large number of particles, the interaction energy will introduce errors. For very large numbers, the gravitational binding energy will do so as well.

Challenge 88, page 97: To write two particles on paper, one has to distinguish them, even if the distinction is arbitrary.

Challenge 89, page 98: Trees, like all macroscopic objects, have a spin value that depends on their angular momentum. Being classical objects whose phase can be observed, the spin value is uncertain. (But even large multiples of $\hbar$ are too small to be measurable in everyday life.) Generally speaking, trees, mountains and people are spin 1 objects. The spin 1 value implies that these objects are unchanged after a full rotation. (How does a block of silver, made of an odd number of silver atoms, each with spin 1/2, relate to this answer?)

Challenge 93, page 103: Twins differ in the way their intestines are folded, in the lines of their hands and other skin folds. Sometimes, but not always, features like black points on the skin are mirror inverted on the two twins.

Challenge 99, page 112: Three.

Challenge 100, page 113: Angels can be distinguished by name, can talk and can sing; thus they are made of a large number of fermions. In fact, many angels are human sized, so that they do not even fit on the tip of a pin.

Challenge 108, page 118: Ghosts, like angels, can be distinguished by name, can talk and can be seen; thus they contain fermions. However, they can pass through walls and they are transparent; thus they cannot be made of fermions, but must be images, made of bosons. That is a contradiction.

Challenge 110, page 122: The loss of non-diagonal elements leads to an increase in the diagonal elements, and thus of entropy.

Challenge 113, page 127: The energy speed is given by the advancement of the outer two tails; that speed is never larger than the speed of light.

Challenge 114, page 129: No, as taking a photo implies an interaction with a bath, which would destroy the superposition.

Challenge 115, page 130: A photograph requires illumination; illumination is a macroscopic electromagnetic field; a macroscopic field is a bath; a bath implies decoherence; decoherence destroys superpositions.

Challenge 118, page 131: It depends. They can be due to interference or to intensity sums. In the case of radio the effect is clearer. If at a particular frequency the signals changes periodically from one station to another, one has a genuine interference effect.

Challenge 120, page 131: Such a computer requires clear phase relations between components; such phase relations are extremely sensitive to outside disturbances. At present, they do not hold longer than a microsecond, whereas long computer programs require minutes and hours to run.

Challenge 121, page 131: A record is an effect of a process that must be *hard to reverse or undo*. The traces of a broken egg are easy to clean on a large glass plate, but hard in the wool of a sheep. Broken teeth, torn clothes, or scratches on large surfaces are good records. Forensic scientists know many additional examples.

Challenge 124, page 138: Any other bath also does the trick, such as the atmosphere, sound vibrations, electromagnetic fields, etc.

Challenge 125, page 139: The Moon is in contact with baths like the solar wind, falling meteorites, the electromagnetic background radiation of the deep universe, the neutrino flux from the Sun, cosmic radiation, etc.

Challenge 126, page 140: Spatially periodic potentials have the property. Decoherence then leads to momentum diagonalization.

Challenge 138, page 166: This is a trick question. A change in α requires a change in c, $\hbar$, e or ε_0. None of these changes is possible or observable, as all our measurement apparatus are based on these units. Speculations about change of α, despite their frequency in the press and in scientific journals, are idle talk.

Challenge 139, page 166: A change of physical units such that $\hbar = c = e = 1$ would change the value of ε_0 in such a way that $4\pi\varepsilon_0 = 1/\alpha = 137.036...$

Challenge 145, page 178: Planck limits can be exceeded for extensive observables for which many particle systems can exceed single particle limits, such as mass, momentum, energy or electrical resistance.

Challenge 147, page 181: Do not forget the relativistic time dilation.

Challenge 148, page 181: The formula with $n - 1$ is a better fit. Why?

Challenge 152, page 186: No, only properties of parts of the universe are listed. The universe itself has no properties, as shown in the last volume..

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Challenge 151, page 186: The slowdown goes *quadratically* with time, because every new slowdown adds to the old one!

Challenge 153, page 188: The double of that number, the number made of the sequence of all even numbers, etc.

Challenge 156, page 192: $|z|^2$ is the *determinant* of the matrix $z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Challenge 160, page 192: Use Cantor's diagonal argument, as in challenge 231.

Challenge 162, page 193: Any quaternion $X = ai+bj+ck$ with $a^2+b^2+c^2 = 1$ solves the equation $X^2 + 1 = 0$; the purely imaginary solutions $+i$ and $-i$ are thus augmented by a continuous sphere of solutions in quaternion space.

Challenge 165, page 195: Any rotation by an angle 2π is described by -1. Only a rotation by 4π is described by $+1$; quaternions indeed describe spinors.

Challenge 167, page 197: Just check the result component by component. See also the mentioned reference.

Challenge 169, page 199: For a Gaussian integer $n + im$ to be prime, the integer $n^2 + m^2$ must be prime, and in addition, a condition on n mod 3 must be satisfied; which one and why?

Challenge 172, page 200: The metric is regular, positive definite and obeys the triangle inequality.

Challenge 177, page 221: This could be solved with a trick similar to those used in the irrationality of each of the two terms of the sum, but nobody has found one.

Challenge 178, page 221: There are still many discoveries to be made in modern mathematics, especially in topology, number theory and algebraic geometry. Mathematics has a good future.

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> No man but a blockhead ever wrote except for money.
>
> Samuel Johnson

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John R. Taylor, *An Introduction to Error Analysis: the Study of Uncertainties in Physical Measurements*, 2nd edition, University Science Books, Sausalito, 1997. Cited on page 181.

150 P. J. Mohr & B. N. Taylor, CODATA recommended values of the fundamental physical constants: 1998, *Reviews of Modern Physics* **59**, p. 351, 2000. This is the set of constants resulting from an international adjustment and recommended for international use by the Committee on Data for Science and Technology (CODATA), a body in the International Council of Scientific Unions, which brings together the International Union of Pure and Applied Physics (IUPAP), the International Union of Pure and Applied Chemistry (IUPAC) and other organizations. The website of IUPAC is www.iupac.org. Cited on pages 181 and 182.

151 Some of the stories can be found in the text by N. W. Wise, *The Values of Precision*, Princeton University Press, 1994. The field of high-precision measurements, from which the results on these pages stem, is a world on its own. A beautiful introduction to it is J. D. Fairbanks, B. S. Deaver, C. W. Everitt & P. F. Michaelson, eds., *Near Zero: Frontiers of Physics*, Freeman, 1988. Cited on page 182.

152 The details are given in the well-known astronomical reference, Kenneth Seidelmann, *Explanatory Supplement to the Astronomical Almanac*, 1992. Cited on page 186.

153 For information about the number π, and about some other mathematical constants, the website oldweb.cecm.sfu.ca/pi/pi.html provides the most extensive information and references. It also has a link to the many other sites on the topic, including the overview at mathworld.wolfram.com/Pi.html. Simple formulae for π are

$$\pi + 3 = \sum_{n=1}^{\infty} \frac{n\,2^n}{\binom{2n}{n}} \qquad (151)$$

or the beautiful formula discovered in 1996 by Bailey, Borwein and Plouffe

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) . \qquad (152)$$

The mentioned site also explains the newly discovered methods for calculating specific binary digits of π without having to calculate all the preceding ones. The known digits of π pass all tests of randomness, as the mathworld.wolfram.com/PiDigits.html website explains. However, this property, called *normality*, has never been proven; it is the biggest open question about π. It is possible that the theory of chaotic dynamics will lead to a solution of this puzzle in the coming years.

Another method to calculate π and other constants was discovered and published by D. V. Chudnovsky & G. V. Chudnovsky, The computation of classical constants, *Proceedings of the National Academy of Sciences (USA)* **86**, pp. 8178–8182, 1989. The Chudnowsky brothers have built a supercomputer in Gregory's apartment for about 70 000 euros, and for many years held the record for calculating the largest number of digits of π. They have battled for decades with Kanada Yasumasa, who held the record in 2000, calculated on an industrial supercomputer. However, the record number of (consecutive) digits in 2010 was calculated in 123 days on a simple desktop PC by Fabrice Bellard, using a Chudnovsky formula. Bellard calculated over 2.7 million million digits, as told on bellard.org. New formulae to calculate π are still occasionally discovered.

For the calculation of Euler's constant γ see also D. W. DeTemple, A quicker convergence to Euler's constant, *The Mathematical Intelligencer*, pp. 468–470, May 1993.

Note that little is known about the basic properties of some numbers; for example, it is Challenge 177 r still not known whether $\pi + e$ is a rational number or not! (It is believed that it is not.) Do Challenge 178 s you want to become a mathematician? Cited on page 187.

154 A good reference is the *Encyclopedia of Mathematics*, in 10 volumes, Kluwer Academic Publishers, 1988–1993. It explains most concepts used in mathematics. Spending an hour with it looking up related keywords is an efficient way to get an introduction into any part of mathematics, especially into the vocabulary and the main connections.

The opposite approach, to make things as complicated as possible, is taken in the delightful text by Carl E. Linderholm, *Mathematics Made Difficult*, 1971. Cited on page 189.

155 An excellent introduction into number systems in mathematics, including hyperreal (or nonstandard) numbers, quaternions, octonions, p-adic numbers and surreal numbers, is the book H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel & R. Remmert, *Zahlen*, 3rd edition, Springer Verlag, 1993. It is also available in English, under the title *Numbers*, Springer Verlag, 1990. For a book explaining how to use hyperreals in secondary school, see Helmut Wunderling, *Analysis als Infinitesimalrechnung*, Duden Paetec Schulbuchverlag, 2007. Cited on pages 190, 199, and 200.

156 A. Waser, *Quaternions in Electrodynamics*, 2001. The text can be downloaded from various websites. Cited on pages 192 and 197.

157 S. L. Altman, *Rotations, Quaternions and Double Groups*, Clarendon Press, 1986, and also S. L. Altman, Hamilton, Rodriguez and the quaternion scandal, *Mathematical Magazine* **62**, pp. 291–308, 1988. See also J. C. Hart, G. K. Francis & L. H. Kauffman, Visualzing quaternion rotation, *ACM Transactions on Graphics* **13**, pp. 256–276, 1994. The latter can be downloaded in several places via the internet. Cited on page 195.

158 See the fine book by Louis H. Kauffman, *Knots and Physics*, World Scientific, 2nd edition, 1994, which gives a clear and visual introduction to the mathematics of knots and their main applications to physics. Cited on page 195.

159 Gaussian integers are explored by G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edition, Clarendon Press, Oxford, 1979, in the sections 12.2 'The Rational Integers, the Gaussian Integers, and the Integers', pp. 178–180, and 12.6 'Properties of the Gaussian Integers' pp. 182–183. For challenges relating to Gaussian integers, look at www.mathpuzzle.com/Gaussians.html. Cited on page 199.

160 About transfinite numbers, see the delightful paperback by Rudy Rucker, *Infinity and the Mind – the Science and Philosophy of the Infinite*, Bantam, 1983. Cited on page 199.

161 E. I. Butikov, The rigid pendulum – an antique but evergreen physical model, *European Journal of Physics* **20**, pp. 429–441, 1999. Cited on page 204.

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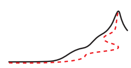

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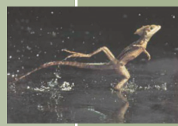

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